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EQUIVARIANT EMBEDDINGS OF G-SPACES

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1. Introduction

Let G denote an infinite topological group with unit e. An *action* of G on a topological space X is a continuous mapping $\pi\colon G\times X \to X$ such that $\pi(e,x) = x$ and $\pi(t,\pi(s,x)) = \pi(ts,x)$ for all $x \in X$ and $s,t \in G$. If π is an action of G on X, the ordered pair $\langle X,\pi\rangle$ is called a *G-space*. If $\langle X,\pi\rangle$ and $\langle Y,\sigma\rangle$ are G-spaces, then a *morphism of G-spaces*, $f\colon \langle X,\pi\rangle \to \langle Y,\sigma\rangle$, is a continuous function $f\colon X \to Y$ such that $f(\pi(t,x)) = \sigma(t,f(x))$ for every $(t,x) \in G\times X$; any mapping satisfying this relation will be called *equivariant*, so that we can speak of equivariant embeddings, etc. In [4], D.H. CARLSON asked whether of each Tychonoff $\mathbb{R}$-space can equivariantly be embedded as a dense subspace in a compact Hausdorff $\mathbb{R}$-space. Motivated by categorical questions, we asked a similar question for G-spaces in [13], and in [14] we characterized the G-spaces for which the answer is "yes", thus generalizing a compactification theorem of R.B. BROOK [3]. In [16] it is shown that this characterization is satisfied by every Tychonoff G-space, provided G is locally compact. In the present paper we shall give a unified approach to this problem and its solution. In particular, the proof will be different from and independent of the results of [13] and [14]. For applications of our compactification theorem, which generalize results from [5], [10] and [11] for certain embedding problems, we refer to [16].

We shall now establish some notation and terminology. If $\langle X,\pi\rangle$ is a G-space, then by $\pi^t x := \pi(t,x) =: \pi_x t$ ($t\in G$, $x\in X$) continuous mappings $\pi^t\colon X \to X$ and $\pi_x\colon G \to X$ are defined. Note that $\pi^e = 1_X$, the identity mapping of X, and that $\pi^{st} = \pi^s \circ \pi^t$ for $s,t \in G$. In particular, it follows that each π^t is a homeomorphism of X onto itself. (Occasionally, we write $\sigma^a(b) := \sigma(a,b) =: \sigma_b(a)$ for arbitrary functions of two variables.)

The symbol $\mathbb{K}$ will always denote either $\mathbb{R}$ or $\mathbb{C}$ (the real or complex number field). If X is any topological space, $C_u(X)$ will denote the Banach algebra of all $\mathbb{K}$-valued *bounded* continuous functions on X, endowed with the supremum norm. A C_1^*-*subalgebra* of $C_u(X)$ is a closed subalgebra of $C_u(X)$ containing the constants and closed under complex conjugation. The constant function on X with value 1 will be denoted u_X.

By a *compactification* of a space X we mean a continuous mapping $f: X \to Y$, where Y is a compact Hausdorff space and $f[X]$ is dense in Y. A *proper compactification* of X is a compactification $f: X \to Y$ such that f is a (dense) embedding of X into Y. Two compactifications $f_i: X \to Y_i$ $(i=1,2)$ are said to be *equivalent* if there is a homeomorphism $g: Y_1 \to Y_2$ such that $f_2 = g \circ f_1$. The following theorem concerning the relationship between C_1^*-subalgebras of $C_u(X)$ and compactifications of X is well-known:

1.1. THEOREM. *Let* X *be a topological space. Then the following statements are true.*

(i) *If* $f: X \to Y$ *is a compactification, then the induced mapping*

$$C(f): h \mapsto h \circ f: C_u(Y) \to C_u(X)$$

is an isometrical isomorphism of the C_1^*-*algebra* $C_u(Y)$ *onto a* C_1^*-*subalgebra of* $C_u(X)$.

(ii) *If* A *is a* C_1^*-*subalgebra of* $C_u(X)$ *then there exists a compactification* $f: X \to Y$ *of* X *such that the range of* $C(f)$ *equals* A; *this condition determines the compactification uniquely, up to equivalence.*

PROOF. (i): easy; see also [8], 4.2.2.
(ii): see [8], 14.2.2. □

We need the following well-known supplements to this theroem:

1.2. PROPOSITION. *A compactification* $f: X \to Y$ *of* X *is proper iff the range* A *of* $C(f)$ *in* $C_u(X)$ *separates points and closed subsets of* X (*i.e. if* $Z \subset X$ *is closed, then* $(\forall x \in X \sim Z)\ (\exists h \in A)\ (h(x)=1 \,\&\, h[Z]=\{0\})$). □

1.3. PROPOSITION. *For* $i = 1,2$, *let* $f_i: X \to Y_i$ *be a compactification of the space* X, *and let* A_i *denote the range of* $C(f_i)$ *in* $C_u(X)$. *The following conditions are equivalent:*

(i) *There exists a continuous mapping* g: $Y_1 \to Y_2$.

(ii) *There exists a linear, multiplicative mapping* T: $A_2 \to A_1$ *such that* $T(u_X) = u_X$.

In that case, g *and* T *are related to each other by*

$$C(f_1)\circ C(g) = T\circ C(f_2),$$

and they determine each other uniquely. In particular, there exists a continuous mapping g: $Y_1 \to Y_2$ *such that* $f_2 = g\circ f_1$ *iff* $A_2 \subseteq A_1$.

PROOF. Use [8], 7.7.1. □

1.4. The uniqueness statement in 1.1(ii) is a direct consequence of the last statement in 1.3 which, in turn, follows from the non-trivial implication (ii) ⇒ (i) in 1.3.

Among the possible applications of 1.1 and 1.3 are the existence proofs of the Stone-Čech compactification for a Tychonoff space X (take $A = C_u(X)$) and of the Bohr compactification for a topological group G (take for A the algebra of all almost periodic functions on G); the universal properties of these compactifications are, of course, consequences of 1.3. In the case of the Bohr compactification of a topological group G, the additional algebraic structure of G is carried over to the compactification by means of 1.3. See [8], §14.7. We shall use a similar procedure for G-spaces in order to obtain (proper) equivariant compactifications.

In accordance with our definitions, an *equivariant compactification* of a G-space $\langle X,\pi\rangle$ is a morphism f: $\langle X,\pi\rangle \to \langle Y,\sigma\rangle$ of G-spaces such that f: $X\to Y$ is a compactification of X; if f: $X \to Y$ is a proper compactification, then we speak of a *proper equivariant compactification*. Following other literature (e.g. [1], [15]), a (proper) equivariant compactification will also be called a *(proper) G-compactification.*

2. Compactifications of G-spaces

2.1. Let $\langle X,\pi\rangle$ be a G-space. Define $\tilde{\pi}$: $G\times C_u(X) \to C_u(X)$ by

$$\tilde{\pi}(t,f) := f\circ\pi^t \quad \text{for } (t,f) \in G\times C_u(X).$$

So $\tilde{\pi}^t = C(\pi^t)\colon C_u(X) \to C_u(X)$, and $\tilde{\pi}^t$ is an isometrical isomorphism of the C_1^*-algebra $C_u(X)$ onto itself such that $\tilde{\pi}^t(u_X) = u_X$. Moreover, $\tilde{\pi}^e$ is the identity mapping of $C_u(X)$ and $\tilde{\pi}^{st} = \tilde{\pi}^t\circ\tilde{\pi}^s$ for $s,t \in G$.

In general, $\tilde{\pi}$ is not continuous on $G\times C_u(X)$ (see 2.3 below). The following lemma gives some information in this respect:

2.2. LEMMA. *Let* $f \in C_u(X)$. *The following conditions are mutually equivalent:*

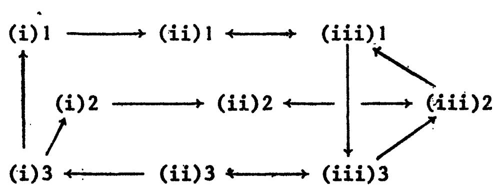
(i) 1. $\tilde{\pi}$ *is continuous at the point* (e,f).
2. $\tilde{\pi}$ *is continuous at some point* (s,f), $s \in G$.
3. $\tilde{\pi}$ *is continuous at all points of* $G\times\{f\}$.

(ii) 1. $\tilde{\pi}_f\colon G \to C_u(X)$ *is continuous at* e.
2. $\tilde{\pi}_f\colon G \to C_u(X)$ *is continuous at some point* $s \in G$.
3. $\tilde{\pi}_f\colon G \to C_u(X)$ *is right-uniformly continuous.*

(iii) 1. $\{f\circ\pi_x\}_{x\in X}$ *is equicontinuous at* e.
2. $\{f\circ\pi_x\}_{x\in X}$ *is equicontinuous at some point* $s \in G$.
3. $\{f\circ\pi_x\}_{x\in X}$ *is right-uniformly equicontinuous on* G.

(In (iii), $C_u(X)$ *has to be considered as a uniform space in the usual way, the uniformity being derived from its norm topology and its additive structure; in* (ii)3 *and* (iii)3, *the right uniformity on* G *has to be considered.)*

PROOF. The following implications are either evident or trivial consequences of the definitions:



□

2.3. EXAMPLE. Consider the G-space $\langle G,\lambda\rangle$, where $\lambda(t,s) := ts$. Then $f \in C_u(G)$ satisfies condition (ii)1 of lemma 2.2 iff f is right-uniformly continuous, that is, iff $f \in RUC(G)$. In general, however, $RUC(G) \neq C(G)$ (cf. [6]).

2.4. Motivated by the terminology which is applicable to example 2.3, we shall say that an element f of $C_u(X)$ is *π-uniformly continuous* whenever it satisfies the conditions of lemma 2.2. The set of all π-uniformly continuous functions on X will be denoted $\pi UC(X)$.

2.5. LEMMA. *If* X *is compact, then* $\pi UC(X) = C(X)$.

PROOF. A straightforward verification of 2.2 (iii)]. □

2.6. PROPOSITION. *Let* $f: \langle X,\pi\rangle \to \langle Y,\sigma\rangle$ *be an equivariant compactification of the* G*-space* $\langle X,\pi\rangle$. *Then the range of* $C(f)$ *is a* $\tilde{\pi}$*-invariant* C_1^**-subalgebra of* $C_u(X)$ *which is contained in* $\pi UC(X)$.

PROOF. Let A be the range of $C(f)$. Then for every $t \in G$,

$$\tilde{\pi}^t \circ C(f) = C(f \circ \pi^t) = C(\sigma^t \circ f) = C(f) \circ \tilde{\sigma}^t.$$

It follows easily, that A is $\tilde{\pi}$-invariant (that is, $\tilde{\pi}^t g \in A$ for every $t \in G$ and $g \in A$). Moreover, $\tilde{\sigma}$ is continuous on $G \times C_u(Y)$ by 2.5 and 2.2, and $C(f)$ is an isometry of $C_u(Y)$ onto A; so the above equalities imply that $\tilde{\pi}$ is continuous on $G \times A$, that is, $A \subseteq \pi UC(G)$. Finally, by 1.1, A is a C_1^*-subalgebra of $C_u(X)$. □

2.7. PROPOSITION. *Let* A *be a* $\tilde{\pi}$*-invariant* C_1^**-subalgebra of* $C_u(X)$, *and suppose* $A \subseteq \pi UC(X)$. *Let* $f: X \to Y$ *be the corresponding compactification of* X *(cf. theorem* 1.1*). Then there exists an action* σ *of* G *on* Y *such that* $f: \langle X,\pi\rangle \to \langle X,\sigma\rangle$ *is an equivariant compactification of* $\langle X,\pi\rangle$.

PROOF. For every $t \in G$, $\tilde{\pi}^t|_A: A \to A$ is a linear and multiplicative isometry of A into itself such that $\tilde{\pi}^t(u_X) = u_X$. By 1.3, there exists a unique continuous mapping $\sigma^t: Y \to Y$ such that $C(f) \circ C(\sigma^t) = \tilde{\pi}^t \circ C(f)$, that is, $\sigma^t \circ f = f \circ \pi^t$. It is easily verified that $\sigma^e = 1_Y$ and that $\sigma^{st} = \sigma^s \circ \sigma^t$ for all $s,t \in G$. It remains to be verified that the mapping $\sigma: (t,y) \mapsto \sigma^t y$: $G \times Y \to Y$ is continuous, and for this it is sufficient to show that $h \circ \sigma: G \times Y \to \mathbb{K}$ is continuous for every $h \in C(Y)$. So fix $(t,y) \in G \times Y$ and $h \in C(Y)$, and note that for any $(s,z) \in G\times Y$

$$|h\circ\sigma(s,z) - h\circ\sigma(t,y)| \leq \|\tilde{\sigma}(s,h) - \tilde{\sigma}(t,h)\| + |\tilde{\sigma}^t h(z) - \tilde{\sigma}^t h(y)|.$$

It is easy to see that $\tilde{\sigma}$: $G\times C_u(Y) \to C_u(Y)$ is continuous (indeed, $\tilde{\pi}$: $G\times A \to A$ is continuous, and $C(f)$: $C_u(Y) \to A$ is an isometry); moreover, $\tilde{\sigma}^t h$: $Y \to \mathbb{K}$ is continuous. Using this, the continuity of $h\circ\sigma$ follows from the above inequality. □

2.8. Let TOP^G be the category of all G-spaces and equivariant continuous mappings. It is not difficult to show that the full subcategory COMP^G of all compact Hausdorff G-spaces is reflective in TOP^G. For details, see [13, subsection 4.3]. This means that for each G-space $\langle X,\pi\rangle$ there exists a "maximal" equivariant compactification f: $\langle X,\pi\rangle \to \langle Y,\sigma\rangle$ with the following universal property: for any equivariant compactification g: $\langle X,\pi\rangle \to \langle Z,\zeta\rangle$ there exists a unique morphism of G-spaces $\bar{g}$: $\langle Y,\sigma\rangle \to \langle Z,\zeta\rangle$ such that $g = \bar{g}\circ f$.
Using 2.6, 2.7 and 1.3, it follows that this maximal G-compactification of $\langle X,\pi\rangle$ corresponds to the largest $\tilde{\pi}$-invariant C_1^*-subalgebra of $C_u(X)$ which is contained in $\pi\mathrm{UC}(X)$. We show that this is the whole of $\pi\mathrm{UC}(X)$:

2.9. PROPOSITION. *The subset* $\pi\mathrm{UC}(X)$ *of* $C_u(X)$ *is a* $\tilde{\pi}$- *invariant* C_1^**-subalgebra of* $C_u(X)$.

PROOF. Obviously, $\pi\mathrm{UC}(X)$ is a subalgebra of $C_u(X)$ containing u_X and invariant under complex conjugation. In order to show that it is a C_1^*-subalgebra of $C_u(X)$ (i.e. that it is closed in $C_u(X)$), consider $f \in C_u(X) \sim \pi\mathrm{UC}(X)$. Now there exists $\varepsilon > 0$ such that for every neighbourhood U of e in G there are $t \in U$ and $x \in X$ with $|f\circ\pi(t,x)-f(x)| \geq \varepsilon$. Then for any $g \in C_u(X)$ with $\|f-g\| < \varepsilon/3$ we have

$$|g\circ\pi(t,x) - g(x)| \geq |f\circ\pi(t,x) - f(x)| - 2\|f-g\| > \varepsilon/3,$$

whence $g \notin \pi\mathrm{UC}(X)$. This shows that $\pi\mathrm{UC}(X)$ is closed in $C_u(X)$. Finally, in order to prove that $\pi\mathrm{UC}(X)$ is invariant, consider $h \in \pi\mathrm{UC}(X)$ and $s \in G$. By 2.2 (ii)2 there is a neighbourhood V of e in G such that

$$|f(\pi(t,x)) - f(\pi(s,x))| < \varepsilon$$

for all $t \in G$ with $t \in Us$. Writing $f(\pi(t,x)) = (\tilde{\pi}^s f)(\pi(s^{-1}t,x))$, and substituting u for $s^{-1}t$, we see that

$$|\tilde{\pi}^s f(\pi(u,x)) - \tilde{\pi}^s f(x)| < \varepsilon$$

for all $u \in s^{-1}Us$. Since $s^{-1}Us$ is a neighbourhood of e, it follows from 2.2 (ii)1 that $\tilde{\pi}^s f \in \pi UC(X)$. □

2.10. PROPOSITION. *If* G *is locally compact Hausdorff and* X *is a non-compact Tychonoff space, then* $\pi UC(X)$ *contains a* $\tilde{\pi}$*-invariant* C_1^**-subalgebra* A *of* $C_u(X)$ *which separates points and closed subsets of* X, *such that* $d(A) \le \max\{d(G), w(X)\}$.
(Here $d(A)$ is the density character of A, i.e. the least cardinal number of a dense subset of A, and $w(X)$ is the weight of X, the least cardinal number of an open (sub)base for X).

PROOF. (outline). Let $\mathcal{B}$ denote a local base at e in G such that each $U \in \mathcal{B}$ has compact closure in G, with cardinality $|\mathcal{B}| = \ell w(G)$, the local weight of G. Fix for every $U \in \mathcal{B}$ a continuous function ψ_U: $G \to [0,3]$ such that $\psi_U(e) = 0$ and $\psi_U(t) = 3$ for $t \in G \sim U$. Clearly, for every $U \in \mathcal{B}$ the set $A_U := \{t \in G : \psi_U(t) \le 2\}$ is compact. In addition, let $F \subseteq C_u(X)$ be a subset which separates points and closed subsets of X; F can be chosen such that $|F| = w(X)$ (use [7], theorem 2.3.8). Set

$$\tilde{f}_U(x) := \inf_{t \in G}\{\psi_U(t) + f(\pi^t x)\}$$

for every $x \in X$, $f \in F$ and $U \in \mathcal{B}$. Clearly, the infimum can be taken over the compact set A_U. It follows that $\tilde{f}_U \in C_u(X)$. It is not difficult to show that for every $t \in G$

$$|\tilde{f}_U(\pi(t,x)) - \tilde{f}_U(x)| \le \max\left\{\inf_{u \in G}\{\psi_U(ut^{-1}) - \psi_U(u)\},\ \inf_{u \in G}\{\psi_U(ut) - \psi_U(u)\}\right\}.$$

Since ψ_U is left-uniformly continuous, it follows that $\tilde{f}_U$ satisfies condition 2.2 (ii)1, so that $\tilde{f}_U \in \pi UC(X)$. Finally, the set $F^* := \{\tilde{f}_U : (f,U) \in F \times \mathcal{B}\}$ separates points and closed subsets of X. Let A be the C_1^*-subalgebra of $C_u(X)$

generated by the set $\cup\{\tilde{\pi}^t[F^*]: t \in G\}$. Then it is not difficult to see that A has all required conditions. In particular, if S is a dense subset of G, the set of all linear combinations with rational coefficients of finite products of elements of $\cup\{\tilde{\pi}_f[S]: f \in F^*\}$ is dense in A. So indeed $d(A) \leq$ $\leq \aleph_0 d(G)|F^*| = d(G)\ \ell w(G)\ w(X) \leq \max\{w(G), w(X)\}$ (since X is non-compact, $w(X) \geq \aleph_0$). □

2.10. THEOREM. *If* G *is a locally compact topological group then every* G*-space* $\langle X,\pi\rangle$ *with* X *a non-compact Tychonoff space has a proper* G*-compactification* $f: \langle X,\pi\rangle \to \langle Y,\sigma\rangle$ *such that* $w(Y) \leq \max\{L(G/G_0), w(X)\}$, *where* G_0 *is the stabilizer of* $\langle X,\pi\rangle$.
(Here $L(Z)$ denotes the Lindelöf (or: covering) degree of the space Z, i.e. the least cardinal number κ such that each open cover of Z has a subcover of cardinality $\leq \kappa$).

PROOF. By definition, $G_0 := \{t \in G: (\forall x \in X)(\pi^t x = x)\}$. Then G_0 is a closed normal subgroup of G, so G/G_0 is a locally compact Hausdorff topological group, which acts in a natural way on X, thus defining a G/G_0-space $\langle X,\pi'\rangle$. By 2.9, 2.7 and 1.2, there exists a proper G/G_0-compactification $f: \langle X,\pi'\rangle \to \langle Y,\sigma'\rangle$ of $\langle X,\pi'\rangle$, and we may assume that $w(Y) = d(C_u(Y)) \leq \max\{w(G/G_0), w(X)\}$ (cf. also [8], 7.6.5). However, G/G_0 acts effectively on X, so $\ell w(G/G_0) \leq w(X)$ (cf. [12]). Since $w(G/G_0) = \max\{\ell w(G/G_0), L(G/G_0)\}$, it follows that $w(Y) \leq$ $\leq \max\{L(G/G_0), w(X)\}$. Finally, Y can easily be turned into a G-space $\langle Y,\sigma\rangle$, and it is then not difficult to show that $f: X \to Y$ is also equivariant with respect to the actions π and σ on X and Y respectively. So $f: \langle X,\pi\rangle \to \langle Y,\sigma\rangle$ has all required properties. □

2.11. REMARK. In a similar way it can be shown that if G is locally compact the "maximal" G-compactification of a Tychonoff G-space (cf. 2.8) is proper.

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