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CARTESIAN-CLOSED COREFLECTIVE SUBCATEGORIES OF UNIF.

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In this paper we consider the problem of determining which coreflective subcategories of Unif, the category of uniform spaces and uniformly continuous maps, are cartesian-closed. Most of the results here are obtained either by imposing additional conditions on the category, or by considering cartesian-closedness with respect to special exponential operations. A few examples are given, but the results indicate that cartesian-closedness is a fairly restrictive condition on coreflective subcategories of Unif.

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Background.

We assume that the reader is familiar with the basic definitions of category theory as presented in the early chapters of [7] or [11].

We assume throughout that all subcategories are full, non-trivial, and isomorphism-closed. The theorem, due to Kennison in [10], that a subcategory of Unif is coreflective if and only if it is closed under the formation of uniform sums and quotient spaces will be used repeatedly. In Unif, a coreflection modifies a given uniform space into a finer space, preserving the underlying set.

If $\mathbb{S} \subseteq$ Unif, the coreflective hull of $\mathbb{S}$ is denoted $\varphi(\mathbb{S})$, and it consists of all quotients of sums of members of $\mathbb{S}$. Let $\mathbf{C} = \varphi(\mathbb{S})$; the coreflection cX of a uniform space X is the quotient of the sum of all $\mathbb{S}$-subspaces of X, under the natural map. If $X \in$ Unif, $S \subseteq X$, and $S \in \mathbb{S}$ (with uniformity on S inherited from X), then S is also a subspace of cX. It follows that cX is projectively generated by the set of all functions with domain X and range in Unif whose restrictions to all $\mathbb{S}$-subspaces of X are uniformly continuous. Also, $X \in \varphi(\mathbb{S})$ if and only if each of these functions is uniformly continuous on the whole space X. Coreflections of this sort are described in [3] and [5].

Definition. A category $\mathbf{C}$ having all finite products is cartesian-closed if, for each object X in $\mathbf{C}$, the functor $P_X\colon \mathbf{C} \to \mathbf{C}$ defined by $P_X(Y) = X \times Y$ has a right adjoint $E_X\colon \mathbf{C} \to \mathbf{C}$.

The values of E_X are denoted by $E_X(Y) = Y^X$. The objects Y^X are called exponential objects, and they satisfy the condition $\mathbf{C}(Z \times X, Y) = \mathbf{C}(Z, Y^X)$, for all X, Y, Z in $\mathbf{C}$, where the equality represents a bijection between the hom sets which is natural in Z and in Y.

We will use a variation of this definition, based on the following fact: the existence of a right adjoint to P_X is equivalent to the existence of objects Y^X, for each Y in $\mathbf{C}$, and of $\mathbf{C}$-morphisms $e_Y\colon Y^X \times X \to Y$, such that $\{e_Y : Y \in \mathbf{C}\}$ is natural in X and such that each map e_Y satisfies the condition:

(u) Given $Z \in \mathbf{C}$ and a $\mathbf{C}$-morphism $f\colon Z \times X \to Y$, there exists a unique $\mathbf{C}$-morphism $g\colon Z \to Y^X$ which makes the following diagram commute:

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\;e_Y\;} & Y \\ {\scriptstyle g \times i_X}\nwarrow & & \nearrow{\scriptstyle f} \\ & Z \times X & \end{array}$$

(Here i_X denotes the identity map on X.)

The primary example of a cartesian-closed category is that of sets and functions. The exponentials X^Y are the hom sets from Y to X, and the maps $e_Y\colon X^Y \times Y \to X$ are the common evaluation maps defined by $e_Y(f,y) = f(y)$.

Now let $\mathbf{C}$ be a coreflective subcategory of Unif. Then $\mathbf{C}$ has finite products, denoted $X \otimes Y$, and these are the coreflections in $\mathbf{C}$ of the usual uniform products $X \times Y$.

Notation. If A is a set, let $|A|$ denote the cardinality of A. It is easy to prove the following:

Proposition. Let $\mathbf{C}$ be a cartesian-closed subcategory of Unif. Then $|X^Y| = |\mathbf{C}(Y,X)|$ for all X, Y in $\mathbf{C}$.

Hence, we assume without loss of generality that if $\mathbf{C}$ is a cartesian-closed coreflective subcategory of Unif, then each X^Y has underlying set $\mathbf{C}(Y,X) = \mathcal{U}(Y,X)$, the set of all uniformly continuous maps from Y to X. Also, the map $e_Y\colon X^Y \otimes Y \to X$ is then the ordinary evaluation map.

A more detailed discussion of cartesian-closedness may be found in [11]. Cartesian-closed topological and uniform categories have been studied by a variety of authors recently, and some of these are listed in the bibliography.

Some results for subcategories of Unif

A uniform space may be thought of as a set X together with a collection of covers satisfying the axioms for a uniformity as presented in Chapter 1 of [8]. Alternatively, we may consider a uniformity to be a collection of pseudometrics, as described in Chapter 15 of [4].

If $\mathcal{U}$ is a cover of a set X and $A \subseteq X$, let $st(A,\mathcal{U}) = \cup\{U \in \mathcal{U}: A \cap U \neq \emptyset\}$. If $x \in X$, we write $st(x,\mathcal{U})$ for $st(\{x\},\mathcal{U})$.

If $\mathcal{S}$ is a family of uniform spaces define the uniformity of uniform convergence on $\mathcal{S}$-sets for $\mathcal{U}(X,Y)$ to have subbase consisting of all covers $\mathcal{U}(S,\mathcal{V})$ for $S \subseteq X$, $S \in \mathcal{S}$, and uniform cover $\mathcal{V}$ of Y, where $\mathcal{U}(S,\mathcal{V}) = \{U_f: f \in \mathcal{U}(X,Y)\}$, and $U_f = \{g \in \mathcal{U}(X,Y): g(x) \in st(f(x),\mathcal{V})$ for all $x \in S\}$.

The subbasic covers $\mathcal{U}(S,\mathcal{V})$ all have star-refinements of the same type ([9], Chapter 7), and in fact they form a base if $\mathcal{S}$ is closed under finite unions.

If $X,Y \in \varphi(\mathcal{S})$, we will assume throughout that $\mathcal{U}(X,Y)$ is equipped with the uniformity of uniform convergence on $\mathcal{S}$-sets. If $\mathcal{C}$ is coreflective and $X,Y \in \mathcal{C}$, and if no mention is made of a generating family $\mathcal{S}$ for $\mathcal{C}$, then $\mathcal{U}(X,Y)$ has the uniformity of uniform convergence on all of X.

Let $\mathcal{P}$ be the class of precompact uniform spaces.

Theorem 1. Let $\mathcal{S} \subseteq \mathcal{P}$ and $\mathcal{C} = \varphi(\mathcal{S})$. Then $e: c\mathcal{U}(Y,X) \otimes Y \to X$ is uniformly continuous for all $X \in \mathcal{C}$ and $Y \in \mathcal{S}$. If $\mathcal{S}$ is quotient-invariant, then e is uniformly continuous for all $X,Y \in \mathcal{C}$.

Proof. Let $X,Y \in \mathcal{C}$. Since $c\mathcal{U}(Y,X) \otimes Y = c(c\mathcal{U}(Y,X) \times Y)$, it suffices to show that $e|_Q$ is uniformly continuous for any $\mathcal{S}$-subspace Q of $c\mathcal{U}(Y,X) \times Y$. Let S be the projection of Q onto $c\mathcal{U}(Y,X)$. If $\mathcal{S}$ is quotient-invariant, let T be the projection of Q onto Y; otherwise, assume $Y \in \mathcal{S}$ and let $T = Y$. In any case, S is precompact, $T \in \mathcal{S}$, and Q is a subspace of $S \times T$.

Now S is a precompact family in $\mathcal{U}(Y,X)$. It follows that $\{f|_T: f \in S\}$ is an equiuniform family on T: Let $\mathcal{W}$ be a uniform cover of X and let $\mathcal{V}$ star-refine $\mathcal{W}$. The cover $\mathcal{U}(T,\mathcal{V})$ is uniform in $\mathcal{U}(Y,X)$, and so its restriction to the precompact subspace S is finite. Let $\mathcal{U}(T,\mathcal{V})|_S = \{U_1,\dots,U_n\}$ and let $f_i \in U_i \cap S$, for $i \leqq n$. Let $\mathcal{Z}$ be a uniform cover of Y which refines $f_i^{-1}(\mathcal{V})$, for all $i \leqq n$. Then $\mathcal{Z}|_T < f|_T^{-1}(\mathcal{W})$ for all $f \in S$: Let $f \in S$ and $Z \in \mathcal{Z}$. There exists $i \leqq n$ such that $f \in U_i$, so that $f(y) \in st(f_i(y),\mathcal{V})$ for every $y \in T$. Choose $V \in \mathcal{V}$ such that $Z \subseteq f_i^{-1}(V)$, since $\mathcal{Z} < f_i^{-1}(\mathcal{V})$. Then $f(Z \cap T) \subseteq st(V,\mathcal{V}) \subseteq W$, for some $W \in \mathcal{W}$. Hence $Z \cap T \subseteq f^{-1}(W)$, so $\mathcal{Z}|_T < f|_T^{-1}(\mathcal{W})$. Therefore,

$\{f|_T : f \in S\}$ is equi-uniform on T.

By Exercise G p. 239 in [9], it now follows that $e|_{S \times T}$ is uniformly continuous. Hence, $e|_Q$ is uniformly continuous.

Corollary 1. Let $\mathcal{S} \subseteq \mathcal{P}$, and suppose $\varphi(\mathcal{S})$ is cartesian-closed with exponential objects X^Y. Then:

(a) The identity $i: c\mathcal{U}(Y,X) \to X^Y$ is uniformly continuous for any $X \in \varphi(\mathcal{S})$, $Y \in \mathcal{S}$.

(b) If $\mathcal{S}$ is quotient-invariant, $i: c\mathcal{U}(Y,X) \to X^Y$ is uniformly continuous for all $X,Y \in \varphi(\mathcal{S})$.

This corollary follows immediately from Theorem 1, using condition (u) for the evaluation maps in $\varphi(\mathcal{S})$.

Theorem 2. Let $\mathcal{C} = \varphi(\mathcal{S})$. Suppose $X \times Y = X \otimes Y$ for all $X,Y \in \mathcal{S}$. If $X,Y,Z \in \mathcal{C}$ and $f: Z \otimes Y \to X$ is uniformly continuous, then the associated map $\bar{f}: Z \to c\mathcal{U}(Y,X)$ is also uniformly continuous. (Here $\bar{f}(z)(y) = f(z,y)$.)

Proof. It suffices to show that $\bar{f}|_S$ is uniformly continuous for any $\mathcal{S}$-subspace S of Z. Let $\mathcal{U}(T,\mathcal{V})$ be a uniform cover of $\mathcal{U}(Y,X)$, where $T \in \mathcal{S}$, $T \subseteq Y$, and $\mathcal{V}$ is a uniform cover of X. If $U_g \in \mathcal{U}(T,\mathcal{V})$, then

$$\bar{f}^{-1}(U_g) = \{z \in S: \bar{f}(z)(y) \in st(g(y),\mathcal{V}) \ \forall\, y \in T\}$$

so

$$\bar{f}^{-1}(U_g) = \{z \in S: f(z,y) \in st(g(y),\mathcal{V}) \ \forall\, y \in T\}.$$

By assumption, $S \times T \in \varphi(\mathcal{S})$, so it is a subspace of $Z \otimes Y$. Hence, $f|_{S \times T}$ is uniformly continuous. Define $f_y: S \to X$ by $f_y(z) = f(y,z)$, for $y \in T$. The family $\{f_y : y \in T\}$ is equi-uniform on S, by Theorem 21, Chapter 3 of [8]. Thus, there is a uniform cover $\mathcal{W}$ of S such that $\mathcal{W} < f_y^{-1}(\mathcal{V})$ for all $y \in T$.

We will show that $\mathcal{W} < \bar{f}|_S^{-1}(\mathcal{U}(T,\mathcal{V}))$: Let $W \in \mathcal{W}$, and choose $z_0 \in W$. Let $k = \bar{f}(z_0)$, and let $U_k \in \mathcal{U}(T,\mathcal{V})$. Then $\bar{f}|_S(W) \subseteq U_k$: To show this, let $z \in W$, and let $y \in T$. Then $\bar{f}(z)(y)$ and $\bar{f}(z_0)(y)$ belong to some $V_y \in \mathcal{V}$, where $f_y(W) \subseteq V_y$. Then $\bar{f}(z)(y) \in st(k(y),\mathcal{V})$. So, $\bar{f}(z) \in U_k$, and $\bar{f}(W) \subseteq U_k$. Therefore, $\mathcal{W} < \bar{f}|_S^{-1}(\mathcal{U}(T,\mathcal{V}))$.

Therefore, $\bar{f}|_S$ is uniformly continuous, so $\bar{f}: Z \to \mathcal{U}(Y,X)$ is uniformly continuous. Finally, since $Z \in \mathcal{C}$, $\bar{f}: Z \to c\mathcal{U}(Y,X)$ is also uniform.

Corollary 2. Let $\mathcal{C} = \varphi(\mathcal{S})$, and suppose $X \times Y = X \otimes Y$ for all $X,Y \in \mathcal{S}$. If $\mathcal{C}$ is cartesian-closed with exponentials X^Y, then the identity map $i: X^Y \to c\mathcal{U}(Y,X)$ is uniformly continuous for $X,Y \in \mathcal{C}$.

Proof. The evaluation $e: X^Y \otimes Y \to X$ is uniformly continuous, so by Theorem 2 the associated map $\bar{e}: X^Y \to c\mathcal{U}(Y,X)$ is also uniform, and $\bar{e} = i$.

Corollary 3. Suppose $\mathcal{S} \subseteq \mathcal{P}$, $\mathcal{S}$ is quotient-invariant, and $X \times Y = X \otimes Y$ for all $X, Y \in \mathcal{S}$. Then $\varphi(\mathcal{S})$ is cartesian-closed with exponentials $X^Y = c\mathcal{U}(Y,X)$.

Examples. $\varphi(\mathcal{P})$ is cartesian-closed, and so is $\varphi(\mathcal{K})$, where $\mathcal{K}$ is the class of all compact uniform spaces. The members of these classes are sometimes referred to as precompactly-generated spaces and compactly-generated spaces, respectively. (These examples are also given in [13].)

A recent theorem of M. D. Rice gives the converse of Corollary 2:

Theorem. (Rice) Suppose $\varphi(\mathcal{S})$ is cartesian-closed and $i: X^Y \to c\mathcal{U}(Y,X)$ is uniformly continuous for all exponentials X^Y. Then $X \times Y = X \otimes Y$ for all $X, Y \in \mathcal{S}$. (Equivalently, $X \times Y \in \varphi(\mathcal{S})$ for all $X, Y \in \mathcal{S}$.)

The results above are combined to give the following statements:

Corollary 4. Suppose $\varphi(\mathcal{S})$ is cartesian-closed with exponentials X^Y. The following are equivalent:

(a) $i: X^Y \to c\mathcal{U}(Y,X)$ is uniformly continuous for all $Y \in \mathcal{S}$, $X \in \varphi(\mathcal{S})$.

(b) $X \times Y = X \otimes Y$ for all $X, Y \in \mathcal{S}$.

(c) $i: X^Y \to c\mathcal{U}(Y,X)$ is uniformly continuous for all $X, Y \in \varphi(\mathcal{S})$.

Corollary 5. Suppose $\mathcal{C}$ is coreflective and cartesian-closed. The following are equivalent:

(a) $\mathcal{C}$ is finitely productive.

(b) $i: X^Y \to c\mathcal{U}(Y,X)$ is uniformly continuous for all exponentials X^Y.

This last statement follows immediately from Corollary 4 by letting $\mathcal{S} = \varphi(\mathcal{S}) = \mathcal{C}$. It should be emphasized that in Corollary 5 $\mathcal{U}(Y,X)$ has the uniformity of uniform convergence on all of Y, while in Corollary 4 $\mathcal{U}(Y,X)$ has the uniformity of uniform convergence on $\mathcal{S}$-subspaces of Y. The latter structure is coarser than the former one.

If $X \in$ Unif, let pX be the precompact reflection of X. Let I be the unit interval with the usual uniformity.

Theorem 3. Suppose $\mathcal{C}$ is coreflective, cartesian-closed, and cI is precompact. Then $D \times pD \notin \mathcal{C}$ for any infinite uniformly discrete space D.

Proof. Suppose $D \times pD \in \mathcal{C}$. Then $pD \in \mathcal{C}$, since pD is a quotient space of $D \times pD$. Let $d: D \times D \to cI$ be the function defined by $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$. Then the associated map $\bar{d}: D \to cI^D$ is uniformly continuous. Since cI is precompact, the image of $\bar{d}$ is contained in cI^{pD}, so that $\bar{d}: D \to cI^{pD}$ is also uniformly continuous on the discrete space D. Then $\bar{d} \in (cI^{pD})^D = cI^{pD \otimes D}$, and $pD \otimes D = pD \times D$, so that $d: pD \times D \to cI$ is uniformly continuous. If $\mathfrak{U}$ is the uniform cover of cI consisting of all spheres of radius $1/2$, then $d^{-1}(\mathfrak{U})$ is not a uniform cover of $pD \times D$, contradicting uniform continuity of d.

Therefore, $D \times pD \notin \mathcal{C}$ if D is discrete.

The hypothesis that cI be precompact in Theorem 3 will be satisfied by any coreflection c: Unif $\to \mathcal{C}$ which preserves precompactness.

Since any non-trivial coreflective subcategory of Unif contains all the discrete spaces, Theorem 3 has the following consequence:

Corollary 6. If $\mathcal{C}$ is a coreflective, finitely productive subcategory of Unif and $\mathcal{P} \subseteq \mathcal{C}$, then $\mathcal{C}$ is not cartesian-closed.

These results imply, for example, that $D \times pD \notin \varphi(\mathcal{P})$ for any infinite discrete space D, so that $\varphi(\mathcal{P})$ is certainly not finitely productive.

Also, it follows from Theorem 3 that the coreflective hull of the class of all separable uniform spaces (spaces with countable bases) is not cartesian-closed.

The class of all uniformly discrete spaces forms a coreflective, finitely productive, cartesian-closed subcategory of Unif, and it might be the only such subcategory. It appears, then, that coreflective, cartesian-closed subcategories of Unif would not, in general, be finitely productive. So, one should try to find an appropriate generating proper subclass $\mathcal{S}$ for a given cartesian-closed category and then describe the exponentials in terms of uniform convergence on $\mathcal{S}$-subspaces, as in $\varphi(\mathcal{K})$ and $\varphi(\mathcal{P})$.

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