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Fixed point theorems for metric space mappings

In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. 173--177.

Persistent URL: `http://dml.cz/dmlcz/700629`

FIXED POINT THEOREMS FOR METRIC SPACE MAPPINGS

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1. Let (X,d) be a complete metric space, $T: X \to X$ be a mapping (not necessarily continuous) such that

$$d(T^p x, T^p y) \leqslant a\, d(x,y)$$

where $p = p(x)$, $0 \leqslant a < 1$. Then [2] T has a unique fixed point ξ and $\lim_{n\to\infty} T^n x = \xi$ for each $x \in X$. The following theorem is a direct generalization of this result.

Theorem 1.

Let (X,d) be a complete metric space, $T: X \to X$ be a mapping such that

$$d(T^p x, T^p y) \leqslant \omega(d(x,y))$$

where $p = p(x)$, ω is a nondecreasing semicontinuous from the right function, $\omega(r) < r$ for $r > 0$, $r - \omega(r)$ is unbounded for $r \to \infty$.

Then T has a unique fixed point ξ and $\lim_{n\to\infty} T^n x = \xi$ for each $x \in X$.

Proof. Let $x \in X$ and $d_0 = \max_{0 \leqslant k \leqslant p} d(x, T^k x)$ where $p = p(x)$. Then

$$d(T^p x, T^{np+k} x) \leqslant \omega(d(x, T^{(n-1)p+k} x))$$

and so

$$d(x, T^{np+k} x) \leqslant d(x, T^p x) + d(T^p x, T^{np+k} x) \leqslant$$
$$\leqslant d_0 + \omega(d(x, T^{(n-1)p+k} x)).$$

Therefore for $0 < k < p$

$$d(x, T^{p+k} x) \leqslant d_0 + \omega(d_0) = d_1$$
$$d(x, T^{2p+k} x) \leqslant d_0 + \omega(d_1) = d_2$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$d(x, T^{np+k} x) \leqslant d_0 + \omega(d_{n-1}) = d_n$$
$$\cdots\cdots\cdots\cdots\cdots$$

Since $r - \omega(r)$ is unbounded for $r \to \infty$ there exists $r_0 > 0$ such that $d_0 \leqslant r_0 - \omega(r_0)$. Since ω is nondecreasing $d_0 + \omega(r) \leqslant r_0$ for each $r \leqslant r_0$. Then

$$d_0 \leqslant r_0$$
$$d_1 = d_0 + \omega(d_0) \leqslant r_0$$

. . . .

$$d_n = d_0 + \omega(d_{n-1}) \leqslant r_0$$

. . . .

and therefore the sequence $(T^n x)_{n=0}^{\infty}$ is bounded. The further reasoning is a slight modification of [2]. It is interesting to note that the conditions of nondecreasing of function ω and unboundedness of $r - \omega(r)$ for $r \to \infty$ are essential, what follows from the examples.

Example 1.

Let $X = \{\ln n\}_{n=1}^{\infty}$, $d(x,y) = |x-y|$

$$\omega(r) = \ln \frac{1+e^r}{2}, \quad p(\ln n) = n$$

$$T(\ln n) = \ln(n+1).$$

It is easy to see that T satisfies all conditions of the theorem 1 but one, $r - \omega(r)$ is bounded for $r \to \infty$ $(r - \omega(r) < \ln 2)$

Example 2.

Let $X = \{n\}_{n=1}^{\infty}$, $d(x,y) = |x-y|$

$$T(n) = n+1.$$

Consider a integer value function λ

$$\lambda(1) = 0$$

$$2^{2^{\lambda(x)-1}} \leqslant x < 2^{2^{\lambda(x)}} \qquad \text{for} \quad x \geqslant 2$$

Let
$$\bar{d}(x,y) = 2^{-2^{-(y-x)+\lambda(x)}} \qquad \text{for} \quad y > x$$

$$d(x,y) = \min_{\substack{x = x_0 \\ y = x_n}} \sum_{i=0}^{n-1} \bar{d}(x_i, x_{i+1})$$

where $x = x_0 < x_1 < \ldots < x_n = y$.

It is easy to see that (X, d) is a complete metric space.

Let

$$\omega(r) = \begin{cases} r^2 & \text{for } 0 < r < 1 \\ \frac{1}{2} r & \text{for } r \geqslant 1 \end{cases}, \quad p(x) = \max_{0 \leqslant z \leqslant y(x)} \left\{ 2^{2^{\lambda(x)}} - z \right\}$$

where $y(x) = 2^{2^{\lambda(x)+1}}$.

Then T satisfies all conditions of the theorem 1 but one, ω is not nondecreasing.

2. In [3] the fixed point theorem has been proved for $T: X\to X$ satisfying the inequalities

$$\alpha d(x,y)+\beta d(Tx,Ty)+\gamma[d(x,Tx)+d(y,Ty)]+$$
$$+\delta[d(x,Ty)+d(y,Tx)]\geqslant 0$$
$$\alpha+\beta+2\gamma<\min(0,-2\delta)$$
$$\beta+\gamma+\delta<0.$$

There exists the following converse of this result.

Theorem 2.

Let (X,d) be a complete metric space, $T: X\to X$ be a mapping such that

1) there exists $\lim_{n\to\infty} T^n x = q(x)$ for each $x\in X$ and is a fixed point for T.

2) for each $A\subset J_T=\{x\,|\,Tx=x\}$, $\varepsilon>0$ there exist n_0, $\eta>0$ such that $n\geqslant n_0 \Longrightarrow T^n(\mathcal{U}_\eta(A))\subset \mathcal{U}_\varepsilon(A)$.

3) there exists a neighbourhood $\mathcal{U}$ of J_T such that

$$\forall\varepsilon\ \exists n_0\ \forall x\in\mathcal{U}\ \ (n\geqslant n_0 \Longrightarrow T^n x\in \mathcal{U}_\varepsilon(J_T)).$$

Then there exist a metric $\tilde{d}$ equivalent to the metric d on X and numbers α, β, γ, δ, such that $(X,\tilde{d})$ is a complete metric space

$$\alpha+\beta+2\gamma<\min(0,-2\delta) \tag{1}$$
$$\beta+\gamma+\delta<0 \tag{2}$$
$$\alpha\tilde{d}(x,y)+\beta\tilde{d}(Tx,Ty)+\gamma[\tilde{d}(x,Tx)+\tilde{d}(y,Ty)]+ \tag{3}$$
$$+\delta[\tilde{d}(x,Ty)+\tilde{d}(y,Tx)]\geqslant 0$$

for each x, y from X.

Proof. Let us consider d^* which has been defined in [5].

It proves to be pseudometric under the conditions of our theorem. Let $\tilde{d}(x,y)=d^*(x,y)+d(q(x),q(y))$. It easy to see that $\tilde{d}$ is a metric on X equivalent to the metric d and $(X,\tilde{d})$ is a complete metric space. Let $\alpha=1$, $\beta=-6$, $\gamma=-6$, $\delta=8$, $\kappa=\frac{1}{20}$ being a number such that [5]

$$d^*(Tx,Ty)\leqslant \kappa\, d^*(x,y).$$

These numbers satisfy (1), (2) and we need to check only the condition (3)

$$\alpha\tilde{d}(x,y)+\beta\tilde{d}(Tx,Ty)+\gamma[\tilde{d}(x,Tx)+\tilde{d}(y,Ty)]+\delta[\tilde{d}(x,Ty)+\tilde{d}(y,Tx)]=$$

$$=\alpha d^*(x,y)+\beta d^*(Tx,Ty)+\gamma[d^*(x,Tx)+d^*(y,Ty)]+$$
$$+\delta[d^*(x,Ty)+d^*(y,Tx)]+(\alpha+\beta+2\gamma)d(q(x),q(y)).$$

Substitute $\alpha=1$, $\beta=-6$, $\gamma=-6$, $\delta=8$ and use inequalities

$$d^*(x,Tx)\leqslant d^*(x,Ty)+d^*(Tx,Ty)$$
$$d^*(y,Ty)\leqslant d^*(y,Tx)+d^*(Tx,Ty)$$
$$d^*(Tx,Ty)\leqslant\tfrac{1}{20}d^*(x,y).$$

Then

$$d^*(x,y)-6d^*(Tx,Ty)-6[d^*(x,Tx)+d^*(y,Ty)]+8[d^*(x,Ty)+$$
$$+d^*(y,Tx)]+11d(q(x),q(y))\geqslant d^*(x,y)-\tfrac{6}{20}d^*(x,y)-$$
$$-6[d^*(x,Ty)+\tfrac{1}{20}d^*(x,y)+d^*(y,Tx)+\tfrac{1}{20}d^*(x,y)]+$$
$$+8[d^*(x,Ty)+d^*(y,Tx)]+11d(q(x),q(y))=\tfrac{2}{5}d^*(x,y)+$$
$$+2d^*(x,Ty)+2d^*(y,Tx)+11d(q(x),q(y))\geqslant 0$$

3. Iterative test (for contractive mappings) is conclusive [6] for (X,d) provided for each contractive selfmap T , if T has a fixed point ξ , then the sequence $(T^n)_{n=0}^{\infty}$ converges for each x (necessarily to the fixed point ξ). There are examples of metric spaces for which iterative test is not conclusive [1]. It is known that for each dense in R set iterative test is conclusive. On the other hand the following result can be proved.

Theorem 3.

For each dense in R^2 countable set iterative test is not conclusive.

Proof. Let $X_0=\{(x,y)\,|\,y\neq 0\}\cup\{(0,0)\}\subset R^2$

Consider co-ordinates u,v defined by equalities

$$\frac{x^2}{1+u^2}+\frac{y^2}{u^2}=1$$

$$\frac{x^2}{v^2}-\frac{y^2}{1-v^2}=1\quad,\quad -1<v<1$$

and define a mapping $T\colon X_0\longrightarrow X_0$ setting

$$T(u,v)=(\tfrac{u}{2},v).$$

It is a contractive mapping on X_0 . For each M on y-axis the sequence $(T^nM)_{n=0}^{\infty}$ converges but for M which does not belong to y-axis the sequence $(T^nM)_{n=0}^{\infty}$ does not converge. Therefore on

(X,d) iterative test is not conclusive.

Let X be a dense in R^2 countable set. Then x-axis can be choosen such that there is no point of X on x-axis but $(0,0)$ and therefore we can suppose $X \subset X_0$. Let $T: X_0 \to X_0$ be the above defined mapping. In general case $T(X) \not\subset X$ but we can change T in a suitable manner and find $T': X_0 \to X_0$ such that $T'(X) \subset X$.

4. The detailed proof can be found in [4].

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