Jiří Vilímovský

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UNIFORMLY CONTINUOUS BANACH VALUED MAPPINGS

J. VILÍMOVSKÝ

Praha

Several coreflective classes of (Hausdorff) uniform spaces connected with multiplying and extension properties of Banach valued mappings are described. The proofs concerning real-valued functions will appear in [1], the proofs of the other results will appear elsewhere.

We shall denote t_f the coreflector onto the class of all topologically-fine uniform spaces. For X, Y uniform spaces we put U(X,Y) the set of all uniformly continuous mappings of X into Y. The space X will be called $Y - t_f$ if $U(X,Y) = U(X,t_fY)$. The class of all spaces having this property is coreflective (see [4]). Denoting R the real line with the usual metrisable uniformity, one can prove the following:

Proposition 1. U(X,R) is a ring under usual operations if and only if $f^2 \in U(X,R)$ for each $f \in U(X,R)$.

If α is a cardinal, we denote by $H(\alpha)$ the hedgehog over α, that is the set of all $\langle a,x \rangle$, $a \in \alpha$, $0 \leq x \leq 1$, where $\langle a,0 \rangle = \langle b,0 \rangle$ for all $a,b \in \alpha$, with the metric $D(\langle a,x \rangle, \langle a,y \rangle) = |x - y|$ and $D(\langle a,x \rangle, \langle b,y \rangle) = x + y$ for $a \neq b$.

ω denotes simultaneously the countable cardinal and the countable uniformly discrete space, I stands for a onedimensional compact interval.

The results concerning mappings into finite-dimensional spaces are contained in the following two theorems:

Theorem 1. The following coreflective properties of a uniform space X are equivalent:

(1) U(Y,R) is a ring for each subspace Y of X.

(2) X is $H(\omega)-t_f$.

(3) X is $H(\alpha)-t_f$ for every cardinal α.

(4) X is hereditarily $R-t_f$.

(5) X is hereditarily R^n-t_f for all natural numbers n.

(6) X is hereditarily $(I \times \omega)-t_f$.

(7) If $\{f_n\}$ is a sequence of bounded uniformly continuous real-valued functions such that the supports of f_n's form a uniformly discrete family in X, then Σf_n is uniformly continuous (or equivalently, the family $\{f_n\}$ is uniformly equicontinuous).

(8) If $\{A_n\}$ is a uniformly discrete countable family in X, for all n, $A_n = \bigcup \{A_{n,i} ; i \leq k_n\}$ and each finite family

$\{A_{n,i} ; i \leqq k_n\}$ is uniformly discrete, then the family $\{A_{n,i}\}$ is uniformly discrete, then the family $\{A_{n,i}\}$ is uniformly discrete. (I.e. countable uniformly discrete unions of finite uniformly discrete families are uniformly discrete.)

Theorem 2. The class $H(\omega)-t_f$ described in Theorem 1 is the largest coreflective class contained in $\mathcal{K}$ where for $\mathcal{K}$ can be taken one of the following classes:

a) The class of all uniform spaces X such that all uniformly continuous real-valued functions from each subspace Y of X extend to uniformly continuous functions on X .
b) The class of all uniform spaces X such that all real-valued functions from countable uniformly discrete subspaces extend to uniformly continuous functions on X .
c) The class of all uniform spaces X such that all Banach valued mappings from uniformly discrete subspaces extend uniformly continuously to the whole space X .

The class described in a) is called RE by J. Isbell. Several sufficient coreflective conditions have been known for a space to be in RE. J. Isbell [3] proved that locally fine spaces have this property and recently Z. Frolík [2] showed that even all sub-inversion-closed spaces are in RE. Here we have described the largest such coreflective class and we note that it is much larger than the class of sub-inversion-closed spaces.

Now we turn our attention to infinite-dimensional case. As a result similar to Proposition 1 we obtain:

Proposition 2. The following coreflective conditions on a uniform space X are equivalent:

(1) $U(X,E)$ is a module over $U(X,R)$ for each Banach space E .
(2) For any cardinal α the mapping $x \mapsto f(x) \cdot \| f(x) \|_\infty$ is uniformly continuous, whenever $f \in U(X, l_\infty(\alpha))$.

We recall that the mapping between two uniform spaces is called distally continuous if the preimages of uniformly discrete families remain uniformly discrete.

Now trying to find a theorem analogous to Theorem 1 for the general case of Banach valued mappings, we obtain the following two results:

Theorem 3. The following coreflective properties of a uniform space X are equivalent:

(1) If $\{A_n\}$ is a uniformly discrete countable family in X and for all n there is $A_n = \bigcup\{A_{n,\iota} ; \iota \in J\}$ with $\{A_{n,\iota} ; \iota \in J\}$ uniformly discrete in X , then the family $\{A_{n,\iota}\}$ is uniformly

discrete in X.

(2) For any Banach space E, $f_n \in U(X,E)$ a sequence of bounded mappings with $\{\mathrm{supp}\, f_n ; n \in \omega\}$ uniformly discrete, the mapping f_n is distally continuous.

(3) For any Banach space E, f_n a sequence of bounded distally continuous E-valued mappings with $\{\mathrm{supp}\, f_n; n \in \omega\}$ uniformly discrete in X, the mapping $\sum f_n$ is distally continuous.

(4) For any Banach algebra E and any subspace Y of X, f^2 is distally continuous for $f \in U(X,E)$.

Theorem 4. The following coreflective conditions on a uniform space X are equivalent:

(1) $U(Y,E)$ is a module over $U(Y,R)$ for every subspace Y of X and E a Banach space.

(2) X is RE and $U(X,E)$ is a module over $U(X,R)$ for each Banach space E.

(3) $U(X,E)$ is a module over $U(X,R)$ for all Banach spaces E and X fulfils one of the conditions in Theorem 1.

(4) For every Banach space E, $f_n \in U(X,E)$ bounded uniformly continuous with $\{\mathrm{supp}\, f_n ; n \in \omega\}$ uniformly discrete in X the mapping $\sum f_n$ is uniformly continuous.

(5) $U(X',E)$ is a ring for each subspace X' of X and E a Banach algebra.

(6) Each cover of the form $\{U_n \cap V_a^n ; n \in \omega, a \in A\}$ is a uniform cover of X provided that $\{U_n\}_n$ is a countable finite-dimensional uniform cover and $\{V_a^n\}_a$ is a uniform cover of X for all n.

We recall that the space X is called sub-metric-fine if for all complete metrisable space M there is $U(X,M) = U(X,t_f M)$. One can show that all sub-metric-fine spaces form a coreflective class which is contained in the class described in Theorem 4, and that these two classes are not equal.

Now we present two sufficient coreflective conditions concerning extensions of uniformly continuous Banach valued mappings. Note that it remains open, whether these coreflections are the largest ones or whether such largest coreflective classes even exist.

Theorem 5. Let X be sub-metric-fine. Then for each Banach space E and uniformly continuous E-valued mapping defined on any subspace Y of X there is a uniformly continuous extension of it on the whole space X.

Theorem 6. Let X be a uniform space enjoying the properties from Theorem 4 and let E be a Banach space having its closed balls uniformly injective (for example spaces $l_\infty(\alpha)$ for some cardinal α). Then all uniformly continuous E-valued mappings extend from arbitrary subspaces on the whole space X .

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Institute of Mathematics
of the Czechoslovak Academy of Sciences
Praha