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SOME PRODUCT THEOREMS

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In [1], it was shown that a product of k-spaces is a k-space if and only if the locally compact modifications of these spaces form a topologically coherent pair. In this paper, we improve upon these results for products of k-spaces, and also generalize them to include other classes of topological spaces which can be characterized as the topological modifications of certain classes of convergence spaces.

We use the term *convergence space* in the filter sense, and ask the reader to refer to [1] for background information, terminology, and definitions not given here. The term *space* will always mean convergence space. If X and Y are spaces, then $X \leq Y$ (read "Y finer than X" or "X coarser than Y") means that X and Y have the same underlying set, and $\mathfrak{F} \to x$ in Y implies $\mathfrak{F} \to x$ in X. The *topological modification* λX of X is the finest topological space coarser than X. A space is said to be *compactly generated* if each convergent filter contains a filter converging to the same point which has a filter base of compact sets. The complactly generated spaces include the class of all locally compact, regular, Hausdorff spaces and, in particular, the locally compact modifications of Hausdorff topological spaces.

1. A pair of spaces X, Y is said to be *topologically coherent* if $\lambda(X \times Y) = \lambda X \times \lambda Y$ (i.e., the topological modification of the product equals the product of the topological modifications).

<u>Proposition 1.1.</u> If X is a Hausdorff space and the pair X, X is topologically coherent, then λX is also Hausdorff.

<u>Proposition 1.2.</u> If X is a compactly generated space and Y is any space, then $X \times \lambda Y \geq \lambda(X \times Y)$.

<u>Proposition 1.3.</u> If X and Y are spaces such that λX and Y are both compactly generated, then the pair X, Y is topologically coherent.

<u>Proposition 1.4.</u> Let X and Y be spaces such that $X \times \lambda Y \geq \lambda(X \times Y)$. Then the pair X, Y is topologically coherent if and only if the pair X, λY is topologically coherent.

<u>Corollary 1.5.</u> If X is compactly generated, then, for any space Y, the pair X, Y is topologically coherent if and only if the pair X, λY is topologically coherent.

Theorem 1.6. A Hausdorff space X has the property that the pair X, Y is topologically coherent for any space Y if and only if X is a locally compact topological space.

2. Let CONV be the category consisting of Hausdorff spaces and continuous maps. A functor T : CONV $\to$ CONV is said to be a *modification functor* if the following conditions are satisfied: (1) For all objects X is CONV, X and TX have the same underlying set; (2) For all maps f in CONV, $Tf = f$. The topological modification functor λ, if defined in the obvious way for maps, is a simple example of a modification functor.

A topological space X is said to be *T-determined* relative to a modification functor T if $X = \lambda TX$. Clearly, all topological Hausdorff spaces are T-determined if $T = \lambda$. A more interesting example is obtained if, for each space X, TX is the locally compact modification of X. In this case, the T-determined spaces are the k-spaces (see [1]).

We shall be interested in modification functors T which satisfy the following additional conditions:

(U) For each X in CONV, $X \leq TX$;

(P) For each pair of spaces X, Y, $T(X \times Y) \geq TX \times TY$;

(LC) For each X in CONV, TX is locally compact.

The topological modification functor satisfies only condition (P), whereas the locally compact modification functor (henceforth denoted by L) satisfies all three conditions. The next theorem is a generalization of Theorem 3.4 of [1].

Theorem 2.1. Let T be a modification functor satisfying (U) and (P), and let X and Y be T-determined topological Hausdorff spaces. Then $X \times Y$ is T-determined if and only if the pair TX, TY is topologically coherent.

Theorem 2.2. Let T be a modification functor satisfying (U), (P), and (LC), and let X and Y be T-determined topological Hausdorff spaces. Then the following statements are equivalent:

(a) $X \times Y$ is T-determined.

(b) The pair TX, TY is topologically coherent;

(c) The pair X, TY is topologically coherent;

(d) The pair TX, Y is topologically coherent.

Theorem 2.3. If T is a modification functor satisfying (U), (P), and (LC), X is a locally compact T-determined topological

Hausdorff space, and Y is any T-determined space, then $X \times Y$ is T-determined.

Theorems 2.1 and 2.3 are proved in [1] in the special case when $T = L$ is the locally compact modification functor and the T-determined spaces are the k-spaces. Theorem 2.2, on the other hand, is a new result, even for k-spaces.

For any space X, let SX be the *sequential modification of* X defined as follows: $\mathfrak{F} \to x$ in SX if and only if there is a filter $\mathcal{S} \to x$ in X such that $\mathfrak{F} \geq \mathcal{S}$, and $\mathcal{S}$ is generated by a sequence. It is easy to see that the sequential modification functor S satisfies conditions (U), (P), and (LC). The S-determined topological spaces are the *sequential topological spaces,* in which the closed sets can be characterized as those sets which contain all of their sequential limit points.

Corollary 2.4. If X and Y are Hausdorff sequential topological spaces (k-spaces), then $X \times Y$ is a sequential space (k-space) if and only if either of the pairs SX, Y or X, SY (LX, Y or X, LY) is topologically coherent.

REFERENCE

[1] D. C. Kent and G. D. Richardson, "Locally Compact Convergence Spaces", *Michigan Math. J.* 22 (1975) 353-360.