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TWO-SIDED NONSINGULAR TRANSFORMATIONS

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## 1. Introduction

Let $(X,\Sigma,\mu)$ and $(Y,\Phi,\nu)$ be finite measure spaces. A subset Z of X is called null if $Z\in\Sigma$ and $\mu(Z) = 0$. We denote by $\mathbb{A} = \mathbb{A}(X)$ and $\mathbb{B} = \mathbb{B}(Y)$ the corresponding quotient Boolean measure algebras modulo the null sets. By the Stone representation theorem, there exist unique to within homeomorphism 0-dimensional compact Hausdorff spaces H, K such that $\mathbb{A}, \mathbb{B}$ are Boolean isomorphic to the algebras $\mathcal{A}, \mathcal{B}$ of all closed and open subsets of H, K, respectively. Since any family of disjoint subsets of positive measure is countable, both H and K have the Souslin property and are Stonian. The measures $\mu$ and $\nu$ are represented in a natural way by positive normal category Radon measures on H and K, respectively ([1], Section 4).

The analogy between the structure of a measure space and the corresponding Stonian space can be extended to comprise also transformations of such spaces. A $(\Sigma,\Phi)$-measurable transformation $\tau: X \longrightarrow Y$ is called <u>nonsingular</u> if $\tau^{-1}(Z)$ is a null set whenever Z is null. Any nonsingular transformation $\tau$ induces an order continuous Boolean homomorphism $T_\tau : \mathbb{B} \longrightarrow \mathbb{A}$ by means of the formula $T_\tau[Z] = [\tau^{-1}(Z)]$, where $[\;]$ denotes the the equivalence class modulo null sets. On the other hand, for any Boolean homomorphism $T: \mathbb{B} \longrightarrow \mathbb{A}$ there exists a unique continuous map $k: H \longrightarrow K$ inducing T in the sense that $T\hat{B} = (k^{-1}(B))^\wedge$, where $^\wedge$ denotes the Stonian isomorphisms $\mathcal{A} \longrightarrow \mathbb{A}$ and $\mathcal{B} \longrightarrow \mathbb{B}$. In case of an order continuous T, the map k is open (see [7], III 9.5). Consequently, to any nonsingular transformation $\tau$ there corresponds a unique continuous open map $k_\tau: H \longrightarrow K$ such that for any B in $\mathcal{B}$ and any Z in $\Phi$ satisfying $[Z] = \hat{B}$, we have $(k_\tau^{-1}(B))^\wedge = [\tau^{-1}(Z)]$.

The nonsingular transformation $\tau: X \longrightarrow Y$ will be called two-sided nonsingular if $\tau(Z)$ is null whenever Z is null. The following weaker property seems to be of interest: a nonsingular transformation $\tau$ will be called essentially two-sided nonsingular if the above condition holds for all null sets Z disjoint with some fixed null set $X_0$.

The measure space X is called Borel if $(X,\Sigma)$ is Borel isomorphic to a Borel subset of the unit interval. Two-sided nonsingular transformations of Borel spaces occurred in author's earlier work [3]. The aim of this note is to describe such transformations in terms of topological properties of the corresponding maps between the associated Stonian spaces (Corollary 2). We also present a related result, a measure theoretic characterization of essentially two-sided nonsingular transformations of Borel spaces with non-atomic measures (Corollary 1).

## 2. Open maps in Stonian spaces

All topological spaces considered in this paper are assumed to be Hausdorff.

The following lemma can be inferred from [7], III 9, yet, for the sake of completeness, we provide a straightforward proof.

Lemma 1. Let H be a 0-dimensional compact space and K a Stonian space. For any continuous map $k: H \longrightarrow K$ the following conditions are equivalent:

(i) $k^{-1}(A)$ is meager whenever A is meager,

(ii) $k^{-1}(B)$ is rare whenever B is rare,

(iii) if $\operatorname{int} U \neq 0$ then $\operatorname{int} k(U) \neq 0$,

(iv) if V is open then $k(V)$ is open.

(The last condition says that k is an open map.)

Proof. (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii) are clear. To prove (iii) $\Rightarrow$ (iv) we can assume that $0 \neq V \subset H$ is closed and open. Then $k(V)$ is closed and $W = \overline{\operatorname{int} k(V)}$ is a closed and open subset of $k(V)$. Therefore $k^{-1}(W)$ is closed and open and so is the difference $V - k^{-1}(W)$. Since the image of the latter is contained in the rare set $k(V) - W$, we must have $V - k^{-1}(W) = 0$, implying $k(V) = W$.

A continuous map $k: H \longrightarrow K$ satisfying the above equivalent conditions can be viewed as topologically nonsingular. If, in addition, $k(A)$ is meager for any meager $A$ then $k$ will be called <u>topologically two-sided nonsingular</u>. If $k(A)$ is meager for meager sets $A$ disjoint with some fixed rare set $M_0$ then $k$ will be called <u>topologically essentially two-sided nonsingular</u>.

<u>Lemma 2.</u> Let $H$ be a compact space and $K$ a Baire space. If a continuous open map $k: H \longrightarrow K$ is topologically nonsingular, then $k(H_1) = k(H)$ implies $k(\overline{\text{int } H_1}) = k(H)$ for any closed subset $H_1$ of $H$.

Proof. Letting $H_0 = \overline{\text{int } H_1}$ we have $k(H_1 - H_0) \supset k(H_1) - k(H_0)$. Since $H_1 - H_0$ is rare, $k(H_1) - k(H_0)$ is meager. Since $H_0$ is compact, $k(H_0)$ is closed. Since $k(H_1) = k(H)$ is open, also $k(H_1) - k(H_0)$ is open, implying $k(H_0) = k(H_1) = k(H)$.

Let us recall that a map $k: H \longrightarrow K$ is called <u>irreducible</u> if $k(H_0) \neq k(H)$ for any closed proper subset $H_0$ of $H$. It is well known and is, in fact, a direct application of the Kuratowski- Zorn lemma that if $H$ is compact then there exists a closed subset $H_0$ of $H$ such that the restriction $k|H_0$ is irreducible and $k(H_0) = k(H)$.

<u>Lemma 3.</u> Suppose $H$ is Stonian and $k: H \longrightarrow K$ is a topologically essentially two-sided nonsingular map. Then there exists a partition of $H$ consisting of a rare subset $H_0$ and closed and open subsets $H_t$, such that, for each $t \neq 0$, $k$ maps $H_t$ homeomorphically onto a closed and open subset $K_t$ of $K$.

Proof. Let $M_0 \subset H$ be the exceptional rare subset for $k$. For any non-empty closed and open subset $M$ of $H - M_0$ there exists a closed subset $M_1$ of $M$ such that $k|M_1$ is irreducible and $k(M_1) = k(M)$. By Lemma 2 we have $k(\overline{\text{int } M_1}) = k(M_1)$ and, by the irreducibility of $k|M_1$, $M_1 = \overline{\text{int } M_1}$. Since $H$ is Stonian, $M_1$ has to be open. By the easy observation that any continuous open irreducible map is one-to-one, k maps $M_1$ homeomorphically onto k(M). By the Kuratowski - Zorn lemma there exists a maximal family of disjoint closed and open subsets of H, such that k restricted to any set in this family is a homeomorphism. By the first part of our proof, the union of this family is dense in H. The complement of this union, $H_0$, is the rare set needed for the partition.

Let us note that if H has the Souslin property (any family of disjoint open subsets is countable) then the partition obtained in Lemma 3 has to be countable. For countable partitions we have also the following converse result.

Lemma 4. Suppose H and K are Stonian spaces and $k: H \longrightarrow K$ is a continuous map. If there exists a countable partition for k as in Lemma 3 then k is topologically essentially two-sided nonsingular.

Proof. Since the counterimage of every rare set is rare, k is an open map by virtue of Lemma 1. Let now $H_0$ the rare set in the partition and $H_n$, n = 1, 2,..., the remaining closed and open sets. If $A \subset H - H_0$ is a meager set then $A_n = A \cap H_n$ is meager for any n, so the $k(A_n)$ are meager and, finally, $k(A) = \bigcup k(A_n)$ is meager.

By the last two lemmas we obtain the following theorem.

Theorem 1. Suppose H, K are Stonian spaces and H has the Souslin property. Then a continuous map $k: H \longrightarrow K$ is topologically essentially two-sided nonsingular if and only if the following condition holds:

(a) there exists a countable partition $H_0$, $H_1$, ... of H with $H_0$ rare and the remaining $H_n$ closed and open, such that, for all $n \neq 0$, k maps $H_n$ homeomorphically onto a closed and open subset $K_n$ of K.

## 3. Measurable transformations

Let us recall that a measurable transformation from X onto Y is called a point isomorphism if $\tau$ is one-to-one and both $\tau$ and $\tau^{-1}$ are measurable and nonsingular. It is worth to note that this notion depends on the ideals of null sets rather than the measures and their numerical values. For any measurable transformation $\tau : X \longrightarrow Y$ we consider the following property of $\tau$, which is analogous to (a) in Theorem 1:

($\alpha$) there exists a countable measurable partition $X_0, X_1, \ldots$ of X with $X_0$ null and the remaining $X_n$ of positive measure, such that, for all $n \neq 0$, $\tau$ maps $X_n$ isomorphically onto a measurable non-null subset $Y_n$ of Y.

The term "measurable" is always used in the sense of the $\sigma$-algebras $\Sigma$ and $\Phi$. It is a routine to show that if $\tau$ satisfies ($\alpha$) then the associated continuous map $k_\tau$: $H \longrightarrow K$ of the corresponding Stonian spaces satisfies (a) of Theorem 1. In order to obtain the converse, we will assume that both X and Y coincide with the measure theoretic product $I^{\mathfrak{m}}$ of $\mathfrak{m}$ copies ($\mathfrak{m}$ any positive cardinal number) of the unit interval endowed with the Borel $\sigma$-algebra and Lebesgue measure. By $M_{\mathfrak{m}}$ we denote the assosiated Stonian space. It should be noted that $M_{\mathfrak{m}}$ are mutually homeomorphic for all $1 \leq \mathfrak{m} \leq \omega_0$.

Theorem 2. Let X = Y = I , H = K = M . A continuous map $k: H \longrightarrow K$ satisfies the condition (a) of Theorem 1 if and only if $k = k_\tau$ for some measurable transformation $\tau: X \longrightarrow Y$ satisfying the condition ($\alpha$).

Proof. We need only to prove the necessity. Let $H_0, H_1, \ldots$ be a partition of H as in (a). To $H_1, H_2, \ldots$ there corresponds a partition $a_1, a_2, \ldots$ of 1 of the quotient Boolean algebra $\mathbb{A}(X)$. Also to every $K_n = k(H_n)$ there corresponds an element $b_n$ of $\mathbb{B}(Y)$. This correspondence is to be understood in the sense of the Stonian isomorphism $\wedge$. For $n \geqslant 1$, the restriction $k|H_n$ induces a Boolean isomorphism between the ideals generated by $b_n$ and $a_n$. Let now $X'$ and $Y'$ be measure theoretic unions of two disjoint copies of X and Y, respectively. The Boolean algebras $\mathbb{A}(X')$ and $\mathbb{B}(Y')$ are still homogeneous and isomorphic to $\mathbb{A}(I^{\mathfrak{m}})$. Therefore there exists an isomorphism $S_n: \mathbb{B}(Y') \longrightarrow \mathbb{A}(X')$ extending $T_n$. By Maharam's theorem [5] there exists a point isomorphism $\sigma_n: X' \longrightarrow Y'$ inducing $S_n$. We choose a measurable subset $Y_n'$ of Y satisfying $[Y_n'] = b_n$. Now by putting $X_n = X \cap \sigma_n^{-1}(Y_n')$ and $Y_n = \sigma_n(X_n)$ we obtain a point isomorphism $\sigma_n | X_n = \tau_n$ that induces $T_n$. The $a_n$ are disjoint, so by an easy inductive argument we can arrange that the $X_n$ be also disjoint. Letting $X_0 = X - \cup X_n$ we obtain the required partition of X. Now it suffices to take any measurable transformation $\tau_0$ from $X_0$ into Y and define $\tau = \cup \tau_n$.

In general $\tau$ is not uniquely determined by $k_\tau$, even to within equivalence modulo almost everywhere and even in case $k_\tau$ is a homeomorphism, see [5], p. 702 . However, in the separable case, i.e. for $1 \leq \mathfrak{m} \leq \omega_0$, it is easyto see that $k_\sigma = k_\tau$ implies $\sigma = \tau$ a.e.

## 4. Separable case

Suppose X and Y are Borel spaces. For the sake of simplicity let us assume that both $\mu$ and $\nu$ are non-atomic. Now, the quotient algebras $\mathbb{A}$ and $\mathbb{B}$ are Boolean isomorphic to the quotient algebra $\mathbb{A}(I)$ of the unit interval. By Sikorski's result [8], 6.2, there exist point isomorphisms of X and Y onto I. Therefore we can simply assume that both X and Y coincide with I. Here the results of the preceding section apply with $\mathfrak{m} = 1$ (and also $1 \leq \mathfrak{m} \leq \omega_0$). The following measure theoretic fact is analogous to Theorem 1.

Theorem 3. Let $\tau: X \longrightarrow Y$ be a measurable transformation, X = Y= I. Then $\tau$ is essentially two-sided nonsingular if and only if $\tau$ satisfies the condition ($\alpha$) of Section 3.

Proof. The sufficiency is clear. In order to prove the necessity we can assume that $\tau$ is two-sided nonsingular. By Mackey's version of von Neumann's selection lemma ([4], 6.3) there exist a null set $Z \subset Y$ and a Borel set $X_1 \subset X - \tau^{-1}(Z)$, such that $\tau|X_1$ is a point isomorphism onto the set $\tau(X_1) = \tau(X - \tau^{-1}(Z))$. Let us note that $X_1$ has to be of positive measure. Now following along the lines the argument of Lemma 3 we obtain a maximal family $X_1, X_2, \ldots$ of disjoint subsets of positive measure, such that the restrictions $\tau|X_n$ are point isomorphisms and $X - \bigcup X_n$ is a null set.

As a corollary we obtain the announced in Section 1 characterization of essentially two-sided nonsingular transformations.

Corollary 1. The following conditions are equivalent for any nonsingular transformation $\tau: I \longrightarrow I$:

(1) there exists two-sided nonsingular transformation $\tau_1 = \tau$ a.e.,

(2) there exists a Borel partition $X_1, X_2, \ldots$ of I and a transformation $\tau_2 = \tau$ a.e. such that $\tau_2|X_n$ is one-to-one for all n,

(3) there exists a countable-to-one transformation $\tau_3 = \tau$ a.e.,

(4) there exists a bimeasurable (i.e. measurable and taking Borel sets into Borel sets) transformation $\tau_4 = \tau$ a.e.

Proof. Any one-to-one nonsingular transformation $\sigma: I \longrightarrow I$ is essentially two-sided nonsingular. Indeed, $\sigma$ is bimeasurable and by the Kuratowski - Zorn lemma there exists a maximal necessarily countable, but possibly void family of pairwise disjoint non-null images $Y_n$ of null sets $Z_n$. Letting $X_0 = \bigcup Z_n$ it is easyto see that $\sigma | I - X_0$ is a two-sided nonsingular transformation.

Now (1) $\Leftrightarrow$ (2) follows from Theorem 3, (2) $\Rightarrow$ (3) is trivial, (3) $\Rightarrow$ (2) is an immediate consequence of the Luzin theorem ([2],V 46, p. 296), and (2) $\Leftrightarrow$ (4) follows from Purves' result on bimeasurable functions [6], p.149.

The following is a consequence of Theorems 2 and 3, and the remark at the end of Section 3.

Corollary 2. Let $\tau$: I $\longrightarrow$ I be a nonsingular transformation. Then $\tau$ is essentially two-sided nonsingular if and only if $k_\tau$ is topologically essentially two-sided nonsingular.

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