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FREDHOLM POINTS OF COMPACTLY PERTURBED BOUNDED LINEAR OPERATORS

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When speaking about compact perturbations of symmetric operators one always has to do with the celebrated H.Weyl's theorem ([4,p.391-395],[2,p.243-244]). Using a definition of the essential (limit) spectrum $L\sigma(T)$ of the operator T given in [4] a version of Weyl's theorem for bounded operators can be stated as follows ([4,p.395]) .

T h e o r e m A . Let V be a bounded symmetric operator and U a compact one both operating on a Hilbert space H . Then $L\sigma(T) = L\sigma(V)$, where $T = U + V$.

Here we used the following definition. We say that $\beta \in L\sigma(A)$ if it fulfils one of the following conditions:

(a) β belongs to the continuous spectrum $C\sigma(A)$.

(b) β is a limit point of the point spectrum $P\sigma(A)$.

(c) β is an eigenvalue having an infinite multiplicity.

If V is assumed to be symmetric Gochberg - Krein present the following result ([1,Theorem 5.1]) .

T h e o r e m B . Let V be a self-adjoint operator and U a relatively compact operator with respect to V . Then the part of the spectrum of $T = U + V$ located outside the real axis consists of isolated eigenvalues to each of which there corresponds a finite-dimensional generalized eigenspace $N(T)$ and the following relation holds

$$H = N(T) \oplus R(T)$$

where $TR(T) \subset R(T)$ and the bounded inverse to $(\beta I - T)$ exists on $R(T)$ for every complex β .

Remarkable is that neither the Weyl's theorem nor the generalized version in [2] and [1] give information concerning the part of the spectrum which is outside $L\sigma(T) = L\sigma(V)$ and concerning the location of it.

The aim of this communication is to present some simple conditions upon the operators V and U in order to ensure the peripheral part $\pi\sigma(T)$ of the spectrum of T to be a discrete set of poles of the resolvent operator. Some sufficient conditions are then derived to guarantee e.g. that the peripheral part of the spectrum of T contains exactly one simple pole of the resolvent operator of T possessing a one-dimensional eigenspace. The latter situation is typical for applications to some problems of classical and quantum physics. We note that some of our results, similarly as in [2] and [1] apply in Banach spaces as well.

In the beginning we show an example which illustrates the fact that some conditions have to be imposed upon the operators V and U in order to guarantee a prescribed form of the set $\pi\sigma(T) \setminus \sigma(V)$.

Let $H = L^2(0,1)$ be the Hilbert space of classes of Lebesgue square integrable functions on $[0,1]$. We define f by setting

$$f(s) = \begin{cases} 0 & \text{if } s \in \left[0,\frac{1}{2}\right], \\ 2s-1 & \text{if } s \in \left[\frac{1}{2},1\right], \end{cases}$$

and

$$U(s,t) = \begin{cases} \frac{1}{2} & \text{for } s,t \in \left[0,\frac{1}{2}\right] \\ 0 & \text{for } s \in \left(\frac{1}{2},1\right] , \ t \in [0,1] \\ & \text{and } s \in \left[0,\frac{1}{2}\right] , \ t \in \left(\frac{1}{2},1\right] . \end{cases}$$

Let

$$Ux = y \Longleftrightarrow y(s) = \int_0^1 U(s,t)x(t)dt$$

and

$$Vx = y \Longleftrightarrow y(s) = f(s)x(s) , \ s \in [0,1] ,$$

and $T = U + V$.

Obviously, $r(T) = r(V) = 1 = \sup \text{ess}\{ f(s):s \in [0,1] \}$. It follows that $\pi\sigma(T) = \{r(T)\} = \{1\}$, however $\beta = 1$ is not isolated with respect to $\sigma(T)$, $1 \in C\sigma(T)$.

<u>Remark.</u> We note that the support supp Ux and supp Vx are disjoint:

$$(Tx)(s) = \begin{cases} (Ux)(s) & s \in \left[0,\frac{1}{2}\right] , \\ (Vx)(s) & s \in \left[\frac{1}{2},1\right] . \end{cases}$$

This example suggests that the value $x'(Ux)$ should be positive for vectors x in a neighborhood of a suitable approximate eigenvector corresponding to $r(V)$, where the functional x' is appropriately chosen.

Let Y be a real Banach space, Y' its dual and $B(Y)$ the space of bounded linear endomorphisms of Y into Y ; the norms in Y' and $B(Y)$ being given by $\|y'\| = \sup\{|y'(x)| : x \in Y, \|x\| \leq 1$ and $\|T\| = \sup\{\|Tx\| : x \in Y, \|x\| \leq 1\}$ respectively.

We assume that $K \subset Y$ is a generating and normal cone in Y, i.e. K satisfies (i) - (vi):

(i) $K + K \subset K$,

(ii) $\beta K \subset K$, β real, $\beta \geqq 0$,

(iii) $K \cap (-K) = 0$,

(iv) $K - K = Y$,

(v) $\exists \delta > 0 : \forall x,y \in K \Longrightarrow \|x + y\| \geqq \delta \|x\|$,

(vi) $x_n \in K$, $\|x_n - x\| \to 0 \Longrightarrow x \in K$.

It follows from (i) - (vi) that

$$K' = \{ y' \in Y' : y'(y) \geqq 0 \ \forall \ y \in K \}$$

is a cone in Y' and K' fulfils the conditions (i) - (vi) as well.

In case that Y is a Hilbert space we denote the inner product in Y by (x,y), $x,y \in Y$, and denote this space by the symbol H. We also consider the dual space H' as identical with H associating to every $y' \in H'$ the representative $y_{y'} \in H$ according to the Riesz representation theorem: $y'(x) = (x, y_{y'})$.

We denote by $Y^{\S}$ the complex extension of Y , i.e. $z \in Y^{\S}$ if and only if $z = x + iy$, $x,y \in Y$, $i^2 = -1$. The norm in $Y^{\S}$ is defined by

$$\|z\| = \sup \{ \|(\cos \beta)x + (\sin \beta)y\| : 0 \leqq \beta = 2\pi \} .$$

If $Y = H$ is a Hilbert space then $H^{\S}$ also may have Hilbert space structure

$$(z,w) = [(x,u) + (y,v)] + i[(y,u) - (x,v)] ,$$

where $z = x + iy$ and $w = u + iv$, $x, y, u, v \in H$.

Let $T \in B(Y)$. We let $T^{\S}z = Tx + iTy$ for $z = x + iy, x,y \in Y$, and call $T^{\S}$ the complex extension of T .

Let $\sigma(T^{\S})$ be the spectrum of $T^{\S}$ and let $r(T^{\S}) = \sup\{|\beta| : \beta \in \sigma(T^{\S})$ be its spectral radius. By definition, we let

$\sigma(T) = \sigma(T^{\S})$ and $r(T) = r(T^{\S})$.

Let $T \in B(Y)$. Then the set $\pi\sigma(T) = \{\beta \in \sigma(T) : |\beta| = r(T)\}$ is never empty and we call it the peripheral (part of) spectrum of T .

By I we denote the identity operator and by $R(\beta,T) = (\beta I-T)^{-1}$ the resolvent operator of T .

We call an operator $T \in B(Y)$ positive (more precisely K-positive) [3] if $Tx \in K$ whenever $x \in K$; a positive operator T is called indecomposable (with respect to the cone K) [5], if to every pair $x \in K \setminus 0$, $x' \in K' \setminus 0$ there is an index $p = p(x,x')$ such that $x'(T^p x) > 0$. An indecomposable operator T is called primitive (K-primitive), if for every $x \in K \setminus 0$ there is a positive integer $p = p(x)$ such that $x'(T^k x) > 0$ for every $x' \in K' \setminus 0$ and $k \geqq p$. Note that I.Sawashima in [5] uses the concept of a semi-non-support operator for an indecomposable one and the strict non-support operator for the primitive one.

We say that y is a quasiinterior element of the cone K if $x'(y) > 0$ for all $x' \in K' \setminus 0$; a linear form $x' \in K'$ is called strictly positive if $x'(x) > 0$ for all $x \in K \setminus 0$.

We write $\langle x,x' \rangle$ in place of $x'(x)$.

As typical we show the following three theorems.

<u>T h e o r e m 1 .</u> Let U and V , both in $B(Y)$, be K-positive operators. Let U be compact and let $t > r(U + V)$. We also assume that for every $\beta > 0$ there exists a $v'_\beta \in K'$ such that

$$R(t,V)v'_\beta = \frac{1}{t-r(V)} v'_\beta + w'_\beta ,$$

such that $(v'_\beta + \beta a w'_\beta) \in K'$ and $(b\beta v'_\beta - w'_\beta) \in K'$, where a and b are positive reals independent of β. Furthermore, let there exist a $d > 0$ independent of β and x such that for every sufficiently small $\beta > 0$ the following relation

$$\langle UR(t,U)x, v'_\beta \rangle \geqq d \langle x, v'_\beta \rangle$$

holds for all $x \in K$. Then $\pi\sigma(T)$ contains only isolated poles of $R(t,T)$.

Remark. Let us note that if U is indecomposable with respect to the cone K then

$$\langle UR(t,U)x, v'_\beta \rangle \geqq d(x) \langle x, v'_\beta \rangle$$

with $d(x) > 0$, however, $\inf \{ d(x) : x \in K \setminus 0 \} = 0$ in general.

Theorem 2. Let $Y = H$ be a Hilbert space. Let U and V both in $B(H)$ be K-positive self-adjoint operators. In addition, let U be compact and let a quasiinterior eigenvector $u_0 \in K$ correspond to the spectral radius $r(U)$: $Uu_0 = r(U)u_0$, $\|u_0\| = 1$. We assume that for every $\beta > 0$ there is a system v_β, $v_\beta \in K$, $v_\beta \neq 0$, such that $Vv_\beta = r(V)v_\beta + y_\beta$ with $\|y_\beta\| = h(\beta) \|v_\beta\|$, $\lim h(\beta) = 0$ as $\beta \to 0$ and that there is a constant a independent of β such that

$$\left| \frac{(y_\beta , u_0)}{(v_\beta , u_0)} \right| \leqq a\beta .$$

Then $\pi\sigma(U + V)$ contains only poles of $R(t,U+V)$.

Remark. The requirement concerning the system of approximate eigenvectors $\{v_\beta\}_{\beta > 0}$ is obviously satisfied if $r(V)$ is an eigenvalue of V with an eigenvector $v_0 \in K$. In this case one can choose $v_\beta = v_0$ and $y_\beta = 0$ for all $\beta > 0$.

<u>Theorem 3.</u> Let U and V both in B(Y) be K-positive operators. Let V satisfy the hypotheses of either Theorem 1 or Theorem 2 and let U be compact and K-primitive. In addition, if V satisfies the assumptions of Theorem 2, let U be self-adjoint. Then $\pi\sigma(U+V) = \{r(U+V)\}$ and the corresponding eigenspace is one-dimensional.

At conclusion we note that the Weyl's theorem has been discovered for the needs of quantum mechanical applications; one can say that quantum mechanics has actually provoked H.Weyl to formulate and prove a result which was "obvious" to theoretical physicists at that time. Our result has a similar property in this sense, it is natural and quite obvious to nuclear reactor physicists. Despite this fact the author feels sorry because he has not found any result of the type discussed in this note in the mathematical literature.

References

[1] Gochberg I.C., Krein M.G. The basic properties on defect numbers, root numbers and indices of linear operators. Usp.mat.nauk XII (1957), N2(74), 43-118 (Russian).

[2] Kato T. Perturbation Theory for Linear Operators. Springer Verlag, Berlin-Heidelberg-New York 1966.

[3] Krein M.G., Rutman M.A. Linear operators leaving invariant a cone in a Banach space. Usp.mat.nauk III (1948), N 1, 3-95 (Russian). English translation in Translations of the AMS, Vol.26 (1952).

[4] Riesz F., Szőkefalvi-Nagy B. Leçons d'analyse fonctionelle. (Russian translation). Izd.Inostrannoj literatury, Moscow 1954.

[5] Sawashima I. On spectral properties of some positive operators. Natur.Sci.Rep.Ochanomizu Univ. 15 (1964), 55-64.

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