Y. Katuta

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MONOTONE INCREASING COVERS AND PARACOMPACTNESS

Y. KATUTA

Tsukuba

For the paracompactness many characterizations have been obtained until now. In this paper we give new characterizations in terms of "monotone increasing cover".

For the countable paracompactness we have the following result which is essentially due to F. Ishikawa [1]:

A space X is countably paracompact if and only if for each countable monotone increasing open cover $\{U_n\}$ of X, there exists a countable open cover $\{V_n\}$ of X such that $\overline{V}_n \subset U_n$ for each n.

Let λ be a limit ordinal. We consider the following property, say $P(\lambda)$, for a space X:

For each well-ordered monotone increasing open cover $\{U_\alpha | \alpha < \lambda\}$ of length λ, there exists an open cover $\mathcal{V} = \{V_{\alpha,n} | \alpha < \lambda, n=1,2,\ldots\}$ satisfying

(1) $V_{\alpha,n} \subset U_\alpha$,

(2) $V_{\alpha,n} \subset V_{\beta,n}$ whenever $\alpha < \beta < \lambda$,

(3) $V_{\beta,n} = \bigcup_{\alpha<\beta} V_{\alpha,n}$ for each limit ordinal $\beta < \lambda$,

(4) $\overline{V}_{\alpha,n} \subset V_{\alpha,n+1}$.

If the cover $\mathcal{V}$ satisfies an additional condition

(5) $\overline{\bigcup_{\alpha<\lambda} V_{\alpha,n}} \subset \bigcup_{\alpha<\lambda} V_{\alpha,n+1}$,

we say that X has the property $P'(\lambda)$.

We have the following implications:

$$P(\lambda) + P(\omega_0) \Rightarrow P'(\lambda) \Rightarrow P(\lambda).$$

Theorem 1. A regular space X is paracompact if and only if for each regular ordinal λ, X has the property $P(\lambda)$ (or $P'(\lambda)$).

Theorem 2. A regular space X is paracompact if and only if each well-ordered monotone increasing open cover of X has an open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that $\mathcal{V}_n$ is cushioned in $\mathcal{V}_{n+1}$ for each n.

A subset G of a space X is said to be perfectly open, if there exists a sequence $\{G_n\}$ of open sets such that $G = \bigcup_{n=1}^{\infty} G_n$ and

$\overline{G}_n \subset G_{n+1}$ for each n. Obviously,

$$\text{cozero} \Rightarrow \text{perfectly open} \Rightarrow \text{open } F_\sigma.$$

As an application of Theorem 1, we have the following theorem.

Theorem 3. Let X be a subspace of a compact Hausdorff space Y. If for each compact set C in $Y - X$ there exists a perfectly open set G of $X \times Y$ such that $X \times C \subset G$ and $G \cap \Delta = \emptyset$, then X is paracompact. Here, $\Delta = \{(x,x) | x \in X\}$.

Corollary 4. Let X be a subspace of a compact Hausdorff space Y. If $X \times Y$ is normal, then X is paracompact.

The corollary has been already obtained by K. Morita [2].

Next, let μ be an infinite cardinal. Corresponding to Theorems 1 and 2, we have the similar characterizations for the μ-paracompactness; in this case, a space must be normal.

Theorem 5. A normal space X is μ-paracompact if and only if for each regular ordinal $\lambda \leqq \mu$, X has the property $P(\lambda)$ (or $P'(\lambda)$).

Theorem 6. A normal space X is μ-paracompact if and only if each well-ordered monotone increasing open cover of X with length $\leqq \mu$ has an open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that $\mathcal{V}_n$ is cushioned in $\mathcal{V}_{n+1}$ for each n.

Let λ be a limit ordinal and let $W(\lambda+1)$ be the space consisting of all ordinals $\leqq \lambda$ with the order topology. The property $P'(\lambda)$ on a space X is characterized in the product space $X \times W(\lambda+1)$ as follows.

Theorem 7. A space X has the property $P'(\lambda)$ if and only if for each open set G of $X \times W(\lambda+1)$ with $X \times \{\lambda\} \subset G$, there exists a perfectly open set H of $X \times W(\lambda+1)$ such that $X \times \{\lambda\} \subset H \subset G$.

Corollary 8 (K. Kunen). Let μ be an infinite cardinal. If $X \times W(\mu+1)$ is normal, then X is μ-paracompact and normal.

The corollary follows from Theorems 5 and 7, and the converse is also true by K. Morita [2, Theorem 2.2].

References

[1] F. Ishikawa, On countably paracompact spaces, Proc. Japan Acad., 31 (1955), 686-687.

[2] K. Morita, Paracompactness and product spaces, Fund. Math., 50 (1962), 223-236.

UNIVERSITY OF TSUKUBA, IBARAKI, JAPAN