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SOME PROBLEMS OF CONVERGENCE IN COUNTABLY MODULARED SPACES

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Poznań

1. Let $\varrho_i : X \to [0,\infty]$, $i=1,2,\ldots$, be a sequence of pseudomodulars in a real linear space X, i.e. $\varrho_i(0)=0$, $\varrho_i(-x)=\varrho_i(x)$, $\varrho_i(\alpha x+\beta y) \leqslant \varrho_i(x)+\varrho_i(y)$ for $\alpha,\beta \geqslant 0$, $\alpha+\beta=1$, $x,y \in X$, and let $\varrho_i(x)=0$ for all i imply $x=0$. By means of this sequence, one may define the following modulars in X :

$$\varrho(x)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{\varrho_i(x)}{1+\varrho_i(x)}\ ,\quad \varrho_0(x)=\sup_i \varrho_i(x)\ ,\quad \varrho_s(x)=\sum_{i=1}^{\infty}\varrho_i(x)$$

(see [1], [6]). Let $\tilde{\varrho}$ be any of the symbols ϱ, ϱ_0, ϱ_s. Then $X_{\tilde{\varrho}}=\{x \in X : \tilde{\varrho}(\lambda x) \to 0 \text{ as } \lambda \to 0\}$ is the modular space generated by the modular $\tilde{\varrho}$, $\|x\|_{\tilde{\varrho}}=\inf\{u>0 : \tilde{\varrho}(x/u) \leqslant u\}$ is an F-norm in $X_{\tilde{\varrho}}$, and $\|x_n - x\|_{\tilde{\varrho}} \to 0$ as $n \to \infty$ with $x, x_n \in X_{\tilde{\varrho}}$ is equivalent to the condition $\tilde{\varrho}(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$. If there exists a $\lambda > 0$ such that $\tilde{\varrho}(\lambda(x_n - x)) \to 0$ as $n \to \infty$, then we shall write $x_n \xrightarrow{\tilde{\varrho}} x$ (see [5]). Obviously, $\|x_n - x\|_{\tilde{\varrho}} \to 0$ implies $x_n \xrightarrow{\tilde{\varrho}} x$.

In this paper we shall establish some conditions in order that convergence in the norm $\|\cdot\|_{\varrho}$ be equivalent to convergence in the norm $\|\cdot\|_{\varrho_0}$, in two important spaces. Also, some completeness problems will be solved.

2. In the first of the above mentioned cases, let μ be a finite measure in a σ-algebra Σ of subsets of an abstract, nonempty set Ω, and let X be the set of Σ-measurable real functions on Ω with equality μ-almost everywhere. Let (φ_i) be a sequence of φ-functions (see [8]), and let

$$\varrho_i(x)=\int_{\Omega}\varphi_i(|x(t)|)\,d\mu .$$

The following condition will be used :

(γ) there exist positive constants $k, c, u_0 > 0$ and an index i_0 such that $\varphi_i(cu) \leqslant k\,\varphi_{i_0}(u)$ for all $u \geqslant u_0$ and all $i \geqslant i_0$.

<u>Theorem 1</u>. If (φ_i) are equicontinuous at 0, (γ) holds and $x_n \in X_\varrho$, then $x_n \xrightarrow{\varrho} 0$ implies $x_n \xrightarrow{\varrho_0} 0$ and $\|x_n\|_\varrho \to 0$ implies $\|x_n\|_{\varrho_0} \to 0$.

<u>Proof</u>. From 2.1 in [1] follows $x_n \in X_{\varrho_0}$. Moreover, (γ) may be written in the form: there exist a positive constant c and an index i_0 such that for every $u' > 0$ there is a $k' > 0$ such that $\varphi_i(u) \leqslant k'\varphi_{i_0}(u/c)$ for all $u \geqslant u'$ and all $i \geqslant i_0$.

Hence

$$(*) \qquad \varrho_i(\lambda x_n) \leqslant k'\,\varrho_{i_0}\left(\frac{\lambda x_n}{c}\right) + \varphi_i(u')\mu(\Omega)$$

for $i \geqslant i_0$ and $\lambda > 0$. Choosing arbitrary $\varepsilon > 0$, we may take $u' > 0$ such that $\varphi_i(u')\mu(\Omega) < \frac{1}{2}\varepsilon$, and a constant $k' > 0$ coresponding to this u'. Now, let us suppose that $x_n \xrightarrow{\varrho} 0$, i.e. $\varrho(\lambda' x_n) \to 0$ for a $\lambda' > 0$. This implies $\varrho_i(\lambda' x_n) \to 0$ as $n \to \infty$ for all i. In particular, $\varrho_{i_0}(\lambda' x_n) \to 0$ as $n \to \infty$. Choosing $\lambda = c\cdot\lambda'$ we may find n_0 such that

$$\varrho_{i_0}\left(\frac{\lambda}{c}x_n\right) \leqslant \frac{\varepsilon}{2k'} \quad \text{for } n \geqslant n_0 .$$

Applying the inequality $(*)$ we get $\varrho_i(\lambda x_n) < \varepsilon$ for $n \geqslant n_0$ and $i \geqslant i_0$. Now, we choose $\bar{n}$ in such a manner that $\varrho_i(\lambda' x_n) < \varepsilon$ for $n \geqslant \bar{n}$ and $i < i_0$. Taking $\lambda_0 = \min(\lambda, \lambda')$, we obtain $\varrho_i(\lambda_0 x_n) < \varepsilon$ for $n \geqslant \max(n_0, \bar{n})$ and all i. Consequently, $x_n \xrightarrow{\varrho_0} 0$. Supposing $\|x_n\|_\varrho \to 0$, we obtain $\|x_n\|_{\varrho_0} \to 0$ in a similar way.

<u>Theorem 2</u>. We suppose the measure μ to be atomless and (φ_i) to be equicontinuous at 0. Then

1° if there exists a $\lambda > 0$ such that for every i there are numbers $\beta_i, \vartheta_i > 0$ for which $\varphi_i(\lambda u) \leqslant \beta_i \varphi_k(u)$ for all $u \geqslant \vartheta_i$ and $k \geqslant i$, and if $x_n \in X_{\varrho_0}$, $x_n \xrightarrow{\varrho} 0$ imply $x_n \xrightarrow{\varrho_0} 0$, then there holds (γ),

2° if for every $\lambda > 0$ and every i there are numbers $\beta_i, \vartheta_i > 0$ such that $\varphi_i(\lambda u) \leqslant \beta_i \varphi_k(u)$ for all $u \geqslant \vartheta_i$ and all $k \geqslant i$, and if $x_n \in X_{\varrho_0}$, $\|x_n\|_{\varrho} \to 0$ implies $\|x_n\|_{\varrho_0} \to 0$, then there holds (γ).

Proof. Let us suppose (γ) does not hold, then there exists an increasing sequence (i_n) of indices and an increasing sequence (u_n) of positive numbers, $u_n \to \infty$, $\varphi_n(u_n) > 1$ for $n=1,2,\ldots$, such that

$$\varphi_{i_n}\left(2^{-n} u_n\right) > 2^n \varphi_n(u_n) \qquad \text{for } n=1,2,\ldots$$

(compare [1]). We choose measurable, pairwise disjoint sets $A_n \in \Omega$ such that $\varphi_n(u_n)\mu(A_n) = 2^{-n}\mu(\Omega)$ and we take

$$x_n(t) = \begin{cases} u_n & \text{for } t \in A_n, \\ 0 & \text{for } t \notin A_n. \end{cases}$$

Then

$$\varrho_i(\lambda x_n) = \varphi_i(\lambda u_n)\mu(A_n) \leqslant \varphi_i(\lambda u_n)\mu(\Omega) \to 0$$

as $\lambda \to 0$, uniformly with respect to i. Hence $\varrho_0(\lambda x_n) = \sup_i \varrho_i(\lambda x_n) \to 0$ as $\lambda \to 0$, i.e. $x_n \in X_{\varrho_0}$.

Now, under the assumptions of 1°, we obtain

$$\varrho_i(\lambda x_n) = \varphi_i(\lambda u_n)\mu(A_n) \leqslant \beta_i \varphi_n(u_n)\mu(A_n) = \beta_i \frac{\mu(\Omega)}{2^n}$$

for a suitable $\lambda > 0$, $n \geqslant i$ and n so large that $u_n \geqslant \vartheta_i$. Hence $\varrho_i(\lambda x_n) \to 0$ as $n \to \infty$ for all i. Consequently, $x_n \xrightarrow{\varrho} 0$. It is easily seen that under the assumptions of 2°, we get $\|x_n\|_{\varrho} \to 0$. Now, we prove that $x_n \xrightarrow{\varrho_0} 0$ does not hold, all the more, also $\|x_n\|_{\varrho_0} \to 0$ does not hold. Indeed, supposing $x_n \xrightarrow{\varrho_0} 0$, there would exist a $\lambda > 0$ such that $\varrho_i(\lambda x_n) \to 0$ as $n \to \infty$ uniformly in i. In particular, $\varrho_{i_n}(\lambda x_n) \to 0$ as $n \to \infty$. On the other hand,

$$\varrho_{i_n}(2^{-n} x_n) = \varphi_{i_n}(2^{-n} u_n)\mu(A_n) \geqslant 2^n \varphi_n(u_n)\mu(A_n) = \mu(\Omega),$$

a contradiction.

Theorem 3. The space X_{ϱ_s} is complete.

Proof. Let (x_n) be a Cauchy sequence in X_{ϱ_s}. Then $\sum_{i=1}^{\infty} \varrho_i(\lambda(x_n - x_m)) \to 0$ as $m,n \to \infty$ for every $\lambda > 0$. Let us fix

λ. There exists an increasing sequence of indices (n_k) such that

$$\sum_{i=1}^{\infty} \varrho_i(\lambda(x_n - x_m)) < \frac{1}{2^k}\varphi_1\left(\frac{1}{2^k}\right) \quad \text{for } m,n \geqslant n_k.$$

In particular,

$$\sum_{i=1}^{\infty} \varrho_i\left(\lambda(x_{n_{k+1}} - x_{n_k})\right) < \frac{1}{2^k}\varphi_1\left(\frac{1}{2^k}\right), \quad k=1,2,\dots .$$

Let us choose

$$A_k = \left\{ t \in \Omega : \sum_{i=1}^{\infty} \varphi_i\left(\lambda(x_{n_{k+1}}(t) - x_{n_k}(t))\right) > \varphi_1\left(\frac{1}{2^k}\right)\right\},$$

then

$$\frac{1}{2^k}\varphi_1\left(\frac{1}{2^k}\right) > \int_{A_k} \sum_{i=1}^{\infty} \varphi_i\left(\lambda|x_{n_{k+1}}(t) - x_{n_k}(t)|\right) d\mu \geqslant \mu(A_k)\varphi_1\left(\frac{1}{2^k}\right),$$

and so $\mu(A_k) < 2^{-k}$. Denoting $A = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k$, we have $\mu(A)=0$.

Hence for any $t \in A' = \Omega \setminus A$ there exist a j such that

$$\sum_{i=1}^{\infty} \varphi_i\left(\lambda|x_{n_{k+1}}(t) - x_{n_k}(t)|\right) \leqslant \varphi_1\left(\frac{1}{2^k}\right) \quad \text{for } k \geqslant j .$$

In particular, $\left|x_{n_{k+1}}(t) - x_{n_k}(t)\right| < \frac{1}{\lambda 2^k}$ for $k \geqslant j$, and so the series

$$x_{n_0}(t) + \sum_{k=1}^{\infty}\left(x_{n_{k+1}}(t) - x_{n_k}(t)\right)$$

is convergent. Denoting its sum by $x(t)$ we obtain $x_{n_k}(t) \to x(t)$ a.e. in Ω. By Fatou lemma,

$$\sum_{i=1}^{N} \int_{\Omega} \varphi_i\left(\lambda|x_{n_k}(t) - x(t)|\right) d\mu \leqslant \varliminf_{j\to\infty} \sum_{i=1}^{\infty} \varrho_i\left(\lambda(x_{n_k} - x_{n_j})\right) <$$

$$< \frac{1}{2^k}\varphi_1\left(\frac{1}{2^k}\right)$$

for every N and $k=1,2,\dots$. Taking $N \to \infty$, we obtain

$$\varrho_s\left(\lambda(x_{n_k} - x)\right) < \frac{1}{2^k}\varphi_1\left(\frac{1}{2^k}\right) \quad \text{for } k=1,2,\dots .$$

Moreover,

$$\varrho_s(\lambda(x_m - x_{n_k})) < \frac{1}{2^k}\varphi_1\left(\frac{1}{2^k}\right) \quad \text{for } m \geqslant n_k .$$

Hence

$$\varrho\left(\frac{1}{2}\lambda(x_m - x)\right) < \frac{1}{2^k}\varphi_1\left(\frac{1}{2^k}\right) \quad \text{for } m \geqslant n_k .$$

Let us choose $\varepsilon > 0$ and $\lambda > 0$ and let us take k so large that $\frac{1}{2^{k-1}}\varphi_1\left(\frac{1}{2^k}\right) < \varepsilon$. We obtain $\varrho_s\left(\frac{1}{2}\lambda(x_m - x)\right) < \varepsilon$ for $m \geqslant n_k$, where k depends both on ε and on λ. Let us remark that the function x is independent of λ. Indeed, let x', x'' correspond to two values λ', $\lambda'' > 0$, i.e.

$$\varrho_s\left(\frac{1}{2}\lambda'(x_m - x')\right) < \varepsilon \quad \text{for } m \geqslant n_k'$$

and

$$\varrho_s\left(\frac{1}{2}\lambda''(x_m - x'')\right) < \varepsilon \quad \text{for } m \geqslant n_k'' ,$$

and let $0 < \lambda' \leqslant \lambda''$. Then

$$\varrho_s\left(\frac{1}{4}\lambda'(x' - x'')\right) \leqslant \varrho_s\left(\frac{1}{2}\lambda'(x_m - x')\right) + \varrho_s\left(\frac{1}{2}\lambda''(x_m - x'')\right) < 2\varepsilon$$

for $m \geqslant \max(n_k', n_k'')$. Hence

$$\varrho_s\left(\frac{1}{4}\lambda'(x' - x'')\right) = 0$$

and consequently, $x'(t) = x''(t)$ a.e. This proves x to be independent of λ, and so $x_m \to x$ in X_{ϱ_s}.

Let us still remark, that both spaces X_ϱ and X_{ϱ_0} are complete in the respective norms $\|\cdot\|_\varrho$ and $\|\cdot\|_{\varrho_0}$. In case of X_ϱ this follows from completenes of the Orlicz spaces $L^*_{\varphi_i}$ for i=1,2,.. (see e.g. [5]). Completeness of X_{ϱ_0} follows from that of X_ϱ and from 1.4 in [1], applying Fatou lemma.

3. Now, we take as X the space of all infinitelly differentiable functions in $]-\infty,\infty[$ and we put

$$\varrho_i(x) = \int_{-\infty}^{\infty}\varphi\left(|x^{(i-1)}(t)|\right)dt , \quad i=1,2,\dots ,$$

where φ is a convex φ-function (see e.g. [8]).

<u>Theorem 4</u>. If $x_n \in X_{\rho_0}$, then $x_n \xrightarrow{\rho} 0$ implies $x_n \xrightarrow{\rho_0} 0$ and $\|x_n\|_\rho \to 0$ implies $\|x_n\|_{\rho_0} \to 0$ as $n \to \infty$.

<u>Proof</u>. Since $x_n \in X_{\rho_0}$, applying the arguments of [4], we get

$$\rho_1(\lambda x_n) \geqslant \rho_2(\lambda x_n) \geqslant \ldots$$

for any $\lambda > 0$ and $n=1,2,\ldots$. Supposing $x_n \xrightarrow{\rho} 0$, there exists a $\lambda > 0$ such that $\rho_1(\lambda x_n) \to 0$ as $n \to \infty$. By the above inequalities, $\rho_i(\lambda x_n) \to 0$ as $n \to \infty$ uniformly in i . Consequently, $x_n \xrightarrow{\rho_0} 0$. Similary, $\| x_n\|_\rho \to 0$ implies $\| x_n\|_{\rho_0} \to 0$.

4. We define now the modulars ρ_i like in 3, but replacing $]-\infty,\infty[$ by the r-dimensional space R^r . Thus, X will mean the space of all infinitely differentiable functions in R^r and we write

$$\rho_i(x) = \int_{R^r} \varphi\left(\left| D^i x(t)\right|\right) dt ,$$

where $i=(i_1, i_2, \ldots, i_r)$ is a multiindex and

$$D^i = \frac{\partial^{i_1+\ldots+i_r}}{\partial t_1^{i_1} \ldots \partial t_r^{i_r}} .$$

In the following, we shall omit the symbol R^r under the sign of the integral.

<u>Theorem 5</u>. The space X_{ρ_s} is complete.

<u>Proof</u>. Let (x_n) be a Cauchy sequence in X_{ρ_s}. Then $\rho_s(\lambda(x_n - x_m)) \to 0$ as $m,n \to \infty$ for every $\lambda > 0$. Hence we get, in particular,

$$\int \varphi\left(\lambda \left| D^i x_n(t) - D^i x_m(t)\right|\right) dt \to 0 \quad \text{as} \quad m,n \to \infty$$

for every $\lambda > 0$. Let P be set of multiindices $p=(p_1, p_2, \ldots, p_r)$ with $p_j=0$ or 1, $j=1,2,\ldots,r$. Applying formula (4) from [2] we obtein

$$\varphi\left(\frac{\lambda}{2^r}\left| D^i x_n(t) - D^i x_m(t)\right|\right) \leqslant \sum_{p \in P} \int \varphi\left(\lambda \left| D^i x_n(v) - D^i x_m(v)\right|\right) dv$$

for every $t \in R^r$. Consequently, the sequence $\left(D^i x_n(t)\right)$ is uniformly convergent in R^r as $n \to \infty$ for every i. Thus, there exists an infinitely differentiable function x such that $D^i x_n(t) \to D^i x(t)$ uniformly in R^r as $n \to \infty$ for every i. Let us choose an $\varepsilon > 0$ and let us fix $\lambda > 0$. There exist an index N such that $\varrho_s\left(\lambda(x_n - x_m)\right) < \varepsilon$ for $m,n \geqslant N$. Let $I = (I_1, I_2, \ldots, I_r)$ be a fixed multiindex, and let $i = (i_1, i_2, \ldots, i_r) \leqslant I$ means that $i_k \leqslant I_k$ for $k=1,2,\ldots,r$. Applying Fatou lemma, we get

$$\sum_{i \leqslant I} \int \varphi\left(\lambda \left| D^i x_n(t) - D^i x(t) \right|\right) dt \leqslant$$

$$\leqslant \varliminf_{m \to \infty} \sum_{i \leqslant I} \int \varphi\left(\lambda \left| D^i x_n(t) - D^i x_m(t) \right|\right) dt \leqslant \lim_{m \to \infty} \varrho_s\left(\lambda(x_n - x_m)\right) \leqslant \varepsilon$$

for $n \geqslant N$. Since I is arbitrary, we obtain $\varrho_s\left(\lambda(x_n - x)\right) \leqslant \varepsilon$ for $n \geqslant N$. Hence $x_n \to x$ in X_{ϱ_s} and $x \in X_{\varrho_s}$.

Let us remark, that completeness of X_ϱ was proved in [3], Lemma 3 and Theorem 10. The problem of completeness of X_{ϱ_0} will be dealt with in another note.

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