Toposym 4-B

Evelyn Nelson
Algebras of continuous functions in universal algebra

In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. [331]--332.

Persistent URL: http://dml.cz/dmlcz/700611

**Terms of use:**

© Society of Czechoslovak Mathematicians and Physicist, 1977

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

# ALGEBRAS OF CONTINUOUS FUNCTIONS IN UNIVERSAL ALGEBRA

EVELYN NELSON
Hamilton, Ontario

This talk was a report on joint work with B. Banaschewski and represents part of a forthcoming paper on Boolean powers [1].

The algebras of continuous functions to which the title alludes are defined as follows: for a topological space X and an algebra A, C(X,A) is the algebra of all functions $X \to A$, continuous with respect to the discrete topology on A, with pointwise defined operations, and $E(X,A) = D(X,A)/\sim$ where D(X,A) is the algebra of functions $X \to A$ continuous on some dense open subset of X, and $\sim$ is the congruence which identifies two functions whenever they coincide on some dense open set.

These algebras, for suitable Boolean spaces X, are (isomorphic with) the Boolean powers, bounded and unbounded, of the algebra A, introduced by A.L. Foster [2].

It is clear that, for any X, the algebra C(X,A) is a subalgebra of a power of A; moreover a subalgebra S of a power $A^I$ is of the form C(X,A) for some Boolean space X iff S contains the constant maps, each $u \in S$ has finite image and $Q(r,s,u,v) \in S$ whenever $r,s,u,v \in S$ where $Q(r,s,u,v)(i) = u(i)$ if $r(i) = s(i)$ and $= v(i)$ otherwise. The situation for E(X,A) is somewhat different: for any space X, E(X,A) is the direct limit of the C(U,A) for the dense open $U \subseteq X$; E(X,A) in general need not be embeddable in a power of A, and decent topological criteria which ensure that E(X,A) is so embeddable are not known. (Of course for the trivial case of a discrete space I, $E(I,A) \simeq E(\beta I,A) \simeq A^I$).

Each algebra A of type $\tau$ determines a (contravariant) functor C(-,A) from the category TOP of topological spaces and continuous maps to the category $Alg(\tau)$ of all algebras of type $\tau$ and their homomorphisms; for a continuous $f: X \to Y$, $C(f,A): C(Y,A) \to C(X,A)$ is given by $u \rightsquigarrow uf$. Moreover, the restrictions of such functors to the category BooS of Boolean spaces and their homomorphisms can be characterized in purely categorical terms: a contravariant functor $F: \text{BooS} \to Alg(\tau)$ is naturally equivalent to C(-,A) for some A iff F copreserves finite coproducts and projective limits, i.e., takes finite coproducts to the corresponding products, and projective limits to the corresponding direct limits. An application, for general linear groups, is that

$GL(n,B) \simeq C(\Omega B, GL(n,\underset{\sim}{2}))$ for each Boolean ring B, $\Omega B$ being the Stone space of B, $\underset{\sim}{2}$ the two-element Boolean ring. An analogous statement holds for rings of $n \times n$ matrices over Boolean rings.

For any space X, there is a complete lattice embedding from $\Phi\mathfrak{R}X$, the filter lattice of the Boolean algebra of regular open subsets of X, to $\mathfrak{C}E(X,A)$, the congruence lattice of $E(X,A)$, given as follows: each filter $\mathfrak{F}$ in $\mathfrak{R}X$ determines a congruence $\Theta_{\mathfrak{F}}$ on $D(X,A)$; $(u,v) \in \Theta_{\mathfrak{F}}$ iff $u|U \cap V = v|U \cap V$ for some dense open $U \subseteq X$ and some $V \in \mathfrak{F}$. Since $\Theta_{\mathfrak{F}} \supseteq \sim$, this in turn determines a congruence on $E(X,A)$. It turns out that this embedding $\Phi\mathfrak{R}X \to \mathfrak{C}E(X,A)$ is an isomorphism for all spaces X iff it is an isomorphism for spaces X of the form $\beta I$ for I discrete. Algebras A with $\Phi\mathfrak{R}X \simeq \mathfrak{C}E(X,A)$ for all X include all fields and all simple finite lattices.

The quotient $E_{\mathfrak{F}}(X,A) = D(X,A)/\Theta_{\mathfrak{F}}$ is called a Boolean filter power of A.

Any two filters $\mathfrak{F}$ in $\mathfrak{R}X$ and $\mathfrak{G}$ in $\mathfrak{R}Y$ determine a filter $\mathfrak{F} * \mathfrak{G}$ in $\mathfrak{R}(X \times Y)$: $S \in \mathfrak{F} * \mathfrak{G}$ iff $I\Gamma I\{x \in X \mid I\Gamma\{y \in Y \mid (x,y) \in S\} \in \mathfrak{G}\} \in \mathfrak{F}$, where I and $\Gamma$ denote interior and closure respectively. Now the familiar isomorphism $(A^Y)^X \to A^{X\times Y}$ for sets X and Y given by $f \rightsquigarrow \underline{f}$, $\underline{f}(x,y) = f(x)(y)$, restricts to an embedding $D(X,D(Y,A)) \to D(X \times Y,A)$ for any spaces X and Y; this in turn factors to produce an embedding $E_{\mathfrak{F}}(X, E_{\mathfrak{G}}(Y,A)) \to E_{\mathfrak{F}*\mathfrak{G}}(X \times Y,A)$. It is not difficult to see that the latter is an isomorphism for all $\mathfrak{F}$ and $\mathfrak{G}$ whenever the induced embedding $E(X,E(Y,A)) \to E(X \times Y,A)$ is an isomorphism. For discrete Y, the latter is equivalent with $\mathfrak{R}X$ satisfying certain infinite distributivity laws; however, criteria for this to hold for more general Y are not known.

For any A, the algebras $E_{\mathfrak{U}}(X,A)$ for ultrafilters $\mathfrak{U}$ in $\mathfrak{R}X$ coincide, up to isomorphism, with the Boolean ultrapowers of A constructed by Mansfield [3] via Boolean-valued models. The above topological approach yields an alternative proof of Mansfield's result that two algebras are elementarily equivalent, i.e., satisfy the same first order sentences, iff they have isomorphic Boolean ultrapowers.

REFERENCES

1. B. Banaschewski and E. Nelson. Boolean powers as algebras of continuous functions.
2. A.L. Foster, Generalized Boolean theory of universal algebra I and II. Math. Z. 58 (1953) 306 - 336 and 59 (1953) 191 - 199.
3. R. Mansfield, Boolean ultrapowers. Ann. Math. Logic 2 (1971) 297 - 323.

McMaster University