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PROJECTIVE COVERS IN CATEGORIES OF TOPOLOGICAL SPACES AND TOPOLOGICAL ALGEBRAS

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Hamilton

Introduction. In certain categories, one has the following situation with respect to *injectivity*:

1. The following are equivalent for an object X:

(i) X is injective.

(ii) Every monomorphism $X \to Y$ has a left inverse.

(iii) Every essential monomorphism $X \to Y$ is an isomorphism, where a monomorphism f is essential iff g is a monomorphism whenever gf is one.

2. The following are equivalent for a morphism $f: X \to Y$:

(i) f is an essential monomorphism and Y is injective.

(ii) f is an essential monomorphism, and, for any g, if gf is an essential monomorphism then g is an isomorphism.

(iii) f is a monomorphism and Y is injective, and if $f = gh$ with monomorphisms g and h where g has injective domain then g is an isomorphism.

3. Every object X has an injective hull, i.e. there exists an essential monomorphism $X \to Y$ with injective Y.

Among the categories in which these conditions hold are the category of all (left) modules over a ring and module homomorphism [7], the category of all Boolean lattices and Boolean homomorphisms [11, 2], the category of all distributive lattices and lattice homomorphisms [3], and, with a certain restriction on the type of monomorphism considered, the category of partially ordered sets and order preserving mappings [2]. Because of the first of these, or else, simply for aesthetic reasons, it seems natural to regard this type of situation as the ideal for injectivity; on the other hand, it is interesting to see that this ideal is in fact attained in categories which are otherwise rather unlike categories of modules.

As to the *dual* of this situation, some aspects of it, especially the question of

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projective covers, the duals of injective hulls, have also been considered in categories of modules [4, 22], but it appears that the most natural setting for projectivity, or certain forms of relative projectivity, to have these ideal features are categories connected with topology. In general topology, the first explicit result in this direction dealt with compact, and locally compact Hausdorff spaces and their continuous, resp. perfect mappings [9]. This was followed by work which established analoguous results for other, particular categories of topological spaces [8, 23]; provided better insight into the formal structure of the existing results [20, 12, 16]; dealt specifically with properties of projective covers [13, 19]; or, finally, presented an axiomatic discussion of categories of topological spaces which obtained the desired properties of projectivity for a large number of categories, including, all those previously considered [1].

The object of the present paper is twofold: On the one hand, it is to provide a detailed presentation of the results summarized in [1], and, on the other, to exhibit the same ideal features of projectivity (with respect to certain mappings) in suitable categories of *topological algebras*. Regarding the latter, the situation is that one has the one classical case, provided by the category of compact abelian groups and their continuous homomorphisms via Pontryagin Duality and the properties of injectivity in the category of abelian groups and group homomorphisms, but it is shown here that this is merely one instance of a widely applicable principle.

In detail, the paper is arranged as follows: The first section deals with the generalities, formulated in categorical terms, which provide the theory for the later applications. Then, the conditions stated in the first section which ensure the desired properties of projectivity (always: with respect to perfect onto mappings) are shown to hold in certain types of categories of topological spaces and, on the basis of this, verified for a number of individual categories. Next, the representation of the projective covers in question as spaces of maximal open filters, in the manner of [13], is discussed in detail. Finally, categories of topological algebras are considered, first at a very general level, and then restricted to the case of topological groups, especially profinite and pro-p groups. The paper ends with some assorted further remarks, mostly about certain duality considerations.

1. Categorical Considerations. Let $\mathbf{K}$ be a category and $\mathbf{P}$ a class of morphisms of $\mathbf{K}$. We shall be concerned with the following five conditions on $\mathbf{P}$ and $\mathbf{K}$:

(P1) $\mathbf{P}$ is closed under composition.

(P2) If $f \in \mathbf{P}$ is a right inverse of a $g \in \mathbf{P}$ then f is an isomorphism; conversely, any isomorphism of $\mathbf{K}$ belongs to $\mathbf{P}$.

(P3) For any $f \in \mathbf{P}$ there exists a $g \in \mathbf{K}$ such that $fg \in \mathbf{P}$ and, for all $h \in \mathbf{K}$, $fgh \in \mathbf{P}$ implies $h \in \mathbf{P}$.

(P4) **K** has pullbacks, and these preserve **P** in the sense that, for any pullback diagram

$$\begin{tikzcd} \cdot \arrow[r, "v"] \arrow[d, "u"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

$v \in \mathbf{P}$ whenever $f \in \mathbf{P}$.

(P5) Any well-ordered inverse system in **P** has a lower bound in **P**, i.e. if I is a well-ordered set and $(X_\alpha, f_{\alpha\beta})$ an inverse system indexed by I all whose morphisms belong to **P** then there exists an $X \in \mathbf{K}$ and morphisms $h_\alpha : X \to X_\alpha$ for all $\alpha \in I$ such that $f_{\alpha\beta} h_\beta = h_\alpha$ for all $\alpha \leqq \beta$, and all $h_\alpha \in \mathbf{P}$.

In the following, $\mathbf{P}^*$ will be the class of those $f \in \mathbf{P}$ for which $fh \in \mathbf{P}$ implies $h \in \mathbf{P}$ (any $h \in \mathbf{K}$); the $f \in \mathbf{P}^*$ are called the *essential* $f \in \mathbf{P}$. The isomorphism of **K** belong to $\mathbf{P}^*$ in view of (P2), and (P3) states that for any $f \in \mathbf{P}$ there exists a $g \in \mathbf{K}$ such that $fg \in \mathbf{P}^*$. Simple calculation shows that the following holds for $\mathbf{P}^*$ under the hypothesis (P1):

Lemma 1. (*i*) $\mathbf{P}^*$ *is closed under composition.* (*ii*) *If* f, $g \in \mathbf{P}$ *and* $fg \in \mathbf{P}^*$ *then* $g \in \mathbf{P}^*$. (*iii*) *If* $f, fg \in \mathbf{P}^*$ *then* $g \in \mathbf{P}^*$.

An object X of **K** is called **P**-*projective* iff the usual projectivity condition holds for X with **P** in place of the class of epimorphisms. If $f : Y \to X$ belongs to $\mathbf{P}^*$ and Y is **P**-projective then f (or, sometimes, Y) is called a **P**-*projective cover* of X.

Concerning **P**-projectivity, one now has:

Proposition 1. *If* (P1) – (P4) *hold then the following conditions are equivalent for* $X \in \mathbf{K}$:

(i) *X is* **P**-*projective.*

(ii) *Any $f : Y \to X$ in* **P** *has a right inverse.*

(iii) *Any $f : Y \to X$ in* $\mathbf{P}^*$ *is an isomorphism.*

Proof. (i) $\Rightarrow$ (ii) holds by the usual argument, involving id_X, the identity on X, which belongs to **P** by (P2).

(ii) $\Rightarrow$ (iii). For $f : Y \to X$ in $\mathbf{P}^*$, one has $fg = \mathrm{id}_X$ with suitable g; then $fg \in \mathbf{P}$, hence $g \in \mathbf{P}$, and therefore g is an isomorphism by (P2). It follows that f is also an isomorphism.

(iii) $\Rightarrow$ (i) Given $f: Z \to Y$ in $\mathbf{P}$ and any $g: X \to Y$, one can embed the diagram cruical for the $\mathbf{P}$-projectivity of X into the following diagram:

$$\begin{tikzcd}
F \arrow[d, "u"'] & \\
E \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[ul, "w"'] \arrow[d, "g"] \\
Z \arrow[r, "f"'] & Y
\end{tikzcd}$$

where the bottom square is a pullback diagram, u is taken such that $pu \in \mathbf{P}^*$, which can be done by (P3) since $p \in \mathbf{P}$ by (P4), and $w = (pu)^{-1}$, which exists by hypothesis. It is clear that $h = quw: X \to Z$ has the required property.

Remark. If (P1) and (P2) holds for $\mathbf{P}$ then one still has (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), as the above proof shows. If, further, (P3) holds one also has (iii) $\Rightarrow$ (ii). There do, however, exist $\mathbf{K}$ and $\mathbf{P}$ satisfying (P1) – (P3) for which (iii) $\Rightarrow$ (i) does *not* hold.

Corollary 1. *If* (P1) *and* (P2) *hold then, for any* $f: A \to X$ *and* $g: B \to X$ *in* $\mathbf{P}^*$ *with* $\mathbf{P}$-*projective* A *and* B, *there exists an isomorphism* $h: A \to B$ *such that* $f = gh$.

Proof. There exists an h of the stated kind merely by the $\mathbf{P}$-projectivity of A. Now, g and gh belong to $\mathbf{P}^*$, hence $h \in \mathbf{P}^*$ by Lemma 1, and thus h is an isomorphism since (i) $\Rightarrow$ (iii) only requires (P1) and (P2).

Remark. The essential uniqueness of $\mathbf{P}$-projective covers, which this corollary asserts, could also be proved for any class $\mathbf{P}$ of epimorphisms containing the identities [14]. I do not know what the relation between a class $\mathbf{P}$ satisfying all the conditions stated above and the epimorphisms is; in all applications discussed below, the $\mathbf{P}$ *are* particular classes of epimorphisms, but whether this is so accidentally or necessarily is left open.

Corollary 2. *If* (P1) – (P4) *hold then, for the following conditions on* $f: X \to Y$ *in* $\mathbf{P}$:

(i) f *is* $\mathbf{P}$-*projective cover of* Y;

(ii) $f \in \mathbf{P}^*$, *and if* $fg \in \mathbf{P}^*$ *then* g *is an isomorphism*;

(iii) X *is* $\mathbf{P}$-*projective, and if* $f = gh$, g *and* h *in* $\mathbf{P}$ *and* g *with* $\mathbf{P}$-*projective domain, then* h *is an isomorphism*; *one has* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), *and* (iii) $\Rightarrow$ (i) *if, further,* Y *has a* $\mathbf{P}$-*projective cover.*

Proof. (i) $\Leftrightarrow$ (ii) By definition, $f \in \mathbf{P}^*$; moreover, if $fg \in \mathbf{P}^*$ then also $g \in \mathbf{P}^*$ by Lemma 1, and hence g is an isomorphism by the proposition. Conversely, if f: $X \to Y$ is in $\mathbf{P}^*$, then for any g: $Z \to X$ in $\mathbf{P}^*$, $fg \in \mathbf{P}^*$ by Lemma 1, and hence g is an isomorphism by hypothesis, which shows that X is $\mathbf{P}$-projective.

(ii) $\Rightarrow$ (iii). That X is $\mathbf{P}$-projective when (ii) holds was just proved. Now, let $f = gh$ as stated; then, by Lemma 1, $h \in \mathbf{P}^*$, and since the domain of g is $\mathbf{P}$-projective, h is an isomorphism.

(iii) $\Rightarrow$ (i). Let g: $Z \to Y$ be a $\mathbf{P}$-projective cover and h: $X \to Z$ such that $f = gh$, by $\mathbf{P}$-projectivity of X. From $g \in \mathbf{P}^*$ and $gh \in \mathbf{P}$ one then has $h \in \mathbf{P}$, and by (iii) h is an isomorphism, hence also $f \in \mathbf{P}^*$.

Let $\mathbf{K}$ now be a subcategory of a category $\mathbf{L}$, $\mathbf{Q}$ a class of morphisms of $\mathbf{L}$, and $\mathbf{P}$ a class of morphism of $\mathbf{K}$ such that

(E1) If $f, fg \in \mathbf{K}$ then $g \in \mathbf{K}$ for any f, $g \in \mathbf{L}$.

(E2) $\mathbf{K}$ is $\mathbf{Q}^*$-left fitting in the sense that f: $Y \to X$ in $\mathbf{Q}^*$ and $X \in \mathbf{K}$ implies $Y \in \mathbf{K}$.

(E3) $\mathbf{P} = \mathbf{K} \cap \mathbf{Q}$, and for any X, $Y \in \mathbf{K}$, if f: $X \to Y$ is in $\mathbf{Q}$ then also $f \in \mathbf{P}$.

Under these hypotheses one has:

Proposition 2. *If* (P1) – (P4) *hold for* $\mathbf{Q}$ *and* $\mathbf{L}$ *then the previous proposition and its corollaries still hold for* $\mathbf{P}$ *and* $\mathbf{K}$; *moreover, the* $\mathbf{P}$*-projectives in* $\mathbf{K}$ *are exactly the* $X \in \mathbf{K}$ $\mathbf{Q}$*-projective in* $\mathbf{L}$, *and for any* $X \in \mathbf{K}$, f: $Y \to X$ *is a* $\mathbf{P}$*-projective cover in* $\mathbf{K}$ *iff it is a* $\mathbf{Q}$*-projective cover in* $\mathbf{L}$.

Proof. To begin with, one readily obtains that $\mathbf{P}$ satisfies the conditions (P1) – – (P3) in $\mathbf{K}$, and that $\mathbf{Q}^* \cap \mathbf{K} \subseteq \mathbf{P}^*$: (P1) is obvious, and (P2) results immediately from the fact that any isomorphism $X \to Y$ in $\mathbf{L}$ where X, $Y \in \mathbf{K}$ belongs to $\mathbf{Q}$ and hence, by (E3), to $\mathbf{P}$; next, $\mathbf{Q}^* \cap \mathbf{K} \subseteq \mathbf{P}^*$ is a direct consequence of $\mathbf{P} = \mathbf{K} \cap \mathbf{Q}$, and from this and (E3), (E2) one obtains (P3).

Now, the first part of the proposition is easily verified, on the basis of these comments, by checking each step in the proofs, the most crucial points being that certain morphisms or objects do, in fact, belong to $\mathbf{K}$ rather than to $\mathbf{L}$.

For the second part, let $X \in \mathbf{K}$ be $\mathbf{P}$-projective in $\mathbf{K}$. Then, for any f: $Y \to X$ in $\mathbf{Q}^*$: $Y \in \mathbf{K}$ by (E2), hence $f \in \mathbf{P}$ by (E3), thus $f \in \mathbf{Q}^* \cap \mathbf{K}$, and so $f \in \mathbf{P}^*$, and therefore f is an isomorphism, first in $\mathbf{K}$ but then also in $\mathbf{L}$. By Proposition 1, this shows X is $\mathbf{Q}$-projective in $\mathbf{L}$. Conversely, let $X \in \mathbf{K}$ be $\mathbf{Q}$-projective in $\mathbf{L}$. Then, in the diagram

$$\begin{tikzcd} & X \arrow[dl, "h"'] \arrow[d, "g"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

where $f \in \mathbf{P}$, $g \in \mathbf{K}$ are given, and $h \in \mathbf{L}$ with $g = fh$ exists by $\mathbf{Q}$-projectivity, one has actually $h \in \mathbf{K}$, by (E1), since $f, fh \in \mathbf{K}$; hence X is $\mathbf{P}$-projective.

Finally, let $f\colon Y \to X$ belong to $\mathbf{P}^*$, with Y $\mathbf{P}$-projective. Then Y is also $\mathbf{Q}$-projective and $f \in \mathbf{Q}$, and it has to be shown that $f \in \mathbf{Q}^*$. Consider, then, any $g \in \mathbf{L}$ such that $fg \in \mathbf{Q}$; now, by the properties of $\mathbf{Q}$, there exists an $h \in \mathbf{L}$ such that $fgh \in \mathbf{Q}^*$ where

$$T \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X,$$

and by (E3) one has $T \in \mathbf{K}$ since $X \in \mathbf{K}$, thus also $fgh \in \mathbf{K}$, again from (E3). It follows that $gh \in \mathbf{K}$ since $f, f(gh) \in \mathbf{K}$, and thus by Lemma 1 $gh \in \mathbf{P}^*$ since $f, f(gh) \in \mathbf{P}^*$. This shows that gh is an isomorphism by $\mathbf{P}$-projectivity, therefore $(gh)^{-1} \in \mathbf{Q}^*$ and hence $f \in \mathbf{Q}^*$; consequently, one has $g \in \mathbf{Q}$, i.e. $fg \in \mathbf{Q}$ implies $g \in \mathbf{Q}$ for any $g \in \mathbf{L}$, thus $f \in \mathbf{Q}^*$. Conversely, if $f\colon Y \to X$, for $X \in \mathbf{K}$, belongs to $\mathbf{Q}^*$, with Y $\mathbf{Q}$-projective, then, clearly, $Y \in \mathbf{K}$ and hence $f \in \mathbf{K}$, which shows that $f\colon Y \to X$ is a projective cover in $\mathbf{K}$.

Concerning the existence of $\mathbf{P}$-projective covers in $\mathbf{K}$, $\mathbf{P}$ and $\mathbf{K}$ as before, one has:

Proposition 3. *If* (P1) – (P5) *hold then a* $\mathbf{P}$*-projective cover exists for any* $X \in \mathbf{K}$ *such that* (i) *the class of all* $Y \to X$ *in* $\mathbf{P}^*$ *is small, and* (ii) *for each* $Y \to X$ *in* $\mathbf{P}^*$ *there exists a set of epimorphisms* $u\colon Y \to Y'$ *such that for any* $f\colon Y \to Z$ *in* $\mathbf{P}^*$ *there exists an isomorphism* $g\colon Y' \to Z$ *with* $f = gu$.

Proof. Let $X \in \mathbf{K}$ be an element with the stated properties. For each Y of a representative set of the class of all $Y \to X$ in $\mathbf{P}^*$, let $\mathfrak{k}_Y$ be the cardinal number of the set of epimorphisms described in (ii), and take any cardinal number $\mathfrak{k}$ larger than the supremum of all $\mathfrak{k}_Y$. Now consider any inverse system $(X_\alpha, f_{\alpha\beta})$ whose indexing set is a segment $[0, \lambda[$ of ordinals, whose morphisms $f_{\alpha\beta}$ are *non-trivial* members of $\mathbf{P}^*$ for $\alpha < \beta$, and whose first term is X. Then, *the cardinal number of* λ *is less than* $\mathfrak{k}$: By (P5), there exists a $B \in \mathbf{K}$ and morphisms $h_\alpha\colon B \to X_\alpha$ in $\mathbf{P}$ such that $f_{\alpha\beta}h_\beta = h_\alpha$ for all $\alpha \leqq \beta$; then, let $g\colon C \to B$ be such that $h_0 g \in \mathbf{P}^*$, and, consequently, by Lemma 1, $g_\alpha = h_\alpha g \in \mathbf{P}^*$ for all α since $f_{0\alpha}(h_\alpha g) = h_0 g$. Suppose now that the cardinal number of λ is greater than $\mathfrak{k}$; then there exist $\alpha, \beta < \lambda$ such that $\alpha < \beta$, and the morphisms $g_\alpha\colon C \to X_\alpha$, $g_\beta\colon C \to X_\beta$ factor through the *same* epimorphism u: $C \to C'$, i.e. $g_\alpha = vu$ and $g_\beta = wu$ with isomorphisms $v\colon C' \to X_\alpha$ and $w\colon C' \to X_\beta$. It follows that $f_{\alpha\beta}wu = f_{\alpha\beta}g_\beta = g_\alpha = vu$, hence $f_{\alpha\beta}w = u$, and thus $f_{\alpha\beta} = vw^{-1}$ – contradicting the fact that $f_{\alpha\beta}$ is nontrivial.

Let $\mathbf{S}$ be a fixed *set* of $f\colon Y \to X$ in $\mathbf{P}^*$ such that for any $g\colon Z \to X$ in $\mathbf{P}^*$ there exists an isomorphism $h\colon Y \to Z$ with $f = gh$, and call an inverse system $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta<\lambda}$ of the type considered above *special* iff, in addition, all $f_{0\alpha}\colon X_\alpha \to X_0 = X$ belong to $\mathbf{S}$. Evidently, the special inverse systems form a *set* Σ. Σ is partially ordered in an obvious way: $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta<\lambda}$ precedes $(X'_\alpha, f'_{\alpha\beta})_{\alpha,\beta<\lambda'}$, iff $\lambda \leqq \lambda'$, $X_\alpha = X'_\alpha$ for all $\alpha < \lambda$, and $f_{\alpha\beta} = f'_{\alpha\beta}$ for all $\alpha \leqq \beta < \lambda$; moreover, this partial order is clearly inductive. Let, then, $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta<\lambda}$ be a maximal member of Σ. Again, by (P5) and the earlier part of this proof, there exists a $C \in \mathbf{K}$ and g_α: $C \to X_\alpha$ in $\mathbf{P}^*$, for each α, such that $f_{\alpha\beta} g_\beta = g_\alpha$ for all $\alpha \leqq \beta$. Suppose now that C is not $\mathbf{P}$-projective. Then there exists a non-trivial h: $D \to C$ in $\mathbf{P}^*$. Let v: $E \to X$ belong to $\mathbf{S}$ such that $v = g_0 hu$ with suitable isomorphism u: $E \to D$; then, for each α, $g_\alpha hu$ belongs to $\mathbf{P}^*$ and is non-trivial, i.e. not an isomorphism, for otherwise h would be an isomorphism, by (P1) and (P2), which was specifically exluded. It follows that by putting $X_\lambda = E$, $f_{\alpha\lambda} = g_\alpha hu$ for $\alpha < \lambda$, and $f_{\lambda\lambda} = \mathrm{id}_E$, the given inverse system is extended to a strictly larger member of Σ – contradicting its maximality. Thus, one obtains that C is projective and g_0: $C \to X_0$ is a projective cover of $X_0 = X$.

In the following, for any category $\mathbf{K}$ and specified class $\mathbf{P}$ of morphism of $\mathbf{K}$, $\mathbf{P}$-projectivity in $\mathbf{K}$ will be said to *behave properly* iff one has Proposition 1 and Corollary 1, the equivalence of the conditions of Corollary 2, and the existence of projective covers for every $X \in \mathbf{K}$, i.e. iff $\mathbf{P}$-projectivity satisfies the duals of the conditions listed for injectivity in the Introduction.

2. Categories of Topological Spaces. This section deals with categories whose objects are topological spaces and whose morphisms are (some or all) continuous mappings from one space to another. All spaces will be Hausdorff, and the classes of mapping to play the role of the above $\mathbf{P}$ will always consist of *onto* mappings which are *perfect*, i.e. continuous, closed, and such that the inverse images of points are compact; these will be called the p.o. mappings.

The basic category in this context is the category $\mathbf{H}$ of all Hausdorff spaces and continuous mappings. Concerning perfect mappings one has the following useful criterion: f: $X \to Y$ in $\mathbf{H}$ is perfect iff every ultrafilter $\mathfrak{U}$ on X for which $f(\mathfrak{U})$ converges is itself convergent.

Lemma 2. *In* $\mathbf{H}$, *the p.o. mappings satisfy* (P1) – (P5).

Proof. (P1) is clear since the composition of perfect mappings produces perfect mappings, and of onto mappings produces onto mappings. (P2) follows directly from the fact that a p.o. mapping which is one-to-one is a homeomorphism.

For (P3) one first has to identify set theoretically the essential p.o. mappings. An onto mapping f: $X \to Y$ in $\mathbf{H}$ is called *minimal* iff, for any closed $A \subseteq X$, $f(A) = Y$ implies $A = X$. Now, *the essential p.o. mappings in* $\mathbf{H}$ *are exactly the minimal ones*: Let f: $X \to Y$ be minimal p.o. and g: $Z \to X$ such that fg is p.o. To show that g is perfect, let $\mathfrak{U}$ be any ultrafilter on Z such that $g(\mathfrak{U})$ converges. Then $f(g(\mathfrak{U}))$ converges, and since fg is perfect, $\mathfrak{U}$ converges; hence g is perfect. Further, one has $f(g(Z)) = Y$, and since $g(Z)$ is closed and f minimal, it follows that $g(Z) = X$. In all, then, g is p.o., and therefore f is essential p.o. Conversely, let f: $X \to Y$ be such a mapping, and consider any closed $A \subseteq X$ such that $f(A) = Y$. Here one has, for the natural embedding $g: A \to X$, that fg is onto and perfect, the latter since f is perfect and A closed, and hence g is by essentialness; it follows that $A = X$. From the characterization of essential p.o. mappings thus obtained one readily derives (P3): Given a p.o. $f: X \to Y$, the compactness of the sets $f^{-1}\{y\}$ shows there exist closed $Z \subseteq X$ minimal with respect to the property that $f(Z) = Y$ (i.e. $Z \cap f^{-1}\{y\} \neq \emptyset$ for each $y \in Y$), and for the natural injection $g: Z \to X$ for any such Z, fg is minimal, and thus essential, p.o.

To check (P4) one has to look at the explicit description of pullbacks in $\mathbf{H}$: If, in the diagram

$$\begin{tikzcd} E \arrow[r, "v"] \arrow[d, "u"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $E = \{(x, z) \mid f(x) = g(z)\} \subseteq X \times Z$, and u and v are the restrictions of the projections $X \times Z \to X$, $X \times Z \to Z$ respectively, f is onto then v is clearly onto; moreover, if f is perfect, and $\mathfrak{U}$ an ultrafilter on E such that $v(\mathfrak{U})$ converges then $u(\mathfrak{U})$ converges since $f(u(\mathfrak{U})) = g(v(\mathfrak{U}))$ does, hence $\mathfrak{U}$ converges in $X \times Z$ and therefore in E since this is a closed subspace of the product. Thus, the pullbacks in $\mathbf{H}$ preserve p.o. mappings.

Finally, for (P5) the explicit description of projective limits in $\mathbf{H}$ has to be considered. Let, then $(X_\alpha, f_{\alpha\beta})$ be an inverse system, indexed, say, by a segment $[0, \lambda[$ of ordinals, where all $f_{\alpha\beta}$ are p.o. mappings, and let $L \subseteq \prod X_\alpha$ be the closed subspace which provides the projective limit, $h_\alpha: L \to X_\alpha$ the restrictions of the natural projections. First, we show that h_0 is onto. Given any $a \in X_0$, let $K = \prod f_{0\alpha}^{-1}\{a\}$. Now, if for any finite $F \subseteq [0, \lambda[$, L_F is the subspace of those $u \in \prod X_\alpha$ with $f_{\alpha\beta}(u_\beta) = u_\alpha$ for all α, $\beta \in F$ such that $\alpha \leqq \beta$, then each L_F is closed, $\{L_F\}$ is a filter basis, and $L = \bigcap L_F$. Also, $K \cap L_F \neq \emptyset$ for each F, for if β is the largest element of F and $a_\beta \in f_{0\beta}^{-1}\{a\}$, then any $u = (u_\alpha)$ in K with $u_\gamma = f_{\gamma\beta}(a_\beta)$, $\gamma \in F$, belongs to this intersection. If follows, by compactness, that $K \cap L \neq \emptyset$, and since $h_0(u) = a$ for any $u \in K \cap L$, $h_0(L) = X_0$. Next, h_0 is perfect: If $\mathfrak{U}$ is an ultrafilter on L such that $h_0(\mathfrak{U})$ converges, then by $f_{0\alpha}(h_\alpha(\mathfrak{U})) = h_0(\mathfrak{U})$ one sees that all $h_\alpha(\mathfrak{U})$ converge, thus $\mathfrak{U}$ converges in $\prod X_\alpha$, and hence in L since L is closed. It follows that h_0 is a p.o. mapping, and the same argument evidently also applies to any h_α, $\alpha > 0$. This shows that L, with the p.o. mappings $h_\alpha: L \to X_\alpha$, provides a lower bound for the given inverse system, i.e. one has (P5).

Lemma 3. *If $f: X \to Y$ is an essential p.o. mapping in* $\mathbf{H}$ *then* card $X \leqq 2^{2^{\operatorname{card} X}}$.

Proof. Since f is minimal X has a dense subspace Z of the same cardinality as Y and every point of X is the limit of some ultrafilter on Z in such a way that no two distinct points are limits of the same filter; thus, X has at most as many points as Z, or Y, has ultrafilters, and this proves the assertion.

These two lemmas, together with the additional observation that, for *any* $X \in \mathbf{H}$, the class of all p.o. $f: X \to Y$ is small, ensure that the results of Section 1 apply to $\mathbf{H}$, with the p.o. mappings constituting the class $\mathbf{P}$. More generally, one has:

Proposition 3. *In a subcategory* $\mathbf{K}$ *of* $\mathbf{H}$, *p.o. projectivity is properly behaved if*

(i) $\mathbf{K}$ *is closed-hereditary*[1]), *closed with respect to pullbacks in* $\mathbf{H}$, *and projective limits in* $\mathbf{H}$ *of well-ordered inverse systems with p.o. mappings; or*

(ii) $\mathbf{K}$ *is a full subcategory of* $\mathbf{H}$ *which is left-fitting with respect to essential p.o. mappings*; *or*

(iii) $\mathbf{K}$ *consists of all objects and all perfect mappings from a category* $\mathbf{L}$ *which satisfies one of these conditions.*

Proof. Regarding (i), it is clear that the stated conditions imply (P1) – (P5) for the class of p.o. mappings in $\mathbf{K}$, and this together with the above remark gives the result. Proposition 2 covers (ii) and the part of (iii) where $\mathbf{L}$ satisfies (i), the latter because for any $f: X \to Y$ and $g: Z \to X$ in $\mathbf{H}$, if fg is perfect then g is, for if $\mathfrak{U}$ is any ultrafilter on Z for which $g(\mathfrak{U})$ converges then $(fg)(\mathfrak{U})$ also converges, and for perfect fg this implies that $\mathfrak{U}$ converges. The remaining case is a corollary to the middle part of the proof of Proposition 2.

Under suitable conditions, the p.o. – projectives in a subcategory $\mathbf{K}$ of $\mathbf{H}$ are exactly the extremally disconnected[2]) members of $\mathbf{K}$. The key to this are the following two lemmas, essentially due to Gleason [9]:

Lemma 4. *In* $\mathbf{H}$, *any minimal continuous closed mapping onto an extremally disconnected space is a homeomorphism.*

Lemma 5. *If the natural mapping* $\Gamma U \oplus \mathbf{C}U \to X$, U *open in* X, *and* ΓU, $\mathbf{C}U$ *its closure and complement respectively, has a right inverse then* ΓU *is open.*

The proofs for these assertions are readily obtained by minor modifications of the relevant proofs in [9].

Proposition 4. *Let* $\mathbf{K}$ *be any subcategory of* $\mathbf{H}$ *which is closed with respect to pullbacks in* $\mathbf{H}$ *and contains, for any* $X \in \mathbf{K}$ *and closed subspaces* A, $B \subseteq X$

---

[1]) For every $X \in \mathbf{K}$ and *closed* subspace $Y \subseteq X$, Y and the natural embedding $X \to Y$ belong to $\mathbf{K}$.

[2]) *Extremaly disconnected* here means merely that every open set has open closure. Such spaces need *not* be regular, a point which is, in fact, of principal importance in the present setting.

the coproduct $A \oplus B$ and its natural injection $A \oplus B \to X$. Then, the p.o. – projectives in **K** *are exactly the extremally disconnected spaces belonging to* **K**.

Proof. The conditions on **K** imply (P1) – (P4) for the p.o. mappings in **K**, (P3) because, as a special case of the second condition, **K** is closed-hereditary; it follows, then, that Proposition 1 applies. If $X \in \mathbf{K}$ is extremally disconnected then, by Lemma 4, any essential p.o. mapping $f\colon Y \to X$ is a homeomorphism, and thus X is p.o. – projective. Conversely, if $X \in \mathbf{K}$ is p.o.-projective then Lemma 5 and the second condition on **K** show that X is extremally disconnected.

Under the same hypotheses for **K**, one evidently has the following further consequences:

Corollary 1. *The p.o.-projectives in* **K** *are exactly the same as the $X \in K$, p.o.-projective in* **H**.

Corollary 2. *As far as they exist in* **K**, *p.o.-projective covers in* **K** *are the same as in* **H**.

Another result, for a different kind of **K**, is:

Corollary 3. *In any full subcategory* **K** *of* **H** *which is left-fitting with respect to essential p.o. mappings the p.o.-projectives are exactly the extremally disconnected spaces belonging to* **K**, *and the same holds for the subcategory of* **K** *with the same objects, but only the perfect mappings from* **K**.

Some subcategories of **H** to which all the considerations of the present section apply are given by the following classes of spaces together with either *all* their continuous mappings, or all their *perfect* mappings:

(1) compact spaces
(2) locally compact spaces
(3) paracompact spaces
(4) σ-compact spaces
(5) Lindelöf spaces
(6) regular spaces
(7) completely regular spaces
(8) zero-dimensional spaces
(9) real-compact spaces
(10) $\mathfrak{k}$-compact spaces[3]

[3]) A space is $\mathfrak{k}$-*compact* iff it is completely regular Hausdorff and every maximal Z-filter in which any fewer than $\mathfrak{k}$ sets have non-void intersection is fixed. For $\mathfrak{k} = \aleph_0$ this means just compactness, for $\mathfrak{k} = \aleph_1$, real-compactness.

Here, the first half are types of spaces which determine subcategories of **H** that are left-fitting with respect to essential p.o. mappings, whereas the second half are classes of spaces which are closed-hereditary and closed under products in **H**.

In some of the categories just described, the p.o. mappings are exactly the *epimorphisms* so that p.o.-projectivity then amounts to ordinary projectivity. Evidently, this is the case for compact Hausdorff spaces and continuous mappings; further instances are the categories given by the *perfect* mappings and the following classes of Hausdorff spaces: all Hausdorff spaces, regular spaces, zero-dimensional spaces, locally compact spaces, and presumably there are several others.

An example of a category in which p.o.-projectivity is rather differently behaved by comparison with the categories listed above is given by the full subcategory **M** of **H** determined by the *metrizable* spaces: In **M**, (P1) – (P4) do hold with respect to the p.o. mappings, and hence Proposition 1 applies; moreover, Proposition 4 also applies, and thus the p.o.-projectives of **M** are exactly the extremally disconnected $X \in \mathbf{M}$; these, however, are *discrete* by a result of Gleason's [9] so that the p.o.-projectives of **M** are exactly the discrete $X \in \mathbf{M}$. This implies that *no non-discrete* $X \in \mathbf{M}$ *can have a p.o.-projective cover in* **M** for if $f\colon Y \to Z$ is such a cover in **M** then Y is discrete, and since f is closed and onto this implies that X is discrete.

We conclude this section with the discussion of a number of particular aspects of p.o.-projectivity. The first concerns the relationship between p.o.-projectivity and certain kinds of reflections.[4]) To begin with, one has:

Lemma 6. *Let* **C** *be any epireflective subcategory of a category* **K** *with reflection* $c_X\colon X \to \gamma X$ *for each* $X \in \mathbf{K}$, *and* **P** *a class of morphisms of* **K**. *Then, for any* **P**-*projective* $X \in \mathbf{K}$, γX *is* $\mathbf{P} \cap \mathbf{C}$-*projective in* **C**.

Proof. For the usual diagram

$$\begin{tikzcd} & \gamma X \arrow[d, "g"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

in **C** where $f \in \mathbf{P} \cap \mathbf{C}$, one has the following enlarged diagram

$$\begin{tikzcd} X \arrow[r, "c_X"] \arrow[d, dashed, "h_0"'] & \gamma X \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

4) The terminology used here is dual to that in [17]. *Epireflective* means all reflections are epimorphisms.

where $fh_0 = gc_X$, h_0 obtained from the **P**-projectivity of X, and $h_0 = hc_0$ from the property of reflections. One thus has $fhc_X = gc_X$, and hence $fh = g$ since c_X is an epimorphism.

Lemma 7. *Let* $f\colon X \to Y$ *belong to* **H** *and* $A \subseteq X$, $B \subseteq Y$ *be dense subspaces such that* $f(A) \subseteq B$ *and the mapping* $g\colon A \to B$ *determined by* f *is perfect. Then* $f(X - A) \subseteq Y - B$.

Proof. Take any $c \in X - A$, and let $\mathfrak{U}$ be an ultrafilter on A converging in X to c. Then $g(\mathfrak{U}) = f(\mathfrak{U})$ converges in Y to $f(c)$; now, if $f(c) \in B$ then $g(\mathfrak{U})$ converges already in B, and by the properness of g $\mathfrak{U}$ must converge in A, hence $c = \lim \mathfrak{U} \in A$, a contradiction.

Let **K** now be a subcategory of **H** and **E** an *extensive* subcategory of **K**, by which is meant a reflective subcategory such that the reflections $e_X\colon X \to \varepsilon X$ with respect to **E** are dense embeddings for each $X \in$ **K**. Given any $f\colon X \to Y$ in **K**, there is a unique $f^\varepsilon : \varepsilon X \to \varepsilon Y$ such that $f^\varepsilon e_X = e_Y f$, by reflectiveness. **E** will be said to *preserve* a given property of mappings if for each $f \in$ **K** with the property in question f^ε also has it. With these concepts, one now has:

Proposition 5. *If* **E** *preserves perfect mappings then, for any* $X \in$ **K**, X *is p.o.-projective in* **K** *iff its* **E**-*reflection* εX *is p.o.-projective in* **E**; *moreover, if* **K** *is also closed-hereditary then, for any p.o.-projective cover* $f\colon X \to Y$ *in* **K**, $f^\varepsilon\colon \varepsilon X \to \varepsilon Y$ *is a p.o.-projective cover in* **E**.

Proof. One part of the first assertion follows directly from Lemma 6, **E** being epireflective since the $e_X\colon X \to \varepsilon X$ are *dense* embeddings. For the converse, the typical diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{f} & Y \end{array}$$

in **K** where f is perfect and onto, is embedded into the diagram

$$\begin{tikzcd} & X \arrow[r, "e_X"] \arrow[d, "g"'] & \varepsilon X \arrow[ddl, "g^\varepsilon"] \\ Z \arrow[r, "f"] \arrow[d, "e_Z"'] & Y \arrow[d, "e_Y"] & \\ \varepsilon Z \arrow[r, "f^\varepsilon"'] & \varepsilon Y & \end{tikzcd}$$

Here, f^ε is perfect by hypothesis; we show it is also onto: For any $a \in \varepsilon Y$, let $\mathfrak{U}$ be an ultrafilter on Y such that $e_Y(\mathfrak{U})$ converges to a, $\mathfrak{W}$ an ultrafilter on Z such that $f(\mathfrak{W}) = \mathfrak{U}$, which exists since f is onto, and then consider $e_Z(\mathfrak{W})$. This is mapped to $e_Y(\mathfrak{U})$ by f^ε, and thus converges by the perfectness of f^ε; for $b = \lim e_Z(\mathfrak{W})$ one then has $f^\varepsilon(b) = a$. Now there exists, by the given p.o.-projectivity of εX, an $h_0: \varepsilon X \to \varepsilon Z$ such that $f^\varepsilon h_0 = g^\varepsilon$. To follow the effect of h_0, take any $a \in X$; if $b = h_0\, e_X(a) \in \varepsilon Z - e_Z(Z)$ then $f^\varepsilon(b) \in \varepsilon Y - e_Y(Y)$ by Lemma 7, whereas $f^\varepsilon(b) = g^\varepsilon\, e_X(a) = e_Y(g(a)) \in e_Y(Y)$; it follows from this that $h_0(e_X(X)) \subseteq e_Z(Z)$, and since all reflection mappings are embeddings here this means there exists an $h: X \to Z$ such that $e_Z h = h_0 e_X$. From this one obtains $f^\varepsilon e_X h = f^\varepsilon h_0 e_X = g^\varepsilon e_X = e_Y g$, and since $f^\varepsilon e_X = e_Y f$ it follows, finally, that $fh = g$, e_Y being a monomorphism.

For the second assertion of the proposition, it suffices to show that f^ε is an essential p.o. mapping whenever $f \in \mathbf{K}$ is. Let $f: X \to Y$, and consider any closed $A \subseteq \varepsilon X$ such that $f^\varepsilon(A) = \varepsilon Y$. Then, by Lemma 7, $f^\varepsilon(A \cap e_X(X)) = e_Y(Y)$, hence $e_Y f(e_X^{-1}(A)) = e_Y(Y)$ and thus $f(e_X^{-1}(A)) = Y$, i.e. fg is again a p.o. mapping for the natural embedding $g: e_X^{-1}(A) \to X$ which belongs to $\mathbf{K}$ by closed-heredetariness. It follows that $e_X^{-1}(A) = X$, and hence $A = \varepsilon X$ since A is closed and e_X has dense image in εX. This shows that f^ε is *minimal*, hence also essential.

Subcategories $\mathbf{K}$ of $\mathbf{H}$ with an extensive subcategory $\mathbf{E}$ which satisfy all hypotheses stated above are given by the completely regular spaces together with the compact spaces, and the zero-dimensional spaces together with the compact zero-dimensional spaces, the reflection being $X \to \beta X$ in the first case, and $X \to \zeta X$, the universal zero-dimensional compactification, in the second. Preservation of perfectness is simply a consequence of the fact that all spaces βX, ζX are actually compact. Another possible example might be provided by the completely regular spaces with the real-compact spaces, or, more generally, $\mathfrak{k}$-compact spaces for arbitrary fixed $\mathfrak{k}$, but we do not know whether perfectness is preserved in this case.

Next, we deal with the question of *rigidity* of p.o.-projective covers. In any category $\mathbf{K}$, a $\mathbf{P}$-projective cover $f: X \to Y$, $\mathbf{P}$ a given class of morphisms of $\mathbf{K}$, is called *rigid* iff the only isomorphism $g: X \to X$ such that $fg = f$ is the identity of X. For the dual concept, rigidity of injective hulls with respect to certain morphisms, there are examples which show this may, but need not occur [3, 14]. Here, the situation is as follows:

Proposition 6. *In any closed-hereditary subcategory* $\mathbf{K}$ *of* $\mathbf{H}$, *all p.o.-projective covers are rigid.*

Proof. We show that *any minimal* p.o. mapping $f: X \to Y$ in $\mathbf{H}$ is rigid; the fact that $\mathbf{K}$ is closed-hereditary then gives the result since it implies that the essential p.o. mappings are minimal.

Let $g: X \to X$ be any homeomorphism such that $fg = f$, and suppose there exists an $a \in X$ such that $g(a) \neq a$. Then, for disjoint neighbourhoods U and V of a

and $g(a)$ respectively, let W be an open neighbourhood of a such that $g(W) \subseteq V$, $W \subseteq U$. It follows that $f(W) = f\,g(W) \subseteq f(V) \subseteq f(X - W)$, and hence $f(X - W) = Y$ which contradicts the minimality of f; thus, $g(x) = x$ for all $x \in X$.

Finally, we consider the use of *free* objects for obtaining p.o.-projective covers. In any subcategory $\mathbf{K}$ of $\mathbf{H}$, an $X \in \mathbf{K}$ is called *free* on a subset S iff any mapping f_0: $S \to Y$, $Y \in \mathbf{K}$, has a unique extension to an f: $X \to Y$ in $\mathbf{K}$. In the full subcategory of $\mathbf{H}$ given by all compact $X \in \mathbf{H}$, for example, the free objects are the Stone-Čech compactifications of discrete spaces. In general, it is clear that any free $X \in \mathbf{K}$ is p.o.-projective, in fact, projective with respect to arbitrary onto mappings in $\mathbf{K}$; hence, if $\mathbf{K}$ is closed-hereditary, and for each $X \in \mathbf{K}$ there exists a p.o. mapping f: $Y \to X$ in $\mathbf{K}$ with Y *free* in $\mathbf{K}$, then each $X \in \mathbf{K}$ has a p.o.-projective cover in $\mathbf{K}$, and, further, the p.o.-projectives in $\mathbf{K}$ are exactly the retracts of the free $Y \in \mathbf{K}$. This is the Rainwater method [20] to obtain projective covers for compact Hausdorff spaces, and there are other situations where this can be employed, e.g. for zero-dimensional compact spaces. In general, however, this approach fails in view of the following observation:

Proposition 7. *If a replete subcategory* $\mathbf{K}$ *of* $\mathbf{H}$ *is hereditary and contains discrete spaces of arbitrary cardinality then the retracts of the free* $X \in \mathbf{K}$ *are discrete and provide only trivial o.p.-projective covers.*

Proof. If $X \in \mathbf{K}$ is free on its subset S, let f_0: $S \to Y$ be a one-one onto mapping for a discrete $Y \in \mathbf{K}$, and f: $X \to Y$ its extension in $\mathbf{K}$. It follows that S, as a *subspace* of X, is discrete, and the mapping g: $Y \to S \to X$, isomorphism followed by natural embedding, belongs to $\mathbf{K}$. Now, gf extends the identity mapping on S to X, and hence $gf = \mathrm{id}_X$; thus f is one-one, and X is discrete. Any retract Y of such X is, of course, again discrete, and for any essential p.o. mapping f: $Y \to Z$, Z is then discrete and f one-one.

Remark. The parts of this section which deal with material presented in [1] are actually slightly more general than what appears there. Placing the perfect (onto) mappings only in the bottom arrow of the projectivity diagram, as it were, seems to give them their right place – there is no need to restrict the entire category to perfect mappings. Incidentally, the greater flexibility gained this way also gets closer to the point of view taken in [13, 16, 19].

3. Projective Covers as Filter Spaces. In [9] and [8], the existence of p.o.-projective covers in the categories considered there is obtained by explicit descriptions of suitable spaces and mappings which are proved to provide the desired covers. In either case, the spaces consist of filters in certain lattices, i.e. the maximal filters in the Boolean lattices of all regular closed or all regular open subsets of the initial space. In analogy with this, we shall now give a similar description of p.o.-projective covers, applicable to any subcategory of **H** which satisfies appropriate conditions. However, we shall follow the approach of [13], rather than that of [9] or [8], and use the lattice of *all* open sets; this appears to have a number of advantages.[5])

To begin with, we summarize some familiar facts. Let X be any space, $\mathfrak{O} = \mathfrak{O}(X)$ its topology, i.e. the collection of its open sets, and $\Omega = \Omega(X)$ the set of all maximal filters $\mathfrak{M} \subseteq \mathfrak{O}$. Then, for any $V \in \mathfrak{O}$, put $\Omega_V = \{\mathfrak{M} \mid V \in \mathfrak{M} \in \Omega\}$; it is immediately obvious that $\Omega_U \cap \Omega_V = \Omega_{U \cap V}$, and the sets Ω_V form the basis of a topology, the usual topology of the maximal filter space of a distributive lattice with zero. The space thus given, again denoted by Ω, is *Hausdorff* since $U \cap V = \emptyset$ implies that $\Omega_U \cap \Omega_V = \emptyset$, and from the fact that $\mathbf{C}\Omega_V = \Omega_{\mathbf{I}\mathbf{C}V}$ one readily deduces that it is *compact*. Moreover, if $\Sigma = \bigcup \Omega_V (V \in \mathfrak{A})$ is any open subset of Ω then its closure is $\Omega_{\mathbf{I}\Gamma U}$, where $U = \bigcup V (V \in \mathfrak{A})$, and thus Ω is *extremally disconnected*. Finally, the closed subsets of Ω are exactly the sets $\Omega_{\mathfrak{F}} = \{\mathfrak{M} \mid \mathfrak{F} \subseteq \mathfrak{M} \in \Omega\}$, $\mathfrak{F}$ any filter in $\mathfrak{O}$ generated by *regular* open sets.

Now, let $\Lambda = \Lambda(X)$ be the subspace of Ω given by all *convergent* $\mathfrak{M} \in \Omega$, i.e. all $\mathfrak{M} \in \Omega$ such that $\mathfrak{M} \supseteq \mathfrak{O}(a)$ for some $a \in X$ where $\mathfrak{O}(a) = \{V \mid a \in V \in \mathfrak{O}\}$. Since every $\mathfrak{O}(x)$, $x \in X$, is contained in some $\mathfrak{M} \in \Omega$ by Zorn's Lemma one sees that Λ is dense in Ω; thus Λ is also extremally disconnected.

An obvious mapping from Λ to X is $\mathfrak{M} \rightsquigarrow \lim \mathfrak{M}$, which will be denoted by lim, or $\lim_X$ if reference to the space is required. It follows from what was just said that lim is an onto mapping; further properties are given in:

Lemma 8. *The mapping* $\lim\colon \Lambda(X) \to X$ *is compact, closed, and minimal onto; moreover, for any* $V \in \mathfrak{O}$, *the image of* $\Lambda_V = \Lambda \cap \Omega_V$ *is* ΓV; *finally,* lim *is continuous iff* X *is regular.*

The proof of this is essentially contained in [8] and in [13]; see also [23].

Remark. As a consequence of this lemma one obtains the well-known theorem of Urysohn's that an H-closed regular Hausdorff space is compact: For such spaces, $\Lambda = \Omega$, and lim is continuous.

In the following, the effect of certain mappings on the spaces Λ is considered.

Lemma 9. *For any minimal p.o. mapping* $f\colon X \to Y$ *in* **H**, *the mapping* f^* *which assigns to each* $\mathfrak{M} \in \Lambda(Y)$ *the filter generated by the sets* $f^{-1}(U)$, $U \in \mathfrak{M}$, *in the topology of* X *is a homeomorphism from* $\Lambda(Y)$ *to* $\Lambda(X)$.

Proof. For any $\mathfrak{M} \in \Omega(Y)$, let $V \subseteq X$ be an open set such that $V \cap f^{-1}(U) \neq \emptyset$ for all $U \in \mathfrak{M}$. Then, $f(V) \cap U \neq \emptyset$ and therefore $(\Gamma \mathbf{C} f(\mathbf{C}V)) \cap U \neq \emptyset$ for all $U \in \mathfrak{M}$ since $f(V) \subseteq \Gamma \mathbf{C} f(\mathbf{C}V))$, the latter being a property of continuous minimal onto mappings [9]. It follows that $\mathbf{C} f(\mathbf{C}V)$ also meets all $U \in \mathfrak{M}$, and thus $\mathbf{C} f(\mathbf{C}V)$,

[5]) The same method is used in [23].

being open, belongs to $\mathfrak{M}$; from this one finally obtains $V \in f^*(\mathfrak{M})$ since $f^{-1}(\mathbf{C}f(\mathbf{C}U)) \subseteq \subseteq U$. This shows that $f^*(\mathfrak{M}) \in \Omega(X)$. Now, take $\mathfrak{M} \in \Lambda(Y)$; then, since $\mathfrak{M}$ is the image of $f^*(\mathfrak{M})$ under f and f is proper, $f^*(\mathfrak{M})$ has a cluster point, and thus converges in view of its maximality; this shows that f^* maps $\Lambda(Y)$ into $\Lambda(X)$. To see this mapping is onto, take any $\mathfrak{M} \in \Lambda(X)$ and consider the filter basis $\mathfrak{B} = \{\mathbf{C}f(\mathbf{C}V) \mid V \in \mathfrak{M}\}$ of open subsets of Y. Now, $\mathfrak{B}$ has $f(\lim \mathfrak{M})$ as a cluster point, and thus there exists an $\mathfrak{N} \in \Lambda(Y)$ such that $\mathfrak{N} \supseteq \mathfrak{B}$; it follows that $f^*(\mathfrak{N}) = \mathfrak{M}$.

That f^* is one-to-one is obvious, and it remains to be shown that it is open and continuous. Openness results from the fact that $V \in \mathfrak{M}$ holds iff $f^{-1}(V) \in f^*(\mathfrak{M})$ for any $\mathfrak{M} \in \Lambda(Y)$, i.e. $f^*(\Lambda_V(Y)) = \Lambda_{f^{-1}(V)}(X)$. To show continuity we prove that $V \in f^*(\mathfrak{M})$, V open in X, holds iff $\mathbf{C}f(\mathbf{C}V) \in \mathfrak{M}$, for any $\mathfrak{M} \in \Lambda(Y)$. $V \in f^*(\mathfrak{M})$ implies that $V \supseteq f^{-1}(W)$ for some $W \in \mathfrak{M}$; from this one obtains that first $\mathbf{C}V \subseteq \mathbf{C}f^{-1}(W)$, then $f(\mathbf{C}V) \subseteq \mathbf{C}W$, hence $\mathbf{C}f(\mathbf{C}V) \supseteq W$, and finally $\mathbf{C}f(\mathbf{C}V) \in \mathfrak{M}$. The converse follows from the fact that $f^{-1}(\mathbf{C}f(\mathbf{C}V)) \subseteq V$.

Let X be any Hausdorff space and $\mathfrak{O}$ its topology. Then, the *regular* $V \in \mathfrak{O}$, i.e. those for which $V = \mathbf{I}\Gamma V$, $\mathbf{I}$ denoting interior, generate a topology $\mathfrak{O}_*$ for which the space composed of the set underlying X and $\mathfrak{O}_*$ is a semi-regular Hausdorff space X_*. The closure operator Γ_* of this new space has the property that $\Gamma_* U = \Gamma U$ for any $U \in \mathfrak{O}$ [15].

Lemma 10. *If X is an extremally disconnected Hausdorff space and $\mathfrak{O}$ its topology then every topology $\mathfrak{O}'$ on the set underlying X for which $\mathfrak{O} \supseteq \mathfrak{O}' \supseteq \mathfrak{O}_*$ determines again an extremally disconnected Hausdorff space X'.*

Proof. Clearly, any such space is Hausdorff since X_* is. Now, for any $V \in \mathfrak{O}'$ the closure $\Gamma' V$ in E' satisfies the condition $\Gamma V \subseteq \Gamma' V \subseteq \Gamma_* V$, hence one has $\Gamma' V = = \Gamma V$; since X is extremally disconnected ΓV is open and therefore regular open in X, i.e. $\Gamma' V \in \mathfrak{O}_*$, and thus $\Gamma' V \in \mathfrak{O}'$.

Lemma 11. *If X is an extremally disconnected Hausdorff space then the mapping $\Lambda(X) \to X_*$ given by lim is a homeomorphism.*

Proof. That this mapping is compact, minimal, and onto follows immediately from the properties of lim. We show that it is also continuous and closed; since this will establish that it is essential p.o. it will then follow that it is a homeomorphism by the p.o.-projectivity of X_* in the category **H**.

X_*, being semi-regular and extremally disconnected, is regular. Let $a = \lim \mathfrak{M}$ for some $\mathfrak{M} \in \Lambda(E)$ and U any neighbourhood of a in X_*; then there exists a neighbourhood V of a in X_* such that $\Gamma_* V \subseteq U$. Now, $\lim (\Lambda_V(X)) = \Gamma V$ by Lemma 8, and from $\Gamma_* V = \Gamma V$ it follows that the neighbourhood $\Lambda_V(X)$ of $\mathfrak{M}$ is mapped into U. Similarly, the image of any closed subset of $\Lambda(X)$ is of the form adh $\mathfrak{F} = \bigcap \Gamma\, V (V \in \mathfrak{F})$, $\mathfrak{F}$ a suitable filter in $\mathfrak{O}$, and since the ΓV are closed in X_*, any such set is closed in X_*.

Combining the three preceding lemmas one obtains the following result for Hausdorff spaces X and Y:

Corollary. *If $f: X \to Y$ is a minimal p.o. mapping and X is extremally disconnected then the mapping $\Lambda(Y) \to X_*$ by $\mathfrak{M} \rightsquigarrow \lim f^*(\mathfrak{M})$ is a homeomorphism.*

To return to p.o.-projective covers, let $\mathbf{K}$ now be a *replete* subcategory of $\mathbf{H}$ which is closed with respect to pullbacks and projective limits of well-ordered inverse systems with p.o. mappings, taken in $\mathbf{H}$, and which contains, for each $X \in \mathbf{K}$ and any closed subspaces $A, B \subseteq X$ the coproduct $A \oplus B$ in $\mathbf{H}$ and the natural mapping $A \oplus B \to X$. The desired description of the p.o.-projective covers in $\mathbf{K}$ is as follows:

Proposition 8. *If all spaces belonging to $\mathbf{K}$ are semi-regular then, for any $X \in \mathbf{K}$, $\Lambda(X)$ and $\lim_X$ belong to $\mathbf{K}$ and $\lim_X: \Lambda(X) \to X$ is a p.o.-projective cover of X in $\mathbf{K}$. In general, a projective cover of X is given by the mapping determined by $\lim_X$ on the space $\Lambda'(X)$ whose underlying set is the same as that of $\Lambda(X)$ and whose topology is generated by that of $\Lambda(X)$ together with $\lim_X^{-1}(\mathfrak{O}(X))$.*

Proof. Let $f: Y \to X$ be a p.o.-projective cover. Since Y is extremally disconnected, the mapping $\lim_Y f^*: \Lambda(X) \to Y_*$ is a homeomorphism. Now, for semi-regular Y one has $Y = Y_*$, hence $\Lambda(X)$ and $\lim_Y f^*$ belong to $\mathbf{K}$; moreover, $f \lim_Y f^* = \lim_X$, and thus $\lim_X: \Lambda(X) \to X$ is a p.o.-projective cover.

If Y is not semi-regular, let $\mathfrak{O} = \mathfrak{O}(Y)$ be its topology, $\mathfrak{O}_*$ as before, and $\mathfrak{O}'$ the topology generated by $\mathfrak{O}_*$ and the sets $f^{-1}(U)$, U open in X. By Lemma 10, the space Y' with the topology $\mathfrak{O}'$ and the same points as Y is again extremally disconnected. Moreover, the mapping $f': Y' \to X$ determined by f is again minimal p.o.: Its continuity is evident from the definition of Y'; the compactness of the inverse images of points follows from the fact that these are compact with respect to $\mathfrak{O} \supseteq \mathfrak{O}'$; a closed $A \subseteq Y'$ is also closed in Y and thus $f'(A) = f(A)$ is closed, and that f' is minimal onto follows the same way. This shows that $f': Y' \to X$ is a p.o.-projective cover of X in $\mathbf{H}$, and Corollary 2 of Proposition 4 implies that $f': Y' \to X$ belongs to $\mathbf{K}$ and is a p.o.-projective cover there. Finally, the composite of the homeomorphism $\lim_Y f^*$ with the mapping $Y_* \to X$ given by f is just $\lim_X$, and hence the description of Y' just corresponds to the description of $\Lambda'(X)$, which proves the second part of the proposition.

As a by-product of the above considerations one has the following remark concerning the spaces $\Lambda(X)$: If $\mathbf{A}$ is any class of semi-regular spaces which is closed with respect to the three types of operations referred to above then $\Lambda(X) \in \mathbf{A}$ for every $X \in \mathbf{A}$. Thus, for instance, $\Lambda(X)$ is real-compact (or, more generally, $\mathfrak{k}$-compact for arbitrary $\mathfrak{k}$) if X is.

Proposition 6 is, primarily, a description of the p.o.-projective covers in $\mathbf{H}$, the category $\mathbf{K}$ just being such that p.o.-projective covers in $\mathbf{K}$ always exist and coincide with the p.o.-projective covers of the $X \in \mathbf{K}$ in $\mathbf{H}$, and for any other type of category satisfying this latter condition the same proposition evidently holds. In particular, this covers all the full subcategories of $\mathbf{H}$, or for that matter, of any $\mathbf{K}$ considered above, which are left-fitting with respect to essential p.o. mappings, as well as the categories one obtains from these by allowing perfect mappings only.

On the other hand, one has the following observation concerning full subcategories of $\mathbf{H}$:

Proposition 9. *If a full subcategory* $\mathbf{S}$ *of* $\mathbf{H}$ *has the property that* $\Lambda'(X)$ *belongs to* $\mathbf{S}$ *for each* $X \in \mathbf{S}$ *then the extremally disconnected* $X \in \mathbf{S}$ *are exactly the p.o.-projectives, and the mapping* $\Lambda'(X) \to X$ *given by* $\lim_X$ *is a p.o.-projective cover in* $\mathbf{S}$ *for each* $X \in \mathbf{S}$.

Proof. By the fullness of $\mathbf{S}$, the extremally disconnected $X \in \mathbf{S}$ are clearly p.o.-projective in $\mathbf{S}$. Conversely, for any p.o.-projective X in $\mathbf{S}$ the mapping $\Lambda'(X) \to$ $\to X$, which by hypothesis belongs to $\mathbf{S}$, has a right inverse $f\colon X \to \Lambda'(X)$ which must be one-to-one, perfect, and onto since $\Lambda'(X) \to X$ is minimal, hence a homeomorphism, and therefore X is extremally disconnected. The second part follows immediately from this.

A category to which this applies is given by the class of all *rim-compact Hausdorff spaces*: For any such space X, one has $\Lambda'(X) = \Lambda(X)$ by its regularity, and $\Lambda(X)$ is evidently rim-compact since it is zero-dimensional. By contrast, a category which does *not* have this property, and which therefore does not satisfy any of the conditions considered above which would imply this property, is given by the *semi-regular Hausdorff spaces*: If $\Lambda'(X)$ is semi-regular then, by the proof of Proposition 6, $\Lambda'(X) = \Lambda(X)$, hence $\lim\colon \Lambda(X) \to X$ is continuous, and therefore X is regular. Since there exist non-regular semi-regular spaces, this proves the assertion.

We conclude this section with a discussion of p.o.-projective covers of extension spaces. The category to be considered is $\mathbf{H}$, although, clearly, all that is going to be said applies to any subcategory of $\mathbf{H}$ in which p.o.-projectivity and p.o.-projective covers are the same as in $\mathbf{H}$.

If E is an extension space of X, i.e. X is a dense subspace of E, it makes sense to ask how their p.o.-projective covers are related to each other. This question will be considered with the aid of suitable filter spaces.

Let $\Lambda(E \mid X)$ be the subspace of $\Omega(X)$ consisting of all those filters which *converge in* E; this is then an extension of $\Lambda(X)$. Also, let ϱ_X be the mapping $\mathfrak{M} \rightsquigarrow \mathfrak{M} \mid X =$ $= \{V \cap X \mid V \in \mathfrak{M}\}$ for $\mathfrak{M} \in \Omega(E)$.

Lemma 12. $\Omega(E)$ *is mapped homeomorphically to* $\Omega(X)$ *by* ϱ_X, *and* $\Lambda(E)$ *corresponds to* $\Lambda(E \mid X)$ *under* ϱ_X.

Proof. It is clear from maximality that ϱ_X maps $\Omega(E)$ one-to-one into $\Omega(X)$; ontoness follows from the fact that for $\mathfrak{M} \in \Omega(X)$, $\mathfrak{M}^\circ = \{U \mid U \cap X \in \mathfrak{M},\, U \in \mathfrak{O}(E)\}$

belongs to $\Omega(E)$, and $\mathfrak{M}^\circ \mid X = \mathfrak{M}$. Concerning its continuity properties, one has $\varrho_X(\Omega_V(E)) = \Omega_{V \cap X}(X)$ for any open $V \subseteq E$, which shows ϱ_X is a homeomorphism. Finally, for any $\mathfrak{M} \in \Lambda(E)$ the trace $\mathfrak{M} \mid X$ converges in E and thus belongs to $\Lambda(E \mid X)$, and if the latter holds for any $\mathfrak{M} \in \Omega(E)$ then $\mathfrak{M}$ has a cluster point in E and, consequently, converges, i.e. $\mathfrak{M} \in \Lambda(E)$.

Let $\Lambda'(E \mid X)$ now be the space obtained from $\Lambda(E \mid X)$ by modifying the topology with respect to the limit mapping in the same way $\Lambda'(X)$ was derived earlier from $\Lambda(X)$. Since $\lim \mathfrak{M} = \lim (\mathfrak{M} \mid X)$ in E for any $\mathfrak{M} \in \Lambda(E)$ one then obtains from Lemma 12 and Proposition 6:

Proposition 10. *The mapping $\Lambda'(E \mid X) \to E$ by taking limits is a p.o.-projective cover of E in* **H**.

The way in which $\Lambda'(E \mid X)$ and $\Lambda'(X)$ are related to each other carries over to arbitrary p.o.-projective covers as follows:

Corollary 1. *If f: $F \to E$ is a p.o.-projective cover of E in* **H** *then g: $Y \to X$, $Y = f^{-1}(X)$ and $g = f \mid Y$, is a p.o.-projective cover of X in* **H**.[6])

In the special case of completely regular Hausdorff spaces and their compactifications one has:

Corollary 2. *For any compactification E of a completely regular Hausdorff space X, the mapping $\Omega(X) \to E$ by taking limits is a p.o.-projective cover of E in* **H**.

Using the fact that, for an extremally disconnected semi-regular Hausdorff space X, $\lim_X$: $\Lambda(X) \to X$ is a homeomorphism and that an extremally disconnected compact Hausdorff space is the Stone-Čech compactification of any of its dense subspaces, one obtaines as a further specialization:

Corollary 3. *For any compactification E of an extremally disconnected semi-regular Hausdorff space X, the continuous mapping $\beta X \to E$ which extends the natural injection $X \to E$ is a p.o.-projective cover of E in* **H**.

4. Categories of Topological Algebras. The categories considered here have as their objects topological algebras of a certain type, i.e. pairs $A = (X, f)$ where X is a topological space and $f = (f_\alpha)_{\alpha \in I}$ a family of continuous mappings f_α: $X^{\mathfrak{k}_\alpha} \to X$, the family $\tau = (\mathfrak{k}_\alpha)_{\alpha \in I}$ of cardinal numbers being the type, and as their morphisms continuous homomorphisms, given by continuous mappings of the underlying spaces which are homomorphisms for the underlying algebras, i.e. $h(f_\alpha(x)) = g_\alpha(h \circ x)$, for all $x \in X^{\mathfrak{k}_\alpha}$ and $\alpha \in I$, for h: $A \to B$ where $A = (X, f)$ and $B = (Y, g)$.

[6]) For $E = \beta X$, this was shown in [16].

The p.o. homomorphisms of such a category **K**, i.e. those which are given by perfect onto mappings, clearly satisfy the conditions (P1) – (P3) of Section 1 whenever **K** is closed-hereditary, i.e. with any $A \in \mathbf{K}$, any *closed subalgebra* of A and its natural embedding into A again belongs to **K**, essentialness being the same minimality as previously. Also, in any such a category **K** whose underlying spaces are Hausdorff there is an upper bound on the cardinality of the algebras A for which there is a p.o. homomorphism $f\colon A \to B$, depending only on B, since any such A has a *dense subalgebra* generated by card B elements, and its cardinality is bounded by a cardinal number which is determined by card B and the type τ in question. Finally, for any topological algebra A, the class of *all* p.o. homomorphisms $f\colon A \to B$ is clearly small since the analogous condition holds for topological spaces.

The conditions (P4) and (P5) are most readily obtained by imposing natural restrictions on each, the underlying algebras and the underlying spaces of the objects of the category. In this vein, let **A** be an *equational class* of algebras of a certain type, **T** a class of Hausdorff spaces, and $\mathbf{C}(\mathbf{A}, \mathbf{T})$ the category of all those topological algebras whose underlying algebra belongs to **A** and whose underlying space to **T**, with all their continuous homomorphisms. The properties of equational classes, i.e. Birkhoff's theorem, then ensure that the conditions one wants $\mathbf{C}(\mathbf{A}, \mathbf{T})$ to satisfy in this context merely depend on **T**, and so one has, on the basis of Section 2 and Proposition 2:

Proposition 11. *If* **T** *is closed-hereditary, and closed with respect to pullbacks and projective limits of well-ordered inverse systems with p.o. mappings, taken in* **H**, *then p.o.-projectivity is properly behaved in* $\mathbf{C}(\mathbf{A}, \mathbf{T})$, *and the same holds for any subcategory of* $\mathbf{C}(\mathbf{A}, \mathbf{I})$ *which is leftfitting in* $\mathbf{C}(\mathbf{A}, \mathbf{T})$ *with respect to essential p.o. homomorphisms.*

Corollary. *If* **T** *is closed-hereditary and closed with respect to products in* **H** *then p.o.-projectivity is properly behaved in any subcategory of* $\mathbf{C}(\mathbf{A}, \mathbf{T})$ *which is closed-hereditary and closed with respect to products in* $\mathbf{C}(\mathbf{A}, \mathbf{T})$.

We now turn to categories of compact algebras in order to exhibit a certain similarity between them and compact Hausdorff spaces [20]. Let **C** be the class of all compact Hausdorff spaces and $\mathbf{A}(\tau)$ the class of all algebras of type τ. Given any completely regular Hausdorff space X, there exist algebras $A \in \mathbf{C}(\mathbf{A}(\tau), \mathbf{C})$ containing X as subspace, e.g. $A = (\beta X, f)$ where $f = (f_\alpha)_{\alpha \in I}$ consists of arbitrarily chosen projections $f_\alpha\colon (\beta X)^{\mathfrak{t}_\alpha} \to \beta X$. One obtains from this, for instance by considering a suitable closed subalgebra of a product, that there exists an algebra $F_\tau(X)$ in $\mathbf{C}(\mathbf{A}(\tau), \mathbf{C})$ which is, in a way, the topological counterpart to the absolutely free algebra of type τ: $F_\tau(X)$ is *free* in $\mathbf{C}(\mathbf{A}(\tau), \mathbf{C})$ *on the generating space* X in that X is a generating subspace, and any continuous mapping $h_0\colon X \to A$, for any $A \in \mathbf{C}(\mathbf{A}(\tau), \mathbf{C})$ has a (unique) extension to a continuous homomorphism $h\colon F_\tau(X) \to A$.

Now, let **A** be any equational subclass of $\mathbf{A}(\tau)$. It is then clear that $\mathbf{C}(\mathbf{A}, \mathbf{C})$ is an epireflective subcategory of $\mathbf{C}(\mathbf{A}(\tau), \mathbf{C})$, and thus, as far as p.o.-projectives go, Lemma 6 applies. With these concepts, one has:

Proposition 12. *For any extremally disconnected regular Hausdorff space X, $F_\tau(X)$ is p.o.-projective in $\mathbf{C}(\mathbf{A}(\tau), \mathbf{C})$, and the p.o.-projectives in $\mathbf{C}(\mathbf{A}, \mathbf{C})$ are exactly the retracts of the reflections of such $F_\tau(X)$ with compact X.*

Proof. Given a p.o.-homomorphism $f\colon A \to B$ and any $g\colon F_\tau(X) \to B$ in $\mathbf{C}(\mathbf{A}(\tau), \mathbf{C})$, consider the diagram

$$\begin{tikzcd} & X \arrow[d, "v"] \\ f^{-1}(g(X)) \arrow[r, "u"'] & g(X) \end{tikzcd}$$

induced by this. Here u is continuous and points have compact inverse images under u; also if $Y \subseteq f^{-1}(g(X))$ is closed then $Y = Z \cap f^{-1}(g(X))$ with closed $Z \subseteq A$, and $g(Y) = f(Z) \cap g(X)$, which is closed in $g(X)$. It follows that u is perfect onto, and hence there exists a continuous mapping $w\colon X \to f^{-1}(g(X))$ such that $uw = v$. For the extension $h\colon F_\tau(X) \to A$ by freeness one then has $fh \mid X = fw = uw = v = g \mid X$, and this implies $fh = g$.

It follows now immediately, by what has already been pointed out, that the retracts of the reflections of these $F_\tau(X)$ are p.o.-projective in $\mathbf{C}(\mathbf{A}, \mathbf{C})$ since retraction always preserves any kind of projectivity. That the $F_\tau(X)$ with compact X already give all results from the fact that the $A \in \mathbf{C}(\mathbf{A}, \mathbf{C})$ are compact.

Remark. If A is the reflection of some $F_\tau(X)$ in $\mathbf{C}(\mathbf{A}, \mathbf{C})$ then one has a continuous mapping $u\colon X \to A$ which is *universal* for all continuous mappings $v\colon X \to B$, $B \in \mathbf{C}(\mathbf{A}, \mathbf{C})$, in that, for any such v, $v = hu$ with a uniquely determined continuous homomorphism $h\colon A \to B$. Whether u is an embedding is a matter which depends on the class **A**.

The remainder of this section will be concerned with rather more special situations, namely, with different categories of topological groups. To begin with, one has the following categories of this type to which Proposition 11 or its corollary apply: First, there is the category of all Hausdorff topological groups and their continuous, or their perfect, homomorphisms, and then there are, among others, the subcategories given by the following classes of groups:

compact groups
locally compact groups
σ-compact groups
zero-dimensional groups

profinite groups
prop-p groups
prodiscrete groups

and in each case the full subcategory given by the abelian groups in question.[7])

For some categories of compact groups the onto (= p.o.) homomorphisms are actually the epimorphisms and hence, in these cases we are dealing with ordinary projectivity. This is evidently so for the compact, profinite, and pro-p *abelian* groups; the resulting proper behaviour of projectivity in these categories is, of course. a well-known fact in virtue of Pontryagin Duality, but it may be worth noting that the present setting provides a proof for this independent from the machinery of representation theory. Non-abelian cases are given by:

Lemma 13. *For profinite and pro-p groups the epimorphisms are exactly the onto homomorphisms.*

Proof. A construction due to Eilenberg and Moore [17] shows that in the category of all *finite* groups and their homomorphisms, subgroups are equalizers. If G is now any profinite group and $H \subseteq G$ a closed proper subgroup then, for each sufficiently small open normal subgroup N of G, $HN \subset G$ and there exist homomorphisms f_N, g_N: $G/N \to G_N$, G_N finite, which coincide exactly on HN/N; the embedding $G \to \prod G/N$ then provides continuous homomorphisms f, g: $G \to \prod G_N$ such that $H = \bigcap HN$ is the subgroup on which f and g coincide, which proves the assertion for profinite groups.

For pro-p groups one has the rather different circumstance that any closed proper subgroup H of such a group G is actually contained in a closed proper normal subgroup of G: One has $HN \subset G$ for some open normal subgroup N of G, and hence NH is contained in some *maximal* open subgroup U of G; U, however, is actually normal since a maximal subgroup of a p-group is normal, and U/N is a maximal subgroup of the p-group G/N.

In some of the categories mentioned above, particular p.o.-projectives are provided by suitable types of free objects. For instance, in the category **HG** of all Hausdorff topological groups and all their continuous homomorphisms there exists, for any completely regular Hausdorff space X, a group $F(X)$, the *free topological group on* X which has X as a generating subspace in such a way that any continuous mapping from X into a $G \in$ **HG** has an extension to a continuous homomorphism $F(X) \to G$ [10]. By an argument analogous to the first part of the proof of Proposition 12 one readily sees that $F(X)$ *is p.o.-projective in* **HG** *for extremally discon-*

[7]) *Profinite* means compact Hausdorff zero-dimensional or, equivalently, projective limit of finite groups. *Prop-p* groups and *pro-discrete* groups are projective limits of p-groups and discrete groups respectively.

nected X. However, none of these $F(X)$ can be non-trivial p.o.-projective covers in **HG**: The underlying group of such an $F(X)$ is a *free* group, hence the underlying group of any normal subgroup of $F(X)$ is again free, and free groups are known not to possess any compact Hausdorff topology (the trivial case, empty set of generators, of course excluded).

For the category $\mathbf{C}(\mathbf{G}, \mathbf{C})$, $\mathbf{G}$ the class of all groups, Proposition 12 is applicable; more specifically, the image of X in the reflection in $\mathbf{C}(\mathbf{G}, \mathbf{C})$ of the absolutely free compact algebra of corresponding type with free generating space X is homeomorphic to X, and the dense subgroup which it generates is a *free* group – facts which are not obvious from its definition as reflection but, rather, require a good deal of additional argument.

In the case of profinite and pro-p groups one has:

Lemma 14. *For any zero-dimensional Hausdorff space X there exists a free profinite and a free pro-p group on X, and these are projective in their respective categories.*

Proof. To construct these groups, let F be the free group generated by the points of the given space X, and let $\mathfrak{N}$ be the collection of all normal subgroups $N \subseteq F$ such that

(1) F/N is finite (is a p-group), and
(2) $(sN) \cap X$ is open-closed in X for each $s \in F$.

$\mathfrak{N}$ is evidently non-void ($F \in \mathfrak{N}$), and one readily sees that it is a filter basis. Now, let $s = x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}$ where $x_i \in X$ and $\varepsilon_i = \pm 1$, be any non-unit element of F, and let $X = U_1 \cup \ldots \cup U_k$ be an open decomposition of X separating the different x_i. Then there exists a homomorphism $h\colon F \to G$, G a finite group (p-group), constant on the U_i, and such that $h(s) \neq e$, by the fact that in a free group the intersection of all normal subgroups with finite (p-power) index is trivial. It follows that $\operatorname{Ker}(h) \in \in \mathfrak{N}$ and $s \notin \operatorname{Ker}(h)$, hence $\bigcap N(N \in \mathfrak{N}) = \{e\}$. Finally, any open-closed $U \subseteq X$ is an $(sN) \cap X$ for some $N \in \mathfrak{N}$; take $N = \operatorname{Ker}(h)$ for any homomorphism $h\colon F \to G$, G appropriate, which is constant on U and on $X - U$, with different values. The group topology with $\mathfrak{N}$ as neighbourhood basis for the unit is Hausdorff and totally bounded, and its restriction to X gives the topology of X; the resulting topological group contains X as a subspace, and its compact completion F^* is a free profinite (pro-p) group on X: For any continuous mapping $f_0\colon X \to G$, G the kind of group in question, the extending group homomorphism $f_1\colon F \to G$ is continuous with respect to $\mathfrak{N}$ and hence extends continuously to a homomorphism $f\colon F^* \to G$.

Now let $f\colon G \to H$ be an epimorphism, and $g\colon F^* \to H$ any homomorphism. Then there exists [21] a continuous section $u\colon H \to G$ of f, and the continuous mapping $h_0 = u(g \mid X)$ extends to a homomorphism $h\colon F^* \to G$ for which $fh = g$.

Corollary 1. *The free profinite (pro-p) group on a one-element space is isomorphic to the additive group of Z-adic (p-adic) integers.*

Remark. It follows from this that the projective cover of a cyclic group of order p in the category of pro-p groups is given by $\mathbf{Z}_p$, the group of p-adic integers; now, $\mathbf{Z}_p$ clearly has automorphisms f distinct from the identity for which $gf = g$, $g\colon Z_p \to \to \mathbf{Z}_p/p\mathbf{Z}_p$ the natural homomorphism (Einseinheiten) – hence projective covers are not rigid here.

Corollary 2. *In the category of profinite groups, the projectives are exactly the retracts of the profinite groups which are free on some subspace.*

For pro-p groups one can actually say a lot more, as will be shown shortly. First, let G be any profinite group and $\Phi(G)$ the intersection of its maximal (proper) open subgroups. $\Phi(G)$ is clearly closed and invariant with respect to *all* (continuous) automorphisms of G. Furthermore:

Lemma 15. *For any closed normal subgroup N of G, the natural homomorphism $G \to G/N$ is essential iff $N \subseteq \Phi(G)$.*

Proof. Let $G \to G/N$ be essential and $H \subseteq G$ any maximal open subgroup. Then $NH \subset G$ by essentialness, hence $NH = H$ and thus $N \subseteq H$; in all this shows $N \subseteq \Phi(G)$. Conversely, let $N \subseteq \Phi(G)$ and $S \subseteq G$ a proper closed subgroup; then there exists an open normal subgroup U of G such that $SU \subset G$, and hence a maximal open subgroup $H \supseteq SU$. From this one has $S, N \subseteq H$ and therefore $SN \neq G$; this expresses the fact that $G \to G/N$ is essential.

Corollary 1. *For profinite groups, any essential epimorphism $f\colon G \to H$ induces an isomorphism $g\colon G/\Phi(G) \to H/\Phi(H)$.*

Proof. For any epimorphism $f\colon G \to H$ one has the commutative diagram of epimorphisms

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ G/\Phi(G) & \xrightarrow[g]{} & H/\Phi(H) \end{array}$$

where u and v are natural, and g is determined by the fact that $\ker(vf) = f^{-1}(\Phi(H)) \supseteq \supseteq \Phi(G)$. Now, if f is essential one readily sees that g is also essential (by checking minimality), and since $\Phi(G/\Phi(G))$ is trivial this shows that g is an isomorphism.

Corollary 2. *For projective profinite groups, $G \cong H$ iff $G/\Phi(G) \cong H/\Phi(H)$.*

Proof. In one direction the implication is obvious. For the other, projectivity of G, H implies that $G \to G/\Phi(G)$ and $H \to H/\Phi(H)$ are projective covers, and the essential uniqueness of these then proves the assertion.

Corollary 3. *For a profinite group G, any projective cover $f\colon H \to G/\Phi(G)$ determines a projective cover $g\colon H \to G$.*

Proof. The projectivity of H provides a homomorphism $h\colon H \to G$ such that $f = gh$, and since g and f are essential epimorphisms, h is one, too.

For a pro-p group G, the maximal open subgroups are normal, as was noted above, and hence have index p. It follows from this that $G/\Phi(G) = (\mathbf{Z}/p\mathbf{Z})^{\mathfrak{k}}$ where $\mathfrak{k}$ is the common cardinal number of the sets $\mathfrak{M}$ of maximal open normal subgroups N of G which are maximal with respect to the condition that $G = N_0 \,.\, \bigcap N(N_0 \neq \neq N \in \mathfrak{M})$ for each $N_0 \in \mathfrak{M}$; $\mathfrak{k}$ will be called the *colength* of G, denoted by $\operatorname{col}(G)$. It is then clear that:

Corollary 4. *For projective pro-p groups, $G = H$ iff* $\operatorname{col}(G) = \operatorname{col}(H)$.

Thus, the projective pro-p groups are distinguished by a single cardinal invariant. In particular, then, it should be possible to determine this invariant for those pro-p groups which are free on a given space. In the following, let $\mathbf{F}_p$ be the field of p elements, $C(X, \mathbf{F}_p)$ the $\mathbf{F}_p$-module of all continuous $\mathbf{F}_p$-valued functions on the space X, and let $\operatorname{Hom}(\ldots)$ stand for continuous homomorphisms.

Lemma 16. *If the pro-p group G is free on the space X then* $\operatorname{col}(G)$ *is the* $\mathbf{F}_p$-*dimension of* $C(X, \mathbf{F}_p)$.

Proof. $\operatorname{Hom}(G, \mathbf{F}_p)$ is isomorphic to $\operatorname{Hom}(G/\Phi(G), \mathbf{F}_p)$ (as $\mathbf{F}_p$-module), and since $G/\Phi(G) \cong \mathbf{F}_p^{\mathfrak{k}}$ as groups, where $\mathfrak{k} = \operatorname{col}(G)$, $\operatorname{Hom}(G/\Phi(G), \mathbf{F}_p)$ is isomorphic to the $\mathbf{F}_p$-module $C_0(I, \mathbf{F}_p)$ of all $\mathbf{F}_p$-valued functions of finite support on a set I with $\operatorname{card} I = \mathfrak{k}$. On the other hand, by the freeness of G, $\operatorname{Hom}(G, \mathbf{F}_p) \cong C(X, \mathbf{F}_p)$ and thus $C(X, \mathbf{F}_p) \cong C_0(I, \mathbf{F}_p)$. Since $\operatorname{col}(G) = \operatorname{card} I$ is clearly the $\mathbf{F}_p$-dimension of the module on the right, this proves the assertion.

Proposition 13. *The projective pro-p groups are exactly the pro-p groups which are free on the one-point compactification of a discrete space.*

Proof. In view of the preceding results it suffices to show that the particular projective groups mentioned take on all possible colengths. However, this is clear since $\dim C(X, \mathbf{F}_p) = \operatorname{card} X$ for the one-point compactification X of a discrete space (which includes the case of finite X, incidentally: being already compact these are equal to their one-point compactifications).

Remark 1. It follows from this proposition, by arguments analogous to some used by Graev [10], that the projective pro-p groups are exactly those which are called free in [21]: Graev considers (in some other categories) what might be called free topological groups on *pointed spaces*, the point, of course, corresponding to the unit of the group, and the "free" pro-p groups of [21] can analogously be described as those which are free on pointed spaces (X, e) where X is the one-point compactification of its discrete subspace $X - \{e\}$. Now, because the space X is not connected such a free group on (X, e) is also free, in the sense used here, on a subspace homeomorphic to X, which one can show by using the proof for the analogous statement in [10]. A different argument for the coincidence of projectivity with "freeness", and hence with freeness, is contained in [21], based on a certain cohomological dimension, where projectivity is equivalent to the dimension being at most 1.

Remark 2. For vastly different spaces can the free pro-p group on them still be the same: For discrete X, the dimension of $C(X, F_p)$ is $2^{\operatorname{card} X}$, and hence a countable discrete space and the one-point compactification of a discrete space of $2^{\aleph_0}$ points have isomorphic free pro-p groups.

Remark 3. A projective *profinite* group need not be free on any subspace. First we show that any projective pro-p group is also projective profinite, by proving that *the pro-p groups are left-fitting, with respect to essential epimorphisms, among the profinite groups*: Let $f\colon G \to H$ be an essential epimorphism, G profinite and H pro-p, and let $K = \operatorname{Ker}(f)$. Then, for any p-Sylow subgroup P of G [21] and any open normal subgroup N of G, one has that the index $(G\colon KN)$ is a p-power since G/KN is a finite quotient of G/K, which is isomorphic to H, and $(G\colon PN)$ is prime to p. It follows that $G = KNPN = KPN$, and by taking the intersection over all N one obtains $G = KP$. Since f is essential this implies $G = P$. Now, a pro-p group has *no* non-trivial homomorphism into a pro-q group for $q \neq p$, and thus any projective pro-p group, although it is projective profinite, fails to be free profinite.

Remark 4. By Duality, one has that an abelian profinite group is projective (in its category) iff it is torsion free, which is a completely internal characterization. This raises the question whether projectivity for arbitrary profinite, or pro-p groups can be characterized by internal conditions.

Remark 5. The projective profinite abelian groups are, again by Duality, described by a *family* of cardinal invariants, the collengths of their p-Sylow subgroups for the different primes p; one wonders whether the same holds for the projective pro-finite groups.

Concluding Remarks. In this final section we collect a number of comments which deal with some further aspects of the preceding work.

(1) As is noted in [9], the dual equivalence between the category of compact zero-dimensional Hausdorff, i.e. compact Boolean, spaces and their continuous

mappings and the category of Boolean lattices (with unit) and Boolean homomorphisms establishes the duals of the results on the former for the latter; in the present setting, going somewhat farther than [9], this means that injectivity is properly behaved, and that the injective Boolean lattices are exactly the complete ones. Now, on the basis of a result in [6] this can be extended to the category **BS** of *all* Boolean, i.e. locally compact zero-dimensional Hausdorff spaces and their perfect mappings. The category of lattices which is dual to this is the category **BL** of all Boolean lattices, in the wider sense of the term, i.e. distributive, relatively complemented lattices with zero, and the zero-preserving lattice homomorphisms $f\colon A \to B$ with the property that for each $b \in B$ there exists $a \in A$ such that $f(a) \geqq b$ [6]. The category of Boolean lattices with unit and Boolean homomorphisms is a full subcategory of this for if f: $A \to B$ belongs to **BL** and A and B have units then f clearly is unitary, and conversely. It follows now that *injectivity is properly behaved in* **BL**; moreover, some additional arguments show that the injective objects in **BL** are exactly the conditionally complete ones.

(2) The category **RC** of all real-compact Hausdorff spaces and their continuous mappings is dually equivalent to a category **CF** of certain algebras over, say, the real number field **R**, via the correspondances $X \rightsquigarrow C(X)$, $A \rightsquigarrow \mathrm{Hom}\,(A, \mathbf{R})$ (Stone-Zariski topology) for their objects and the usual associated correspondances for their morphisms. Consequently one obtains, from the results on p.o.-projectivity in **RC**, statements about injectivity in **CF** with respect to certain types of *embeddings*, and the $A \in$ **CF** which are injective in the given sense are known to be exactly those which are conditionally complete in their usual partial ordering. There is some reason to expect that the associated essential extensions, especially the maximal ones, may be of independent interest: In [18] it is shown that, *for any commutative semi-simple ring A with unit and Hausdorff maximal ideal space $\Omega(A)$ (Stone-Zariski topology) the Utumi maximal ring of quotients of A [24] has the projective cover of $\Omega(A)$ as its maximal ideal space.* Now, if $f\colon X \to Y$ is a projective cover in **RC** then one has for the corresponding injective hull $f^*\colon C(Y) \to C(X)$ that $\Omega(C(X))$ is the projective cover of $\Omega(C(Y))$ since these spaces are homeomorphic to βX and βY respectively. In this situation it may be that the largest Utumi ring of quotients of $f^*(C(Y))$ in $C(X)$ (which need *not* be all of $C(X)$) has X as its real-maximal ideal space and is somehow a "best" ring of quotients of $C(Y)$ with respect to not loosing any real maximal ideals of $C(Y)$; moreover, this may only be a special case of a much more general situation involving algebras over arbitrary fields.

(3) In order to obtain an application, and illustration, of Proposition 11 involving algebras other than groups one might think of compact (Hausdorff) Boolean rings with unit and continuous unitary ring homomorphisms as a simple and manageable case to look at, but unfortunately this happens to be too simple a case to be of any illustration value: Any Boolean ring with unit is semi-simple, and hence any compact ring R of this type is a product of finite simple rings, by a general structure theorem of Kaplansky's; moreover, the factors must again be Boolean, and hence they are

all isomorphic to the field $\mathbf{F}_2$. Thus, R is essentially the ring of all $\mathbf{F}_2$-valued functions on some set X, where card X is the number of its open maximal ideals. From this and a few additional elementary considerations one obtains that *the category of all compact Boolean rings with unit and continuous unitary ring homomorphisms is dually equivalent to the category of sets and hence consists entirely of projectives.*[8]) The same holds if one takes the commutative rings in which $x^p = x$ holds for some fixed prime p other than 2 in place of the Boolean ones: Such a ring R has zero prime-radical, and since $x^p = x$ also holds in any homomorphic image of R the prime ideals P of R are all maximal and $R/P \cong \mathbf{F}_p$. In particular, R is semi-simple, and from here on the argument continues as above, with $\mathbf{F}_p$ in place of $\mathbf{F}_2$.

(4) The *duals* of the results of Section 1 can be used to show that *injectivity with respect to (norm-preserving) embeddings is properly behaved in the category of all Banach spaces*[9]) *and norm-decreasing linear mappings.* This provides an alternative proof to [5] for the existence of injective hulls in this category, and otherwise complements the results of [5].

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[8]) This has also been observed by F. E. J. Linton, based, however, on an entirely different argument.

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