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ONE-TO-ONE CONTINUOUS IMAGES OF A LINE

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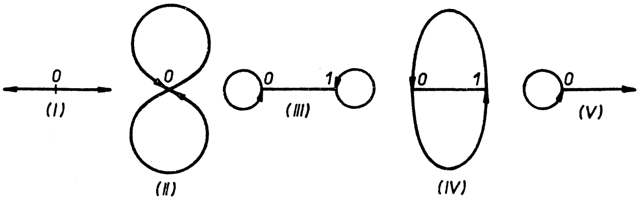
One of the simplest and most intuitively satisfying propositions of elementary topology is that a one-to-one continuous image of the closed interval [0, 1] into Euclidean n-space is a simple arc and the mapping is reversibly continuous (i.e., a homeomorphism). That the proposition is false for such mappings of the open interval (0, 1), when first encountered, requires some thought (especially when the image is required to be a bounded subset of 2-space). While a one-to-one continuous image of (0, 1) or (equivalently) the reals must be arcwise connected it may fail to be locally arcwise connected at any point. As curious as this may seem certain flows in topological dynamics [2] have orbits which are precisely one-to-one continuous images of the reals. For these and other reasons, there is considerable interest in mappings of this type.

Konstantinov has shown [7] that there exists a one-to-one bounded continuous image X of the reals in the plane such that X is a smooth curve of bounded curvature but X is not connected im kleinen at any point. (An earlier example having these properties has been given by Bing [1]). Actually X and its closure in the plane may be quite pathological. In fact, such an image in 3-space, even if required to be locally connected, may still fail to be locally arcwise connected at any point [3]. Hence, the following theorem by Lelek and McAuley [9] is of interest.

Theorem 1. *If a one-to-one continuous image X of the (real) line is a locally compact, locally connected, metric space, then X is homeomorphic to one of the five elementary plane figures:* (I) *an open interval,* (II) *a figure eight,* (III) *a dumbbell,* (IV) *a theta curve, or* (V) *a noose (figure) nine.*



The fact that their proof is rather simple (as well as elementary) led me to suspect that a better theorem might be possible. This is indeed the case.

As a generalization of connectedness im kleinen at a point I introduced [5] the notion of aposyndesis.

Definition. *A topological space X is said to be aposyndetic at $x \in X$ if for each $y \in X - x$ there exists a closed and connected subset of $X - y$ which contains x in its interior. If X is aposyndetic at each of its points, X is said to be aposyndetic.*

Theorem 2. *If a one-to-one continuous image X of the (real) line is a locally compact, aposyndetic, metric space, then X is homeomorphic to one of the plane figures*: (I)–(V). [6]

One can hardly imagine how anything weaker than aposyndesis can be substituted for local connectedness. But Theorem 1 might possibly be improved by weakening the local compactness requirement, for example, by substituting completeness or (more generally – in the presence of local connectivity) local arcwise connectivity. Actually the argument in [9] for Theorem 1 proves the following.

Theorem 3. *If a one-to-one continuous image X of the (real) line is a locally arcwise connected metric space, then X is homeomorphic to one of the five figures*: (I)–(V).

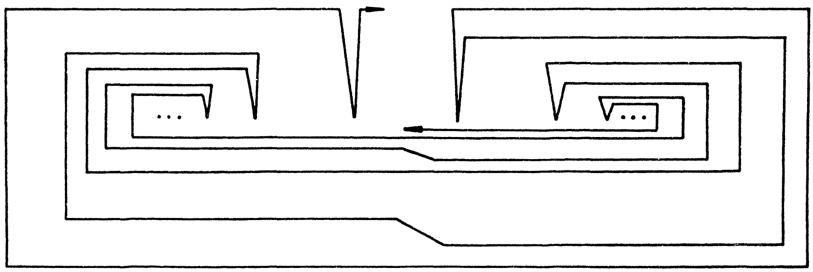
But let us again consider the problem of weakening the requirement of local compactness in Theorem 1. We have remarked above (and at the end of this paper) that we cannot discard it completely even when X lies in 3-space (and satisfies rather stringent geometric conditions). But what if X is required to lie in 2-space [9]? The answer is: "yes", it can be completely discarded because of the following lemma.

Lemma. *If a one-to-one continuous image X of the (real) line is a locally connected subspace of the 2-sphere, then X is a locally compact space.* [3, 6]

It follows from Theorem 1 that

Theorem 4. *If a one-to-one continuous plane image of the (real) line is locally connected, then it is one of the five curves*: (I)–(V). [3, 6]

Now here (in Theorem 4) the proof in [6] is so difficult that one would not suspect that locally connectivity could be weakened. And one can easily construct one-to-one continuous plane images of a line which are aposyndetic but not one of the curves (I)–(V), that is, Theorem 2 would not hold true without local compactness even in 2-space. But one should look more closely at the consequences of discarding local compactness. If a space is locally compact and aposyndetic then it is semi-locally-connected (= each point is contained in an arbitrarily small open set whose complement has finitely many components) [10]. However, in the absence of local compactness (or local peripheral compactness) this is no longer the case [4]. So let us assume that the one-to-one continuous plane image X of a line is both aposyndetic and semi-locally-connected then must X be one of the five curves? The answer is: NO. The figure below indicates how to construct such a plane image.



But can 2-aposyndesis plus semi-local-connectedness be substituted for local connectivity in Theorem 4? (It adds some interest to know that if a plane continuum is 2-aposyndetic it is locally connected. A space is 2-aposyndetic if when given three arbitrary points, x, y, and z, there exists a closed and connected subset which contains x in its interior and misses both y and z.) Again what would be the effect of requiring that X be a smooth plane curve of bounded curvature and aposyndetic or semi-locally-connected?

Returning to 3-space, in [9] Lelek and McAuley raise the question P 616. Is it true that if a set X in a Euclidean space is a smooth one-to-one image of the (real) line, X is of bounded curvature and X is locally connected, then X is locally compact?

The answer to this question is YES by the lemma if X is in 2-space but it is NO for X in 3-space. Consider the following example.

Let $U_1, U_2, U_3, \ldots$ denote a countable collection of spherical regions in 3-space which form a basis for the usual 3-space topology. Now let $T_1, T_2, T_3, \ldots$ denote a simple well-ordering of all pairs $\{U_i, U_j\}$ with $i < j$ and $U_i \cap U_j = \emptyset$. Draw a topological (open) ray which starts at a point in the first member of T_1 and runs straight to a point in the second member of T_1, then to a point in the first member of T_2 and straight to a point in the second member of T_2, then to a point in the first member of T_3, etc. This can be done so that the curve is smooth and of bounded curvature. Furthermore, multiple points may be avoided. It is not difficult to prove that the result of the construction is locally connected and clearly it is a one-to-one continuous image of a line.

There is still left a rather interesting question. Konstantinov has recently shown [8] that if X is a one-to-one bounded plane image of a ray, X is nowhere locally connected, and X is a smooth curve of bounded curvature, then the complement of $\overline{X}$ has at least four components. So if X is a bounded smooth curve of bounded curvature whose closure has less than four complementary domains, must X be one of the five curves: (I)—(V)?

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