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Praha

Introduction. In this paper a brief exposition of convergence spaces and convergence groups and their basic properties is given to the extent necessary for understanding the still unsolved problems which are mentioned in this survey article.

Section 1 deals with closure spaces, and the definitions are given in such a way as to make evident analogous properties of closure spaces and topological spaces. Section 2 is concerned with a special closure space, i.e. with the convergence space. The closure operation for convergence spaces is defined by means of a convergence, i.e. of a special kind of mapping lim fulfilling the axioms $\mathscr{L}_0$, $\mathscr{L}_1$, $\mathscr{L}_2$. Section 3 is devoted to continuous real-valued functions on closure spaces in general and especially on convergence spaces. In the Section 4 the notion of sequential envelope is mentioned and in the last Section 5 a convergence group is defined. The exposition is accompanied by a number of examples (E_1 – E_{10}). Furthermore 15 unsolved problems are given.

1. Each topological space fulfils the axiom of the closed closure. There are, however, spaces of interest in which the axiom of the closed closure is not satisfied. This will be shown by the example E_1 [10]:

E_1. A real-valued function $f(x, y)$ of two real variables x and y is called partially continuous (in variables x, y) if $\lim x_n = x_0$ and $\lim y_n = y_0$ implies that

$$\lim f(x_n, y_0) = \lim f(x_0, y_n) = f(x_0, y_0)\,.$$

From the definition of partial continuity it follows that the ε-neighbourhoods of a point (x_0, y_0) have the form of crosses:

$$U_\varepsilon(x_0, y_0) = \{(x, y_0) : |x - x_0| < \varepsilon\} \cup \{(x_0, y) : |y - y_0| < \varepsilon\}\,,$$

ε being any positive number. Denote by λ a set operation such that $(x_0, y_0) \in \lambda A$ iff $U_\varepsilon(x_0, y_0) \cap A \neq \emptyset$ for each $U_\varepsilon(x_0, y_0)$. Denote

$$A = \{(x, y) : 0 < x < 1, 0 < y < 1\}\,,$$
$$B = \{(x, y) : 0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}\,,$$

then $\lambda A = B - \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and $\lambda\lambda A = B$ so that $\lambda\lambda A \neq \lambda A$ and so the axiom of the closed closure is not valid.

For cases like this it is necessary to generalize the notion of topology. This is done in the following manner [2], [3]:

Let P be a point set. A *closure operation* (or simply a closure) for P is a map v on the system 2^P of all subsets of P into 2^P such that the following axioms are satisfied:

1. $v\emptyset = \emptyset$,
2. $A \subset vA$,
3. $v(A \cup B) = vA \cup vB$.

P is called a *closure space* and denoted (P, v) or simply P. A set A is *closed* if $vA = A$; it is *open* if $P - A$ is closed.

If v_1 and v_2 are two closure operations for the same point set P, then we say that v_1 is *finer* than v_2 (or v_2 is *coarser* than v_1) if

$$A \subset P \quad \text{implies} \quad v_1 A \subset v_2 A .$$

Let A be any set in a closure space (P, v). Form the successive closures [6]

$$v^0 A = A \subset v^1 A = vA \subset v^2 A = vvA \subset \ldots \subset v^\xi A \subset \ldots,$$

ξ being an ordinal and[1])

$$(1) \qquad \begin{aligned} v^\xi A &= vv^{\xi-1} A \quad \text{if} \quad \xi - 1 \text{ exists} \\ v^\xi A &= \bigcup_{\eta<\xi} v^\eta A \quad \text{if} \quad \xi - 1 \text{ does not exist and } \xi > 0 . \end{aligned}$$

It can be proved that for each fixed ordinal ξ the map v^ξ fulfils the axioms 1., 2., 3. Consequently, v^ξ is a closure operation for P which is coarser than v. We have the following statement [3]:

Let (P, v) be a closure space. Then there is an ordinal π such that $\operatorname{card} \pi \leqq \aleph_{\alpha+1}$, $\aleph_\alpha$ *being the power of P, and such that (P, v^π) is a topological space.*

The topology v^π is called the *topological modification* and the topological space (P, v^π) the *topological modification of the closure space* (P, v) [2], [3].

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[1]) Notice that (1) can be written in the form of one formula

$$v^\xi A = \bigcup_{\eta<\xi} vv^\eta A , \quad \xi > 0 .$$

M. Dolcher [4] defines the closure v_D^ξ by the formula

$$v_D^\xi A = v_D \bigcup_{\eta<\xi} v_D^\eta A , \quad \xi > 0 .$$

Evidently $v_D^\xi A = v^\xi A$ for each finite ξ and $v_D^\xi A = v^{\xi+1} A$ for each infinite ξ.

A *v-neighbourhood* of a point x in a closure space (P, v) is any set $V \subset P$ such that

$$x \in P - v(P - V). \tag{2}$$

The collection of all neighbourhoods of a point x is a filter. The least power of a base for the neighbourhood system of x is called the *character* of the point x.

A T_1 *closure* space is a closure space (P, v) such that each one-point subset[2]) is closed, i.e. $vx = x$ for each $x \in P$. A convergence space (L, λ) is *separated*, if for any two distinct points $x, y \in L$ there are disjoint neighbourhoods $V(x) \cap V(y) = \emptyset$. A convergence space (L, λ) is *regular*, if, for any point x and each neighbourhood $V(x)$, there is a neighbourhood $V_0(x)$ such that $\lambda\, V_0(x) \subset V(x)$.

2. The set mapping λ in the example E_1 is a special closure, so-called convergence closure. It is defined by means of convergence as follows:

Let L be a point set. A *convergence* for L is a map which assigns to some sequences $\{x_n\}$ of points $x_n \in L$ points[3]) $\lim x_n \in L$ such that the *axioms of convergence* are fulfilled:

$\mathcal{L}_0$) $\lim a_n = a$ and $\lim a_n = b$ implies that $a = b$,

$\mathcal{L}_1$) if $a_n = a$ for each n then $\lim a_n = a$,

$\mathcal{L}_2$) if $\lim a_n = a$ then $\lim a_{n_i} = a$.

The *closure* (more precisely *λ-closure*) of a set $A \subset L$ is defined as the set

$$\lambda A = \{x : x = \lim x_n, \bigcup x_n \subset A\}. \tag{3}$$

It is easy to prove that (L, λ) is a T_1 closure space. It is called a *convergence space*. The example E_1 shows that a convergence space need not be a topological space. If a convergence closure λ is a topology, then (L, λ) is called a *Fréchet space*.

According to (2) a neighbourhood of a point $x \in L$ is any set $U \subset L$ such that $x \in U - \lambda(L - U)$. It is easy to see that the following statement is true:

$U \subset L$ is a λ-neighbourhood of a point $x \in L$ if and only if $\lim x_n = x$ *implies that $x_n \in U$ for nearly all n.*

Now it is possible to define a new convergence, the so called *star convergence* $\lim^*$, by means of λ-neighbourhoods, viz: $\lim^* x_n = x$ if each λ-neighbourhood $U(x)$ of x contains x_n for nearly all n.

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[2]) For the sake of simplicity a one point set (x) will be denoted by x. In this sense $\bigcup x_n$ denotes the point set consisting of all points x_n.

[3]) Instead of usual symbols f, g, φ, etc. for maps, the sign lim, for historical reasons, is used. For the sake of brevity we write $\lim x_n$ instead of $\lim(\{x_n\}_{n=1}^{\infty})$. The convergence, as a map, can also be defined as the set $\mathfrak{L}$ of all pairs $(\{x_n\}, x)$ such that $\lim x_n = x$.

Evidently the star convergence lim* fulfils axioms $\mathscr{L}_0$, $\mathscr{L}_1$, $\mathscr{L}_2$. In such a way we have a "*star closure*" λ^*:

$$\lambda^* A = \{x : x = \lim^* x_n, \bigcup x_n \subset A\} .$$

It can be proved that

$$\lambda^* A = \lambda A \quad \text{for each} \quad A \subset L .$$

Let lim be a convergence for L. Then lim = lim* if and only if the *Urysohn – Alexandrov axiom* is fulfilled [1]:

$\mathscr{L}_3$) If $\{x_n\}$ does not converge to x, then there is a subsequence $\{x_{n_i}\}$ no subsequence of which converges to x.

In such a way to each convergence lim for L we can assign the star convergence lim* fulfilling $\mathscr{L}_3$ and such that both convergence closures λ and λ^* are identical[4]).

If we denote[3]) by $\mathfrak{L}$ the set of all pairs $(\{x_n\}, x)$ such that $\lim x_n = x$ and by $\mathfrak{L}^*$ the set of all $(\{x_n\}, x)$ such that $\lim^* x_n = x$, then evidently $\mathfrak{L} \subset \mathfrak{L}^*$. From this it easily follows that the star convergence is maximal in the sense that if $\mathfrak{M}$ is any convergence defined for L such that the convergence closure μ induced by $\mathfrak{M}$ is equal to λ, then $\mathfrak{M} \subset \mathfrak{L}^*$. Consequently lim* is sometimes called the *maximal* or *largest convergence* [6], [12].

Now, let us give some examples illustrating the star convergence.

E$_2$. Let R be the set of all real numbers and lim the usual convergence for R. Define another convergence $\lim_0$ for R:

$$\lim_0 x_n = x \quad \text{whenever} \quad \sum |x_n - x| < \infty .$$

The convergence $\lim_0$ does not fulfil $\mathscr{L}_3$. It is easy to see that $\lim = \lim_0^*$ for R.

E$_3$. Let X be a point set and 2^X the system of all subsets of X. Define

$$\operatorname{Lim} A_n = A \quad \text{whenever} \quad \operatorname{Lim\,sup} A_n = \operatorname{Lim\,inf} A_n = A .$$

Then Lim is a star convergence (fulfilling $\mathscr{L}_3$) and $(2^X, \lambda)$ is a convergence space.

E$_4$. Let X be a non void point set and $\mathscr{F}(X)$ a class of all real-valued functions on X. Define

$$\lim f_n = f \quad \text{whenever} \quad \lim f_n(x) = f(x) \quad \text{for each} \quad x \in X .$$

Then lim is a pointwise convergence which fulfils $\mathscr{L}_3$ and $(\mathscr{F}(X), \lambda)$ is a convergence space. If $X = R$ and if $\mathscr{C}$ denotes the class of all continuous real-valued functions

4) P. Urysohn [15] calls lim the convergence "*a priori*" and lim* the convergence "*a posteriori*".

on R, then $\lambda\mathscr{C} = \mathscr{B}_1$, $\mathscr{B}_1$ denoting the class of all continuous functions and of all Baire functions of the first class.

Problem 1. *Let (L, λ) be a convergence space.*

a) *What are the necessary and sufficient conditions such that the following statement $(+)$ is true?*

$(+)$ *If $A_n \subset L$ and $z \in L - \bigcup\lambda A_n$ is a point each neighbourhood of which contains points of A_n for nearly all n, then there is a sequence of $x_n \in A_n$ converging to z.*

b) *Does there exist a convergence space such that its convergence is the star convergence and such that $(+)$ is not true?*

c) *Is $(+)$ true if the convergence is the star convergence and (L, λ) is first countable?*

Let (L, λ) be a convergence space. Then

$$A \subset L \quad \text{implies that} \quad \lambda\lambda^{\omega_1}A = \lambda A\,,$$

ω_1 being the first uncountable ordinal. In such a way we have $\lambda^{\omega_1}\lambda^{\omega_1}A = \lambda^{\omega_1}A$. Therefore, λ^{ω_1} is a topology for L. λ^{ω_1} is the topological modification of the convergence closure λ. The space (L, λ^{ω_1}) is a sequential space [5] and each sequential space can be obtained in the way described above.

Let (L, λ) be a convergence space. Let $\varphi(A)$, $A \in 2^L$, be a mapping on the system 2^L of all subsets of L into the set of all ordinals $\xi < \omega_1$ such that $\varphi(A)$ is the least ordinal, for which $\lambda^{\varphi(A)}A$ is a λ-closed set. Denote $\varphi(2^L)$ the set of all such ordinals $\varphi(A)$, $A \in 2^L$.

Problem 2. *Let L be an infinite point set and H a set of ordinals. What are necessary and sufficient conditions (for H) such that there is a convergence closure λ for L with the property $H = \varphi(2^L)$?*

Problem 3. *What conditions must a convergence closure λ satisfy in order that λ^{ω_1} be a regular space? (more generally: replace λ by a closure v and λ^{ω_1} by v^π).*

3. Let (P_1, v_1), (P_2, v_2) be closure spaces. A map $\varphi : P_1 \to P_2$ is continuous [2], [3] on (P_1, v_1) if the implication (4) is true:

$$\text{(4)} \qquad \text{If} \quad A \subset P_1 \quad \text{then} \quad \varphi(v_1 A) \subset v_2\, \varphi(A)\,.$$

It can easily be proved that the condition (4) can be replaced by the following equivalent condition:

$$\text{(5)} \qquad \text{If} \quad x \in P_1 \quad \text{and} \quad V_2(\varphi(x)) \subset P_2\,, \quad \text{then} \quad \varphi(V_1(x)) \subset V_2(\varphi(x))$$

for a suitable neighbourhood $V_1(x)$ of x.

Since each convergence space (L, λ) is a closure space, the definition of continuity (4) or (5) can be applied. From (5) it follows [12] (Cf. also [8, p. 85]) that a map φ on a convergence space (L_1, λ_1) into a convergence space (L_2, λ_2) is continuous if and only if the following condition is satisfied:

(6) If $x, x_n \in L$, $\lim x_n = x$ then there is a subsequence of naturals $n_1 < n_2 < \ldots$ such that $\lim \varphi(x_{n_i}) = \varphi(x)$.

If $\lim = \lim^*$ in P_2 then φ is continuous if and only if

$$x, x_n \in L,\ \lim x_n = x \quad \text{implies that} \quad \lim \varphi(x_n) = \varphi(x)\,.$$

Consequently we have the special result:

A real-valued function f on a convergence space (L, λ) is continuous if and only if

$$x, x_n \in L,\ \lim x_n = x \quad \text{implies that} \quad \lim f(x_n) = f(x)\,.$$

It is worth noting that from above the statement follows that a

real-valued function f on a convergence space (L, λ) is continuous if and only if it is continuous on the sequential space (L, λ^{ω_1}) [2]. Consequently

$$\mathscr{C}(L, \lambda) = \mathscr{C}(L, \lambda^{\omega_1})$$

where $\mathscr{C}$ denotes the class of all real-valued continuous functions on a given space.

Let (P, v) be a closure space. Denote $\mathscr{C} = \mathscr{C}(P, v)$ and define topologies u_1, u_2, u_3 for P as follows:

$$u_1 A = \{x\colon \text{if } f \in \mathscr{C} \text{ then there are } x_n \in A \text{ such that } \lim f(x_n) = f(x)\}$$
$$u_2 A = \{x\colon \text{if } f \in \mathscr{C}, f(A) = 0 \text{ then } f(x) = 0\}\,, \quad [7]$$

u_3 is defined by means of neighbourhoods $U_{f,\varepsilon}$, where $f \in \mathscr{C}$ and $\varepsilon > 0$:

$$U_{f,\varepsilon}(x_0) = \{x\colon |f(x) - f(x_0)| < \varepsilon\}\,, \quad [2]\,.$$

It can be proved that u_2 and u_3 are topologies [2], [7]. Also u_1 is a topology and $u_1 = u_2 = u_3$. This topology will be denoted $v_{\mathscr{C}}$. If for each couple of two distinct points $x_1, x_2 \in P$ there is a continuous function f on $(P, v_{\mathscr{C}})$ such that $f(x_1) \neq \neq f(x_2)$, then $v_{\mathscr{C}}$ is a separated completely regular topology which is the finest within all completely regular topologies coarser than $v_{\mathscr{C}}$ [7].

Problem 4. *Let (L, λ) be a sequential space. What are necessary and sufficient conditions (for λ) such that*

$$\lambda^{\omega_1} = \lambda_{\mathscr{C}}\,.$$

4. Let (L_1, λ_1) and (L_2, λ_2) be convergence spaces ($L_1 = L_2$ resp. $\lambda_1 = \lambda_2$ is not excluded), $(L_1 \times L_2, \lambda_{12})$ be their convergence product the closure λ_{12} of which is defined by convergence:

$$\lim (x_n, y_n) = (x, y) \quad \text{whenever} \quad x = \lim x_n \text{ in } L_1 \quad \text{and} \quad y = \lim y_n \text{ in } L_2 .$$

For the Cartesian product $L_1 \times L_2$ we have another closure w defined by means of neighbourhoods $W(x, y)$:

$$W(x, y) = V_1(x) \times V_2(y)$$

where $V_1(x)$ denotes any λ_1-neighbourhood of x in L_1 and $V_2(y)$ any λ_2-neighbourhood of y in L_2. In such a way we have a closure product $(L_1 \times L_2, w)$ of convergence spaces L_1 and L_2.

Notice the relation between closures λ_{12} and w. If $\lim (x_n, y_n) = (x_0, y_0)$, then each neighbourhood $W(x_0, y_0)$ contains (x_n, y_n) for nearly all n, so that $\lambda_{12}A \subset wA$ for each $A \subset L_1 \times L_2$. Consequently, λ_{12} is finer than w. If there is a countable base of neighbourhoods at each point of L_1 and L_2, then $\lambda_{12} = w$ [14]. However, the example E_6 shows that $\lambda_{12} = w$ even when there is a point in L_1 with an uncountable character:

E_6. Let L_1 be an uncountable system of disjoint sets including the empty set. Let the closure λ_1 be defined by the set convergence for L_1. Put $L_1 = L_2$, $\lambda_1 = \lambda_2$. Then $\lambda_{12} = w$ even when the character of the element $\emptyset \in L_1$ is uncountable (it equals the cardinality of L_1). Notice that both spaces L_1 and $L_1 \times L_2$ are topological spaces.

Now we are going to give an example E_7 showing that $\lambda_{12} \neq w$.

E_7. Let L_1 be the union $\bigcup A_\iota$ of an infinite collection of disjoint infinite point sets A_ι, $\iota \in I$. Choose a point $x_0 \in L_1$ and define: each point $x \neq x_0$ is isolated; a set $U \subset L_1$ is a neighbourhood of x_0 provided that $A_\iota - U$ is finite for each $\iota \in I$. It is easy to show that L_1 is Fréchet space and $\lambda_{12} \neq w$ in $L_1 \times L_2$, so that $L_1 \times L_2$ fails to be Fréchet.

Remark. The first example of Fréchet spaces L_1 and L_2 such that $L_1 \times L_2$ is not Fréchet was given in [9] (Cf. also [5]).

Problem 5. *What are the necessary and sufficient conditions (in terms of convergence) for* $\lambda_{12} = w$?

Problem 6. *Let* (L_1, λ_1), (L_2, λ_2) *be convergence spaces. There are two definitions* (D_1) *and* (D_2) *of continuity* [11] *of real-valued function* f *of two variables* $x \in L_1$ *and* $y \in L_2$:

(D_1) *If* $(x_0, y_0) \in L_1 \times L_2$ *and* $\lim (x_n, y_n) = (x_0, y_0)$, *then*

$$\lim (x_n, y_n) = f(x_0, y_0) .$$

(D_2) *If* $(x_0, y_0) \in L_1 \times L_2$ *and* ε *is any positive number, then there are neighbourhoods* $V_1(x_0)$ *of* x_0 *in* L_1 *and* $V_2(y_0)$ *of* y_0 *in* L_2 *such that*

$$|f(x, y) - f(x_0, y_0)| < \varepsilon$$

for each point $(x, y) \in V_1(x_0) \times V_2(y_0)$.

What are the necessary and sufficient conditions that both definitions (D_1) *and* (D_2) *are equivalent?*

Remark. Since λ_{12} is finer than w it follows (D_2) implies (D_1). If $\lambda_{12} = w$ then, evidently, (D_1) and (D_2) are equivalent. On the other side, V. Koutník proved that definitions (D_1) and (D_2) need not be equivalent even in the case when both convergence spaces L_1 and L_2 are Fréchet spaces [7].

5. Now, let us define a sequential envelope [12] which is a convergence analogue to the well known Čech-Stone compactification. Let us start with the notion of sequential regularity which corresponds to the notion of complete regularity of topological spaces:

Let (L, λ) be a convergence space and $\mathscr{C}_0$ a subclass of the class $\mathscr{C}(L, \lambda)$ of all continuous real-valued functions on (L, λ). The space (L, λ) is $\mathscr{C}_0$ sequentially regular[5]) provided that the following condition is fulfilled [13]:

If x *is a point and* $\{x_n\}$ *a sequence of points of* L *no subsequence of which converges to* x, *then there is a continuous function* $f \in \mathscr{C}_0$ *such that the sequence* $\{f(x_n)\}$ *does not converge to* $f(x)$.

Definition. *A topological space* (P, u) *is* $\mathfrak{m}$ *completely regular if for each set* $A \subset P$ *of power* $\leqq \mathfrak{m}$ *and each point* $x_0 \in P - uA$ *there is a continuous function on* P *such that* $0 \leqq f(x) \leqq 1$ *for each* $x \in P$ *and* $f(x_0) = 0, f(A) = 1$.

Problem 7. *Is each sequentially regular Fréchet space* $\aleph_0$ *completely regular?*

Problem 8. *Is there an* $\aleph_1$ *completely regular topological space of power* $> \aleph_1$ *which is not* $\aleph_2$ *completely regular? (More generally: consider the cardinals* $\aleph_\alpha$ *and* $\aleph_{\alpha+1}$.)

A convergence space (L_2, λ_2) *is said to be a* $\mathscr{C}$ *sequential*[5]) *envelope of a* $\mathscr{C}$ *sequentially regular space* (L_1, λ_1) *if*

(i) (L_1, λ_1) *is embedded into* (L_2, λ_2) *and* $\lambda_2^{\omega_1} L_1 = L_2$,

(ii) *each continuous function* $f \in \mathscr{C}(L_1)$ *has a continuous extension* $\bar{f} \in \mathscr{C}(L_2)$,

[5]) If $\mathscr{C}_0 = \mathscr{C}$ then instead of "$\mathscr{C}$ sequential" we often use the shortened form "sequential".

(iii) *there is no* $\mathscr{C}$ *sequentially regular space* (L, λ) *containing* (L_2, λ_2) *as a proper subspace fulfilling* (i) *and* (ii) *with respect to* L_1 *and* L.

It can be proved [12] that each sequentially regular space has essentially a unique sequential envelope.

E_8. The system 2^X of all subsets of a given set X is a sequentially regular space. It can be proved [13] that the subsystem $\mathbf{F}$ consisting of all finite subsets of X is a sequential envelope of $\mathbf{F}$ itself. There are, however, systems of sets which differ from their sequential envelopes [7].

E_9. The class of all real-valued functions $\mathscr{F}$ on a given point set X with the pointwise convergence for $\mathscr{F}$ is a sequentially regular space. Consequently each subclass of $\mathscr{F}$ has a sequential envelope, sequential regularity being a hereditary property [12].

For some important applications it is convenient to define $\mathscr{C}_0$ sequential envelope of $\mathscr{C}_0$ sequentially regular spaces L, $\mathscr{C}_0$ being a subclass of the class $\mathscr{C}$ of all continuous functions on L. Here we give the definition:

A convergence space (L_2, λ_2) *is said to be* $\mathscr{C}_0$ *sequential envelope of a* $\mathscr{C}_0$ *sequentially regular space* (L_1, λ_1) *where* $\mathscr{C}_0 \subset \mathscr{C}(L_1, \lambda_1)$, *if*

(i) (L_1, λ_1) *is embedded into* (L_2, λ_2) *and* $\lambda_2^{\omega_1} L_1 = L_2$,

(ii)$_0$ *each continuous function* $f \in \mathscr{C}_0$ *has a continuous extension* $\bar{f} \in \mathscr{C}(L_2, \lambda_2)$; *the space* (L_2, λ_2) *is* $\overline{\mathscr{C}}_0$ *sequentially regular, where*

$$\overline{\mathscr{C}}_0 = \{g \colon g \in \mathscr{C}(L_2),\ g \mid L_1 \in \mathscr{C}_0\}\,,$$

(iii) *there is no convergence space* (L, λ) *containing* (L_2, λ_2) *as a proper subspace and fulfilling* (i) *and* (ii)$_0$ *with regard to* (L_1, λ_1) *and* (L, λ).

E_{10}. Let (A^X, λ) be a convergence space of subsets of a given set X. Let $\mathbf{A} \subset 2^X$ be a set algebra. To simplify the notation, let λ denote also the relative closure for $\mathbf{A}$. Let $\mathscr{P}$ be the class of all probability measures on $\mathbf{A}$. It can be proved [13] that $\mathbf{A}$ is a $\mathscr{P}$ sequentially regular space and that $\lambda^{\omega_1}\mathbf{A}$ is a $\mathscr{P}$ sequential envelope of $\mathbf{A}$.

Problem 9. *Let* R *be the set of all real numbers,* $\mathscr{F}$ *the class of all real-valued functions on* R *and* λ *the closure defined by the pointwise convergence of continuous functions on* R. *Is there a subclass* $\mathscr{C}_0 \subset \mathscr{C}(\mathscr{F}, \lambda)$ *such that* $\lambda^{\omega_1}\mathscr{F}$ *is a* $\mathscr{C}_0$ *sequential envelope of* $\mathscr{F}$?

A set function φ on $\mathbf{A}$ is uniformly continuous, if

$$A_n, B_n \in \mathbf{A},\ \mathrm{Lim}\,(A_n \div B_n) = \emptyset \quad \text{implies that} \quad \lim\,(\varphi(A_n) - \varphi(B_n)) = 0\,.$$

Problem 10. *Let* $\mathbf{A}$ *be a set algebra and* $\mathscr{U}$ *the class of all bounded uniformly continuous functions on* $\mathbf{A}$. *Is* $\lambda^{\omega_1}\mathbf{A}$ *a* $\mathscr{U}$ *sequential envelope of* $\mathbf{A}$?

Problem 11. *Is each uniformly continuous function on a set ring (closed set ring) bounded?*

6. A convergence group L is a group and a convergence space such that the map $x\,y^{-1}$ on $G \times G$ onto G is sequentially continuous, i.e. if [14]

$$\text{(6)} \qquad \text{If} \quad \lim x_n = x\,, \quad \lim y_n = y\,, \quad \text{then} \quad \lim x_{n_i} \,.\, y_{n_i}^{-1} = xy^{-1}\,,$$

$\{n_i\}$ *being a suitable subsequence of* $\{n\}$.

It will be denoted $(L, \lambda, .)$. If $\lim = \lim^*$, then (6) is equivalent to the following condition:

$$\text{(7)} \qquad \text{If} \quad \lim^* x_n = x\,, \quad \lim^* y_n = y\,, \quad \text{then} \quad \lim^* x_n \,.\, y_n^{-1} = x\,y^{-1}\,.$$

A convergence group need not be regular [14]. There arises a

Problem 12. *Is each convergence group separated?*

Let $(L, \lambda, +)$ be a convergence commutative group the closure λ of which is defined by a maximal convergence lim. If x_0 and x_n are points of L, then we say that x_0 is an infinite sum of x_n and denote $x_0 = \sum_1^\infty x_n$, if

$$x_0 = \lim \sum_1^k x_n\,.$$

If $\sum_1^\infty x_n \in L$ then $\lim x_n = 0$. As a matter of fact, since $\sum_1^{k+1} x_n - \sum_1^k x_n = x_{k+1}$ and $\lim \sum_1^k x_n = \lim \sum_1^{k+1} x_n = \sum_1^\infty x_n \in L$ and since lim is maximal we have $\lim x_{k+1} = 0$, by (7), and hence $\lim x_k = 0$.

Problem 13. *Give the characterization of convergence commutative groups* $(L, \lambda, +)$ *for which the following statement is true*

$$\text{(8)} \qquad \lim x_n = 0 \quad \textit{iff} \quad \sum_1^\infty x_n \in L\,.$$

Remark. It is known that each convergence set group $(2^X, \lambda, \div)$ satisfies (8).

Problem 14. *Is there a convergence commutative group* $(L, \lambda, +)$ *and a sequence of points* $x_n \in L$ *such that* $\lim x_n = 0$ *and* $\sum_1^\infty x_{n_i}$ *does not exist for any subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$?

Problem 15. *Is there a sequence of points of a convergence commutative group such that in each subsequence of it there is a subsequence the infinite sum of which exists and another subsequence the infinite sum of which does not exist?*

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