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ON STRONG CONTINUITY OF A PARTIAL ORDER

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1. Introduction

Let X be a set consisting of at least two points, together with a partial order P on it, that is, P is a reflexive, anti-symmetric, transitive relation on X. (For the definitions, see [4].) We write $(x, y) \in P$ equivalently as xPy or $x \leqq y$, so that

$$P = \{(x, y) : x \leqq y\} \subset X \times X .$$

We also write $x < y$ to mean $x \leqq y$ but $x \neq y$. For $a \in X$, let $Pa = \{x \in X : x \leqq a\}$ and $aP = \{x \in X : a \leqq x\}$. Then clearly,

$$P = \{(Px, x) : x \in X\} = \{(x, xP) : x \in X\}$$

where

$$(Px, x) = \{(y, x) : y \in Px\}$$

and similarly for (x, xP).

If X is also a topological space, a natural problem is to relate the topological structure on X with its order structure. Treating a partial order P on X as a multi-valued function, where image of a point $x \in X$ is the set Px (or xP) one may define the "*continuity*" of the partial order P as is customarily done for a multi-function by requiring that the graph of the partial order, namely the subset P of $X \times X$ be closed. This is done for example in [5], [6] and some consequences of this definition are investigated there.

Recently, another definition of the continuity of P was proposed and some of the consequences of the new definition were investigated in [3]. The purpose of the present note is to carry out this analysis further.

2. Preliminary Results

Let X be a topological space consisting of at least two points and let P be a partial order on X. We adopt the notation introduced in § 1. Let Δ be the *diagonal* of X, defined to be the set $\Delta = \{(x, x) : x \in X\} \subset X \times X$. By a neighbourhood we always mean an open neighbourhood. The closure of a set A will be written A^*. We write $a \parallel b$ to mean that the elements a and b in X are mutually incomparable.

Definition 1. *The partial order P (or $\leqq$) is said to be continuous on X if for arbitrary a and b in X*

(i) $a < b$ implies that there exist neighbourhoods G_a and G_b of a and b respectively such that if $x \in G_a$ and $y \in G_b$ then $y \nleqq x$,

(ii) $a \parallel b$ implies that there exist neighbourhoods G_a and G_b of a and b respectively such that if $x \in G_a$ and $y \in G_b$ then $x \parallel y$.

In case P is a linear order, part (ii) of the above definition is inoperative and we get the definition of the continuity of a linear order, as given by Kelley [4]. In general, we have the following

Proposition 1. *A partial order P on a topological space X is continuous on X if and only if P is closed in $X \times X$.*

Proof. Suppose P is continuous and let $(a, b) \in X \times X \setminus P$ be arbitrary.

Case (i) $b < a$. Then by continuity of P, there exist neighbourhoods G_b and G_a of b and a respectively such that $x \in G_b$, $y \in G_a$ imply $y \nleqq x$ ie. $(y, x) \in G_a \times G_b$ imply $(y, x) \notin P$.

Case (ii) $a \parallel b$. Then using continuity of P again, we find neighbourhoods G_a and G_b of a and b respectively such that if $x \in G_a$ and $y \in G_b$ then $x \parallel y$ i.e. $(x, y) \notin P$.

Thus in either case, (a, b) is an interior point of $X \times X \setminus P$. Since (a, b) was arbitrarily chosen in $X \times X \setminus P$ this shows that $X \times X \setminus P$ is open or that $P = P^*$.

Conversely, suppose $P = P^*$ and that $a < b$. Then $(b, a) \notin P$, a closed set. Hence there is a neighbourhood U of (b, a) such that $U \cap P = \emptyset$. Without loss of generality one may take $U = G_b \times G_a$ where G_a and G_b are neighbourhoods of a and b respectively. Then G_a and G_b satisfy (i) of the definition.

If $a \parallel b$ then $(a, b) \notin P$ and $(b, a) \notin P$. Since P is closed, $(a, b) \in P$ implies the existence of neighbourhoods G'_a and G'_b of a and b respectively such that $(G'_a \times G'_b) \cap$ $\cap P = \emptyset$, while $(b, a) \notin P$ implies the existence of the neighbourhoods G''_b and G''_a of b and a respectively such that $(G''_b \times G''_a) \cap P = \emptyset$. Then $G_a = G'_a \cap G''_a$ and $G_b = G'_b \cap G''_b$, satisfy (ii) of the definition.

Proposition 2. *A topological space X with a continuous partial order P on it is Hausdorff.*

Proof. It suffices to prove that Δ is a closed subset of $X \times X$. Since $P = P^*$ and $\Delta \subset P$, we have $\Delta \subset \Delta^* \subset P$. Let $(a, b) \in P \setminus \Delta$ i.e. $a < b$. By continuity of P, there exist neighbourhoods G_a and G_b of a and b respectively such that $x \in G_a$, $y \in G_b$ imply $y \nleqq x$ i.e. $(G_a \times G_b) \cap \Delta = \emptyset$, so that $\Delta = \Delta^*$.

Definition 2. *A partial order P (or $\leqq$) on a topological space X is said to be strongly continuous on X if for arbitrary $a, b \in X$,*

(i) $a < b$ *implies that there exist neighbourhoods* G_a *and* G_b *of* a *and* b *respectively such that if* $x \in G_a$ *and* $y \in G_b$, *then* $x < y$.

(ii) $a \parallel b$ *implies that there exist neighbourhoods* G_a *and* G_b *of* a *and* b *respectively such that if* $x \in G_a$ *and* $y \in G_b$, *then* $x \parallel y$.

It may be noted that part (ii) of both the definitions is the same, and the only difference is in part (i). It is clear that the strong continuity of P implies its continuity in the usual sense (Definition 1). In case P is a linear order, both the definitions are identical. Simple examples can be given of a partial order that is continuous but not strongly continuous. For example, the usual partial order on the Euclidean plane is continuous but not strongly continuous.

Since the strong continuity of P implies its continuity, it follows from Proposition 2 that if P is strongly continuous on X, then X is Hausdorff. The following Proposition gives some conditions on P equivalent to its strong continuity. For any set A of X, $\mathrm{Int}(A)$ denotes its interior and $\mathrm{F}(A)$ its frontier i.e. $\mathrm{F}(A) = A^* \cap \cap (X \setminus A)^*$.

Proposition 3. *For a partial order* P *on a topological space* X, *the following are equivalent*

(1) P *is strongly continuous*

(2) $P = P^*$ *and* $P \setminus \Delta = \mathrm{Int}(P \setminus \Delta)$

(3) $\mathrm{F}(P) \subset \Delta$.

The proof is given in [3].

This result can be improved if X has no isolated points. Recall that a point p in a topological space X is *isolated* if $\{p\}$ is open in X.

Proposition 4. *Let* X *be a topological space with a partial order* P *on it. A point* $p \in X$ *is isolated if and only if* $(p, p) \in \mathrm{Int}\, P$.

Proof. Since P is reflexive, we have $(p, p) \in \Delta \subset P$. If p is isolated, $\{p\}$ is open in X so that $(p, p) \in \{p\} \times \{p\} \subset P$. Hence $(p, p) \in \mathrm{Int}\, P$. Conversely, let $(p, p) \in \in \mathrm{Int}\, P$. Then we can find a neighbourhood G of p such that $(p, p) \in G \times G \subset P$. If $x \in G$ is arbitrary, $(x, p) \in G \times G \subset P$ i.e. $x \leqq p$ but also $(p, x) \in G \times G \subset P$ i.e. $p \leqq x$. Thus $x = p$. In other words, $G = \{p\}$.

Corollary. *If* X *has no isolated points*, $\Delta \subset \mathrm{F}(P)$.

Proof. Since $\Delta \subset P$, obviously $\Delta \cap (\mathrm{Int}(X \times X \setminus P)) = \emptyset$, and since X has no isolated points, $\Delta \cap \mathrm{Int}\, P = \emptyset$. The proof is completed by noting that

$$X \times X = (\mathrm{Int}\, P) \cup \mathrm{F}(P) \cup \mathrm{Int}(X \times X \setminus P)$$

so that $\Delta \subset \mathrm{F}(P)$. One may note that no reference was made to the continuity of P in this proposition and the corollary. The proof of the following proposition is now evident.

Proposition 5. *If P is a partial order on a topological space X which has no isolated points, then P is strongly continuous on X if and only if* $\mathrm{F}(P) = \Delta$.

3. Partial Order From a Semi-lattice

A semi-lattice $\wedge$ on a set X is idempotent, commutative, associative binary operation on X i.e. $\wedge : X \times X \to X$ is such that $x \wedge x = x$, $x \wedge y = y \wedge x$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge y \wedge z$ for all $x, y, z \in X$. For the details one may refer to [2]. If we define P on X by xPy i.e. $x \leqq y$ if and only if $x = x \wedge y$, it is easy to see that P is a partial order on X. A lattice $(X; \wedge, \vee)$ is a set X with two semilattices $\wedge$ and $\vee$ on it related by the absorption law: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$. It is easily seen that both $\wedge$ and $\vee$ in a lattice yield the same partial order P. If a semi-lattice X is a Hausdorff space and if $\wedge$ is continuous on $X \times X$, that is if X is a *topological semi-lattice*, it is natural to ask if P is also continuous or strongly continuous and whether the converse holds. We will provide some of the answers here.

Proposition 6. *If $(X, \wedge)$ is a topological semi-lattice, then the associated partial order P is continuous* (cf. [1]). *The converse is false.*

Proof. Define $f : X \times X \to X \times X$ by $f(x, y) = (x, x \wedge y)$. Then f is continuous and $f^{-1}(\Delta) = P$. Since X is Hausdorff, Δ and therefore P is closed. Hence by Proposition 1, P is continuous.

Now suppose that the converse is true. This would mean: "If P is continuous, then $\wedge$ is continuous". In particular, if X is a lattice, this would mean:

"If P is continuous, then both $\wedge$ and $\vee$ are continuous."

Suppose that X is a lattice and a topological space such that one of the operations say $\wedge$, is continuous. Then by the part of the proposition just proved, this would mean that P is continuous. And if the converse is also true, this would mean that the other operation $\vee$ is also continuous. In other words, we will have proved (by exhibiting a counter-example) the converse to be false if we can exhibit a lattice where one operation is continuous and the other is discontinuous. Lattices of this type abound. The following example is a modification of an example from Anderson and Ward [1].

Example. For $n = 1, 2$ let

$$A_n = \{(x, y) : (n-1)/n \leqq x \leqq 1 \text{ and } y = (n-1)/n\}$$

and

$$B = \{(x, y) : 0 \leqq x \leqq 1 \text{ and } y = x\}$$

Let $X = B \cup \bigcup_{n=1}^{2} A_n$, together with the subspace topology and the partial order inherited from the Euclidean plane.

It may be noted that the inf $\wedge$ of any two elements in X is the same as their inf in the Euclidean plane. Hence $\wedge$ is continuous on X. However, this is not the case with sup operation $\vee$. Thus, for example, if $\{x_\alpha : \alpha \in A\}$ is a net converging from right to $(\frac{3}{4}, \frac{1}{2})$ and $\{y_\alpha : \alpha \in A\}$ converges to $(\frac{1}{2}, \frac{1}{2})$ from above, then $\{x_\alpha \vee y_\alpha : \alpha \in A\}$ converges to $(\frac{3}{4}, \frac{3}{4})$ and not to $(\frac{3}{4}, \frac{1}{2})$ as it should, if $\vee$ were continuous.

The example of the unit square in the Euclidean plane with the usual topology and the usual order shows that in general, the continuity of the lattice operations does not ensure the strong continuity of the partial order, even if X is compact. The converse however, holds when X satisfies some additional conditions.

A partially ordered set X is said to be *order-dense* if $x < y$ implies the existence of $z \in X$ such that $x < z < y$. A subset S of X is said to be *diverse* if no two elements in S are mutually comparable. The set X is said to be of *finite diversity* if every maximal diverse set in X is of cardinality $< \aleph_0$. Equivalently, every diverse subset of X can be extended to a maximal, finite, diverse set say $\{d_1, d_2, \ldots, d_m\}$ such that every $x \in X$ is comparable with some d_i.

Proposition 7. *Let $(X, \wedge)$ be a semi-lattice where X is a compact, Hausdorff space and let $\leqq$ be the associated partial order on X. If X is either order-dense or of finite diversity in $\leqq$, then the strong continuity of $\leqq$ implies that X is a topological semi-lattice.*

Proof. Let $\{(x_\alpha, y_\alpha) : \alpha \in A\}$ be a net in $X \times X$ converging to (a, b). The theorem will be proved if it is shown that the net $\{x_\alpha \wedge y_\alpha : \alpha \in A\}$ in X converges to $a \wedge b$. The proof breaks down in three parts, according as (i) $a < b$ (ii) a is incomparable with b or (iii) $a = b$.

For the cases (i) and (ii), the proof is the same whether X is assumed to be order-dense or of finite diversity and is given as the proof of theorem 2 of [3]. Indeed, neither order-denseness nor finite diversity of X need be assumed so far. The proof of case (iii) when X is orderdense is given as the proof of theorem 2 of [3].

Now suppose that X is of finite diversity in $\leqq$ and that $a = b$. The net $\{x_\alpha \wedge y_\alpha : \alpha \in A\}$ has a cluster point, say z. Clearly $z \leqq a$. We show that the assumption that $z < a$ leads to a contradiction.

Extend $\{a\}$ to a maximal, finite diverse set $\{a, d_1, d_2, \ldots, d_m\}$. Suppose $z < a$. By repeated application of the definition of the strong continuity of $\leqq$, we find neighbourhoods U and G' of z and a respectively and the neighbourhoods $G_0, G_1, \ldots$ $\ldots, G_m$ of $a, d_1, \ldots, d_m$ respectively such that $x \in U$ and $y \in G'$ imply $x < y$; and if $x_i \in G_i$ and $x_j \in G_j$ with $i \neq j$ $(i, j = 0, 1, \ldots, m)$, then x_i is incomparable with x_j. Let $G = G' \cap G_0$. There is $\alpha_0 \in A$ such that for $\alpha \geqq \alpha_0$, $x_\alpha \in G$, $y_\alpha \in G$ i.e. for $\alpha \geqq \alpha_0$, x_α and y_α are incomparable with every d_i $(i = 1, \ldots, m)$. Hence, for $\alpha \geqq \alpha_0$, x_α and y_α are comparable with α, since the set $\{a, d_1, \ldots, d_m\}$ is maximally diverse.

If $a \leqq x_\alpha$ and $a \leqq y_\alpha$ for $\alpha \geqq \alpha_0$, this would mean that $a \leqq x_\alpha \wedge y_\alpha$ for $\alpha \geqq \alpha_0$, which in turn would imply that $x_\alpha \wedge y_\alpha$ is not frequently in U, contradicting the fact that z is a cluster point of the net $\{x_\alpha \wedge y_\alpha : \alpha \in A\}$.

Hence there is at least one x_α or y_α in G, say x_β, such that $z < x_\beta < a$. Using the strong continuity of $\leqq$ again, we can find neighbourhoods V_1, V_2, V_3 of z, x_β and a respectively, such that, for every $x_i \in V_i$ $(i = 1, 2, 3)$; $x_1 < x_2 < x_3$ holds. Since the nets $\{x_\alpha : \alpha \in A\}$ and $\{y_\alpha : \alpha \in A\}$ are eventually in V_3, it follows that $x_\beta \leqq x_\alpha \wedge y_\alpha$ eventually, so that the net $\{x_\alpha \wedge y_\alpha : \alpha \in A\}$ cannot be frequently in V_1, which contradicts the fact that z is its cluster point. Hence z must equal a. Since X is compact and a is the unique cluster point of the net $\{x_\alpha \wedge y_\alpha : \alpha \in A\}$, it follows that the net converges to a.

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