SOME PROPERTIES OF TOPOLOGICAL GROUPS

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Introduction. Three problems about topological groups are considered in this note. These problems are:

(1) *Is a locally compact Hausdorff group whose underlying topological space is also sequential, metrizable?*

(2) *Is an extremally disconnected Hausdorff topological group which is also a k-space discrete?*

(3) *If for a locally compact Hausdorff group G chain nets or even co-chain nets are enough to determine its topology, then is G metrizable?*

The first one arose in a discussion with Franklin. We answer it in the affirmative. The second is reported to have been solved by Arhangelskii in the affirmative. But we could not have access to his proof so far.[1]) We got an independent proof of this result which we present here. The answer to the third problem is also affirmative.

Definitions and Notations. *A topological space X is said to be sequential if every sequentially open set of X is open in X. A subset $A \subset X$ is called sequentially open in X if whenever a sequence $x_1, x_2, \ldots$ of elements of X converges to an element $x_0 \in A$ then $x_n \in A$ for all $n > n_0$ for some n_0.*

A topological space X is said to be extremally disconnected if the closure of each open set in X is again open. A topological space X is said to be a k-space if whenever $A \subset X$ we have that A is closed if and only if $A \cap L$ is closed relative to L for all compact subsets $L \subset X$. A net (x_d) in a set X and directed by a set $(D, \geqq)$ is said to be a chain net if D is totally ordered under $\geqq$. The net (x_d) is said to be a co-chain net, if D satisfies the following condition:

Every countable subset of D is either bounded above or is cofinal.

We note that every chain net is a co-chain net but not conversely. We say that chain nets (co-chain nets) are enough to determine the topology of X if given a subset $A \subset X$ and an element x_0 belonging to the closure of A there is a chain net (co-chain net) x_d in A which converges to x_0. All topological spaces considered here are taken

[1]) Added after the conference: A. Arhangelskii, Groupes topologiques extrémalement discontinus, C. R. Acad. Sc. Paris, t. *265* (1967), 822–825.

to be Hausdorff. For each of the problems we outline the main steps in the form of propositions.

Proposition 1. *Let G be a locally compact totally disconnected group. If G is sequential, then G is metrizable.*

Proposition 2. *Let G be a locally compact Abelian group which is sequential. Then G is metrizable.*

Proposition 3. *Let G be a compact connected sequential group. Then G is metrizable.*

Proposition 4. *Let G be a locally compact connected sequential group. Then G is metrizable.*

Theorem 1. *A locally compact group G which is sequential is metrizable.*

Lemma 1. *Let G be a nondiscrete topological group. Then there is a cardinal number λ such that (1) if $\{O_\alpha\}_{\alpha\in I}$ is a well-ordered decreasing family of open sets such that $|I| < \lambda$ then $\bigcap O_\alpha$ is also open and (2) there is a well ordered decreasing family $\{O_\beta\}_{\beta\in J}$ of open sets such that $\bigcap_{\beta\in J} O_\beta$ is not open and $|J| = \lambda$.*

Lemma 2. *Let G be a nondiscrete topological group. Let λ be the cardinal number satisfying the conditions of Lemma 1. Then there exists a closed subgroup H of G such that H is not open and $H = \bigcap O_\alpha$ of open sets O_α, $\alpha \in J$ where J is a well-ordered set and $O_\alpha \supset O_\beta$ if $\alpha \leqq \beta$ for every α and β in J and $|J| = \lambda$ and $O_\alpha H = O_\alpha$ for all $\alpha \in J$.*

Lemma 3. *Let G be a nondiscrete topological group which is extremally disconnected. Let λ be the cardinal number as in Lemma 1 and H be as in Lemma 2. Then every compact set of G/H is finite.*

Theorem 2. *Suppose G is an extremally disconnected topological group which is also a k-space. Then G is discrete.*

Observation 1. *Let G be a topological group for which co-chain nets are enough to determine the topology and let H be a closed subgroup of G. Then co-chain nets are enough to determine the topology of H too.*

Observation 2. *Let G be a locally compact group and G_0 be the component at the identity. Then there is a closed and open subgroup H of G containing G_0 such that H/G_0 is compact.*

Observation 3. *Let G be a locally compact group, G_0 the connected component at the identity of G, such that G/G_0 is compact. Then there is a compact subgroup H of G such that G/H is separable metric.*

Observation 4. *For the space $\{0, 1\}^m$, co-chain nets are not enough to determine the topology if $m \geqq \aleph_1$.*

Observation 5. *Let G be a compact totally disconnected group. The co-chain nets are enough to determine the topology if and only if G is metrizable.*

Observation 6. *Let G be a compact group and H a normal closed subgroup. If co-chain nets are enough to determine the topology of G then co-chain nets are enough to determine the topology of G/H too.*

Observation 7. *Let G be a compact Abelian group. If co-chain nets are enough to determine the topology of G then G is metrizable.*

Observation 8. *Let G be a compact connected group. If co-chain nets are enough to determine the topology of G then G is metrizable.*

Theorem 3. *Let G be a locally compact group. If co-chain nets are enough to determine the topology of G then G is metrizable.*

Addendum. Added in proof, November 2, 1970.

Definition 1. *A subset A of a topological space X is called chain net closed (co-chain net closed) if whenever a chain net (a co-chain net) from A converges to a point $x \in X$ then $x \in A$.*

Definition 2. *A topological space X is called a chain net space (co-chain net space) if each chain net closed (co-chain net closed) subset of X is a closed subset of X.*

We have the following results:

Theorem 1. *If $\{0, 1\}$ is the two point discrete space then $\{0, 1\}^m$, $m \geqq c$ is not a chain net space (co-chain net space).*

Theorem 2. (Assuming continuum hypothesis). *If G is locally compact group such that the underlying space is a chain net space (co-chain net space) then G is metrizable.*

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