M. P. Berri; Jack R. Porter; Robert M., Jr. Stephenson

A survey of minimal topological spaces

In: Stanley P. Franklin and Zdeněk Frolík and Václav Koutník (eds.): General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Kanpur topological conference, 1968. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1971. pp. [93]--114.

Persistent URL: http://dml.cz/dmlcz/700571

A SURVEY OF MINIMAL TOPOLOGICAL SPACES

M. P. BERRI (1), J. R. PORTER (2)[1], AND R. M. STEPHENSON, JR. (3)

New Orleans

1. Introduction. Given a topological property P and a set X, we let $P(X)$ denote the set of topologies on X with property P and note that $P(X)$ is partially ordered by inclusion. A topological space (X, τ) is *minimal* P (or P-minimal) provided τ is a minimal element in $P(X)$. The study of minimal topological spaces is a study of minimal P spaces.

Closely associated with minimal P are P-closed (or P-complete) spaces and Katětov P spaces. A P-space (X, τ) is *P-closed* provided X is a closed set in every P-space in which it can be embedded. A P-space (X, τ) is said to be *Katětov P* provided τ is finer than some minimal P topology on X.

P-minimal or P-closed spaces have been investigated for a variety of properties: P = Hausdorff, admissible, Banach, etc. We confine ourselves here to the cases in which P denotes various separation properties, and we mainly consider four types of problems: characterizing P-minimal, and P-closed spaces; embedding P-spaces in P-minimal or P-closed spaces; determining for which P it is true that P-minimality or P-closure is productive; and discovering which subspaces of P-minimal (P-closed) spaces are P-minimal (P-closed).

The terminology we use coincides with that in Bourbaki [Bo2], Dugundji [D], and Gillman and Jerison [GJ]; in particular, as used here, the properties of regularity, complete regularity, etc. include the T_1 separation property.

A well-known topological fact is that the topology of a compact Hausdorff space is minimal Hausdorff i.e., it is not strictly finer (stronger, larger) than any other Hausdorff topology. This fact was proven first by A. S. Parkhomenko [Pa]. He also proved that a minimal Hausdorff space is Hausdorff-closed (shortened to H-closed in the literature). H-closure was defined by P. Alexandroff and P. Urysohn in [AU]. They also proved that a regular space is H-closed if, and only if, it is compact. In the same paper, they defined regular-closed and asked whether every regular-closed space is compact. Katětov [K1] was the first to characterize minimal Hausdorff spaces, and he proved that a Urysohn space is minimal Hausdorff if, and only if,

[1]) This research was partially supported by a University of Kansas research grant No. 3416—5038.

it is compact. He also proved that every Hausdorff space can be densely embedded in an H-closed space which possesses properties similar to the Stone-Čech compactification. In 1941, N. Bourbaki [Bo1] characterized minimal Hausdorff and H-closure in terms of filters. He cited an example by Urysohn [U] which was also an example of a noncompact minimal Hausdorff space. Bourbaki inquired whether the naturla topology of the space of rational numbers is finer than some minimal Hausdorff topology.

In his book [Va] published in 1947, R. Vaidyanathaswamy proved that a compact Hausdorff space is minimal Hausdorff and asked if there existed a non-compact, minimal Hausdorff space. In two papers [R1; R2] in 1947, A. Ramanathan answered Vaidyanathaswamy's question in the affirmative by using (like Bourbaki) Urysohn's example. He derived the same characterization of minimal Hausdorff that Katětov [K1] derived. In 1950, F. Obreanu [O1, O2, O3] proved that the H-closure and minimal Hausdorff properties are productive. He gave a partial solution to Cartan's [Bo1] question by showing that the natural topology of the space of rational numbers is not finer than any compact topology.

In 1955, B. Banaschewski [Ba1] investigated minimal P and P-closed spaces for $P =$ Hausdorff, semiregular Hausdorff, regular, zero dimensional, locally compact, and completely regular. For the latter three, he proved that minimal P, P-closed, and compact are equivalent. He proved that minimal semiregular Hausdorff, semiregular Hausdorff-closed, and minimal Hausdorff are equivalent; proved that a minimal regular space is regular-closed; and remarked that a regular space in which every closed set is regular-closed is minimal regular. He asked whether a regular space in which every closed set is regular-closed is compact. In 1961, Banaschewski [Ba4] proved that a Hausdorff space can be densely embedded in a minimal Hausdorff space if, and only if, it is semiregular.

In 1963, N. Smythe and C. A. Wilkens [SW] proved that minimal normal spaces and normal, minimal regular spaces are compact. They asked whether a minimal regular space is necessarily compact. In 1963, M. P. Berri and R. H. Sorgenfrey [BS] answered a question by Alexandroff and Urysohn [AU], Banaschewski [Ba1], and Smythe and Wilkins [SW] by exhibiting a minimal regular space which is not compact. In 1964, Berri [Be3] proved, for certain P, that minimal P and P-closed spaces are of second category and asked if every minimal Hausdorff space is of second category. In addition Berri answered the question posed by N. Bourbaki [Bo1] by showing that the natural topology of the space of rational numbers is not finer than any minimal Hausdorff topology. In 1965, H. Herrlich [He1] answered several questions posed in [BS] and [SW]. Since 1965, numerous papers and dissertations have appeared; most of them are listed in the bibliography.

In this paper, N will denote the set of natural numbers. The symbol $\bigcup[A \mid A \in \Lambda]$ will be used to denote $\bigcup_{A \in \Lambda} A$.

2. Techniques and Methods. We list five basic methods of obtaining coarser topologies-filters, cotopologies, compactness modulo an ideal, semiregularization, and weak topologies.

The key in using the filter method is to define certain filter bases (called P-filter bases) in a P-space; a filter is a P-filter if the filter is generated by a P-filter base. Altering the terminology of [GJ] a filter (or filter base) $\mathscr{F}$ is called *free* if $\bigcap[\bar{F} \mid F \in \mathscr{F}] = \emptyset$ (i.e., $\mathscr{F}$ has no adherent point) and is called *fixed* if $\bigcap[\bar{F} \mid F \in \in \mathscr{F}] \neq \emptyset$ (i.e. $\mathscr{F}$ has an adherent point). For each P-space (X, τ), if $\mathscr{F}$is a *free* P-filter base in (X, τ) and $p \in X$, there is a coarser P-topology $\tau(\mathscr{F}, p)$ generated by the neighborhood base

$$\mathscr{B}(x) = \begin{cases} \mathscr{V}(x), \text{ the } \tau\text{-neighborhood system of } x & \text{for} \quad x \neq p \\ \{V \cup F \mid V \in \mathscr{V}(x) \text{ and } F \in \mathscr{F}\} & \text{for} \quad x = p\,. \end{cases}$$

The filter method can also be used to obtain a P-space extension (Y, σ) of a P-space (X, τ) by letting $Y - X = \{\mathscr{F} \mid \mathscr{F}$ is a maximal element in the set of free P-filters$\}$, where σ is either the simple or strict extension topology on Y [Ba5].

If $\mathscr{M}$ is a collection of open filters on a space $(X, \mathscr{V})$, we denote the disjoint union of X and $\mathscr{M}$ by $X_{\mathscr{M}}$ and define $\mathscr{V}_{\mathscr{M}''}$ to be the topology on $X_{\mathscr{M}}$ generated by the sets $V^* = V \cup \{\mathscr{F} \in \mathscr{M} \mid V \in \mathscr{F}\}$, $V \in \mathscr{V}$, and $\mathscr{V}_{\mathscr{M}'}$ to be the topology on $X_{\mathscr{M}}$ generated by $\{V \mid V \cap X \in \mathscr{V}$, and $\mathscr{F} \in \mathscr{M} \cap V$ implies that $V \cap X \in \mathscr{F}\}$. $(X_{\mathscr{M}}, \mathscr{V}_{\mathscr{M}'})$ is called the *simple extension space of* $(X, \mathscr{V})$ *with filter trace* $\mathscr{M}$ *and* $(X_{\mathscr{M}}, \mathscr{V}_{\mathscr{M}''})$ *is called the strict extension space of* $(X, \mathscr{V})$ *with filter trace* $\mathscr{M}$.

The method of cotopologies was first used in obtaining coarser topologies by J. Aarts and J. de Groot [AdeG], G. Strecker [Str], and G. Viglino [Vi1]. For each basis $\mathscr{B}$ for a space (X, τ), let $\tau(\mathscr{B})$ be the topology on X generated by $\{X - \bar{B} \mid B \in \mathscr{B}\}$. $(X, \tau(\mathscr{B}))$ is called a *co-space* of (X, τ); clearly, $\tau(\mathscr{B}) \subset \tau$. If (X, τ) is a P-space, it is not necessarily true that each *co-space* is a P-space; however, there are enough P-cospaces to characterize minimal P and P-closed for certain P. For example in [Vi2], Viglino establishes the following result: A P-space (X, τ) is minimal P if, and only if, $\{\tau(\mathscr{B}) \mid \tau(\mathscr{B})$ is P; $\tau(\mathscr{B}) \neq \tau\} = \emptyset$, where P denotes any of the following properties:

(i) Hausdorff, (ii) Urysohn, (iii) regular, (iv) completely regular, (v) normal, (vi) locally compact, (vii) completely normal.

If some co-space of (X, τ) has property P, then (X, τ) is called co-P. If every co-space of (X, τ) has property P, then (X, τ) is called *totally co-P*.

The technique of compactness modulo an ideal was first defined and studied by R. Newcomb [N] in 1967. An *ideal* in a set X is a family of subsets of X satisfying:

1. $\emptyset \in \mathscr{I}$ *and* $X \in \mathscr{I}$,

2. *$A, B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$, and*

3. *$A \in \mathscr{I}$ and $B \subset A$ implies $B \in \mathscr{I}$.*

Thus, an ideal is the Boolean algebraic dual to a filter in the set of all subsets of X. An ideal $\mathscr{I}$ in a space (X, τ) is a τ-boundary provided $\tau \cap \mathscr{I} = \emptyset$. The set $I(\tau)$ of nowhere dense sets is a τ-boundary ideal. Let $\mathscr{I}$ be an ideal in (X, τ). (X, τ) is *compact modulo* $\mathscr{I}$ means that every open cover $\mathscr{U}$ of X contains a finite subfamily $U_1, \ldots, U_n$ such that $X + \bigcup[U_i \mid i = 1, \ldots, n] \in \mathscr{I}$ where $A + B$ is the symmetric difference (i.e., $A + B = (A - B) \cup (B - A)$). Let O_U denote $\bigcup[T \mid T \in \tau$ and $T + U \in \mathscr{I}]$; $\tau(\mathscr{I})$ is defined to be the topology generated by $\{O_U \mid U \in \tau\}$ (which is a base for $\tau(\mathscr{I})$). Clearly, $\tau(\mathscr{I})$ is coarser than τ. As in the cotopology method, $(X, \tau(\mathscr{I}))$ is not necessarily a P-space if (X, τ) is a P-space. We note that if $\mathscr{I}$ is a τ-boundary ideal, then $\tau \supset \tau(\mathscr{I}) \supset \tau(I(\tau))$.

The semiregularization method was first defined and studied by M. H. Stone [Sto]. Let (X, τ) be a space. A set $U \subset X$ is *regular-open* if and only if $U = (\overline{U})^0$. The topology τ_s generated by the regular-open sets is called the *semiregularization* of τ and is coarser than τ. A space (X, τ) is *semiregular* if and only if $\tau = \tau_s$. A space (X, τ) is *semiregular at a point* p in X if and only if $\{(\overline{U})^0 \mid p \in U \in \tau\}$ is a neighborhood base at p.

The weak topology method is determined by the set $C(X)$ of real-valued continuous function on a space (X, τ). The *weak-topology* τ_w on X is the smallest topology on X such that all functions in $C(X)$ are continuous. Clearly, τ_w is coarser than τ. The space (X, τ_w) is completely regular if and only if (X, τ) is completely Hausdorff, i.e., for each pair x, y of distinct points, there exists a real-valued continuous function f such that $f(x) \neq f(y)$.

3. P = Hausdorff and Semiregular. In this section, we report the results of minimal Hausdorff spaces, H-closed spaces, semiregular Hausdorff-closed, minimal semiregular Hausdorff and Katětov Hausdorff spaces.

Theorem 3.1. (a) [Pa, K1, Bo1, Va]. *A compact Hausdorff space is minimal Hausdorff.*

(b) [Pa, K1, Bo1, R1]. *A minimal Hausdorff space is H-closed.*

Definition. *A family of open sets $\mathscr{U}$ in a space (X, τ) is a proximate cover of (X, τ) if and only if $\bigcup[U \mid U \in \mathscr{U}]$ is a dense subset of X.*

Definition. *A filter base in (X, τ) is an open filter base if every element is open. An open filter is a filter which is generated by some open filter base. An open ultrafilter is a maximal element in the set of open filters of (X, τ).*

Theorem 3.2. *Let (X, τ) be a Hausdorff space. The following are equivalent*:

(a) *(X, τ) is H-closed.*

(b) [AU]. *Every open cover contains a finite proximate subcover.*

(c) [Bo1]. *Every open filter base is fixed.*

(d) [Bo2]. *Every open ultrafilter converges.*

(e) [Str]. *(X, τ) is totally co-semiregular.*

(f) [Str]. *All the co-spaces of (X, τ) are homeomorphic.*

(g) [Str]. *(X, τ) is totally co-Hausdorff.*

(h) [Str.]. *(X, τ) is totally co-H-closed.*

(i) [Str]. *(X, τ) is totally co-minimal Hausdorff.*

(j) [N]. *(X, τ) is compact modulo some τ-boundary ideal in (X, τ).*

Theorem 3.3.

(a) [AU]. *An H-closed, regular space is compact.*

(b) [K1]. *H-closure is preserved by continuous functions onto Hausdorff spaces.*

(c) [O2, CF, He1]. *A product of non-empty Hausdorff spaces is H-closed if and only if each coordinate space is H-closed.*

(d) [K1]. *If (X, τ) is H-closed and $U \in \tau$, then $\overline{U}$ is H-closed.*

(e) [K1]. *Every non-empty decreasing chain of non-empty H-closed subsets has a non-empty intersection.*

(f) [K1, Sto]. *If every closed subset of a space is H-closed, then the space is compact.*

(g) [Li]. *A Hausdorff space can be embedded as a closed subset in an H-closed space.*

Definition. *A subring K of $C^*(X)$ (the ring of bounded, real-valued functions on a space (X, τ)) separate points if and only if for each pair of distinct points x and y in X, there exists $f \in K$ such that $f(x) \neq f(y)$. A space (X, τ) has the Stone-Weierstrass property if and only if (X, τ) is completely Hausdorff and each subring K which separates points and contains all constant functions has the property that each $f \in C^*(X)$ is the uniform limit of a sequence of functions in K.*

Theorem 3.4. *Let (X, τ) be a Hausdorff space. The following are equivalent*:

(a) *(X, τ) is H-closed and Urysohn.*

(b) [K1]. *(X, τ_s) is compact.*

(c) [Po1]. *(X, τ) is completely Hausdorff and H-closed.*

(d) [Pol, Pr]. *(X, τ) is H-closed and for every pair of disjoint H-closed subsets M, N of (X, τ), there is a real-valued continuous function f defined on (X, τ) such that $f(M) \subset \{1\}$ and $f(N) \subset \{0\}$.*

(e) [Po1]. *(X, τ) is H-closed and has the Stone-Weierstrass property.*

(f) [Ste1]. *(X, τ) is H-closed and $\tau_s = \tau_w$.*

(g) [Str]. *Every filter base contained in $\{\overline{U} \mid U \in \tau\}$ is fixed.*

(h) [Str]. *(X, τ) is totally co-compact.*

(i) [Str]. *(X, τ) is totally co-compact Hausdorff.*

(j) [Str]. *(X, τ) is totally co-normal.*

(k) [Str]. *(X, τ) is totally co-regular.*

(l) [Str, Vi1]. *(X, τ) is totally co-Urysohn.*

Theorem 3.5. [K1]. *Each Hausdorff space (X, τ) can be densely embedded in an H-closed space (X^*, τ^*) satisfying the following properties:*

(a) *If f: $(X, \tau) \to (Y, \sigma)$ is a continuous function where (Y, σ) is Hausdorff and $f(X)$ is dense in Y, then there is a set $X \subset M \subset X^*$ and a continuous, onto extension F: $(M, (\tau^*)_M) \to (Y, \sigma)$ of f.*

(b) *If (Y, σ) is also compact in (a), then we can pick M to be X^*.*

(c) *If (X, τ) is densely embedded in a Hausdorff space (X_0, τ_0) and (X_0, τ_0) satisfies (a), then there is a homeomorphism $(X^*, \tau^*) \to (X_0, \tau_0)$ which leaves X pointwise fixed.*

(d) *(X^*, τ^*) is a simple extension of (X, τ).*

(e) *If $A \subset X$ is a closed nowhere dense subset in (X, τ), then $\mathrm{Cl}_{X^*}(A) \subset X$.*

(f) [Li, PT]. *(X^*, τ^*) is a projective maximum in the set of all H-closed extensions of (X, τ).*

Definition. *The H-closed extension (X^*, τ^*) of a Hausdorff space in 3.5 is called a Katětov extension.*

Theorem 3.6.

(a) [K3, Ba3, I1]. *Let (X, τ) be Hausdorff and not compact. The Katětov extension is not compact.*

(b) [Ba3]. *Every noncompact, completely regular space (X, τ) can be densely embedded in a noncompact extension which is a subspace of the Katětov extension of (X, τ) and which has the Stone-Weierstrass property.*

Definition. *Let (X, τ) be a space. A subset $A \subset X$ is regular nowhere dense closed if and only if there are a pair of disjoint open sets of U and V such that $\overline{U} \cap \overline{V} = A$.*

The next theorem contains solutions to questions posed by Alexandroff [A1].

Theorem 3.7.

(a) [F12, PT, Ve1]. *Every noncompact, completely regular space is densely embeddable in a noncompact, H-closed space which satisfies the Stone-Weierstrass property.*

(b) [PT]. *The Katětov extension of a Hausdorff space has the Stone-Weierstrass property if and only if every regular nowhere dense closed set in the semiregularization is compact.*

We note that many other types of H-closed extensions and their properties have been developed in [F11, F12, Fo1, HS, K2, K4, Li, 02, 04, Po2, Po3, Pr, PT, Sto, StW, T, Ve1, Ve2].

Theorem 3.8. [Bo2]. *The set of isolated points is a denumerable. H-closed space is dense.*

The above theorem is a solution to Bourbaki's [Bo1] problem and shows that the natural topology of the space of rational numbers is not stronger than any minimal Hausdorff topology. This problem was solved independently in [Be3, He1, Po1].

Theorem 3.9. *Let (X, τ) be a Hausdorff space. The following are equivalent*:

(a) (X, τ) *is minimal Hausdorff.*

(b) [K1, R2]. (X, τ) *is H-closed and semiregular.*

(c) [Bo1]. *Every open filter base with a unique adherent point is convergent.*

(d) [Str]. (X, τ) *is H-closed and* $\tau(\mathscr{B}) = \tau$ *for some co-topology* $\tau(\mathscr{B})$.

(e) [Str]. (X, τ) *is totally co-minimal Hausdorff and semiregular.*

(f) [Vi2]. $\tau = \tau(\mathscr{B})$ *for every co-topology* $\tau(\mathscr{B})$.

(g) [Vi2]. *Every Hausdorff co-topology* $\tau(\mathscr{B}) = \tau$.

(h) [N]. (X, τ) *is compact modulo some τ-boundary ideal $\mathscr{I}$ and* $\tau = \tau(\mathscr{I})$.

(i) [Ba1]. (X, τ) *is semiregular Hausdorff-closed.*

(j) [Ba1]. (X, τ) *is minimal semiregular Hausdorff.*

Definition. *A Hausdorff space (X, τ) is minimal Hausdorff at a point p in X if and only if for each Hausdorff topology* $\sigma \subset \tau$, $\{N \mid p \in \operatorname{int}_\sigma N\} = \{N \mid p \in \operatorname{int}_\tau N\}$.

Theorem 3.10.

(a) [K1, Bo1, R2]. *A minimal Hausdorff, Urysohn space is compact.*

(b) [03, Ik, He1]. *A product of non-empty Hausdorff spaces is minimal Hausdorff if and only if each coordinate space is minimal Hausdorff.*

(c) [Sc1]. *If (X, τ) is minimal Hausdorff, A is an open subset in X, and the boundary of A is compact, then $\bar{A}$ is minimal Hausdorff.*

(d) [Ba4]. *A Hausdorff space can be densely embedded in a minimal Hausdorff space if and only if the space is semiregular.*

(e) [Str, StW]. *A Hausdorff space can be embedded as a closed, nowhere dense subset of a minimal Hausdorff space.*

(f) [R2, Po1]. *A space (X, τ) is minimal Hausdorff at a point p in X if and only if (X, τ) is semiregular at p and H-closed.*

Definition. *A Hausdorff space (X, τ) is locally H-closed at a point p if and only if p has a neighborhood which is H-closed.*

Theorem 3.11.

(a) [Po1]. *A Hausdorff space which is locally H-closed except at most one point is Katětov Hausdorff.*

(b) [N]. *Katětov Hausdorff is hereditary on the complements of compact subsets.*

(c) [Ho]. *Let (X, τ) be a Hausdorff space. If (X, τ) is the countable union of nowhere dense, compact subsets and if (X, τ) is densely embeddable in a Baire space (a Baire space is one which is not the countable union of closed, nowhere dense subsets), then (X, τ) is not Katětov Hausdorff.*

Example 3.12. In response to a question posed by Berri [Be3], Herrlich [He2] gives an example of a minimal Hausdorff space which is not of second category. Let R_0 be the set of irrational numbers in $I = [0, 1]$ and R_1 be the set of rational numbers in I. Let $X = R_0 \times \{0\} \cup R_1 \times \{1\} \cup R_1 \times \{2\}$, and define τ on X by a set $U \in \tau$ if

(a) $(x, 0) \in U$ implies there is an open set V in I (usual topology) such that $(x, 0) \in (V \times \{0, 1, 2\}) \cap X \subset U$, and

(b) for $i = 1$ or 2, $(x, i) \in U$ implies there is an open set V in I such that $(x, i) \in (V \times \{i\}) \cap X \subset U$.

Example 3.13. [He1]. This is an example of an H-closed, Urysohn space (X, τ) which is not minimal Hausdorff. Let N be the set of natural numbers and $X = ([0, 1] \times N) \cup \{a\}$ where $a \notin [0, 1] \times N$. The topology τ is generated by the following neighborhood base:

(a) for $x \in [0, 1] \times N$, the neighborhoods of x in the space $[0, 1] \times N$ where N has the discrete topology, and

(b) for a, the sets $U^m = \{a\} \cup \{(x, n) \mid x \in (0, 1]$ and $n > m\}$ where $m \in N$.

Example 3.14. [U, R1, Bo1]. This is an example of a countable minimal Hausdorff space (X, τ) which is not compact. Let $X = \{a\} \cup \{b\} \cup \{a_{ij}\} \cup \{b_{ij}\} \cup \cup \{c_i\}$ where $i, j \in N$. The topology τ is generated by the following neighborhood base:

(a) for a_{ij} and b_{ij}, the sets $\{a_{ij}\}$ and $\{b_{ij}\}$,

(b) for c_i, $U^n = \{c_i\} \cup \{a_{ij}, b_{ij} \mid j = n, n+1, \ldots\}$ where $n \in N$,

(c) for a, $V^n = \{a\} \cup \{a_{ij} \mid j \in N \text{ and } i = n, n+1, \ldots\}$ where $n \in N$, and

(d) for b, $W^n = \{b\} \cup \{b_{ij} \mid j \in N \text{ and } i = n, n+1, \ldots\}$ where $n \in N$.

4. P = Urysohn, Regular and Completely Hausdorff. In this section we report the results of minimal P, P-closed, and Katětov P for P = Urysohn, regular, and completely Hausdorff.

Definition. [Sc2]. *An open filter base $\mathfrak{F}$ in a space (X, τ) is a Urysohn filter base if and only if for each $p \notin a(\mathfrak{F})$ (the set of adherent points of $\mathfrak{F}$), there is an open neighborhood U of p and $V \in \mathfrak{F}$ such that $\overline{U} \cap \overline{V} = \emptyset$.*

Definition. *Let $\mathscr{U}$ and $\mathscr{V}$ be open covers of a space (X, τ). $\mathscr{V}$ is a shrinkable refinement of $\mathscr{U}$ if and only if for each $V \in \mathscr{V}$, there is a $U \in \mathscr{U}$ such that $\overline{V} \subset U$. An open cover is Urysohn if and only if it has a shrinkable refinement.*

Theorem 4.1. *Let (X, τ) be a Urysohn space. The following are equivalent:*

(a) *(X, τ) is Urysohn-closed.*

(b) [He1, Sc2]. *Every Urysohn filter base has nonvoid adherence.*

(c) [He1]. *Every Urysohn cover of (X, τ) has a finite proximate subcover.*

Theorem 4.2. *Let (X, τ) be a Urysohn space. The following are equivalent:*

(a) *(X, τ) is a minimal Urysohn.*

(b) [He1, Sc2]. *Every Urysohn filter base with at most one adherent point is convergent.*

(c) [Vi2]. *Each Urysohn co-topology $\tau(\mathscr{B}) = \tau$.*

Theorem 4.3.

(a) [He1]. *A compact Hausdorff space is minimal Urysohn, and a minimal Urysohn space is Urysohn-closed.*

(b) [He1]. *Urysohn-closure is preserved by a continuous function onto a Urysohn space.*

(c) [Sc2, Ste1, Vi1]. *A minimal Urysohn space is semiregular.*

(d) [He1]. *A regular, Urysohn-closed space is compact.*

(e) [He1]. *The space of rational numbers with the usual topology is not Katětov Urysohn.*

Theorem 4.4. *Let (X, τ) be a minimal Urysohn space. The following are equivalent:*

(a) *(X, τ) is compact.*

(b) [Ste1]. *(X, τ) is completely Hausdorff.*

(c) [Ste1, Vi1]. *(X, τ) is H-closed.*

Theorem 4.5.

(a) [He1, Ste1, Sc2]. *If a product of nonvoid spaces is minimal Urysohn, then each coordinate space is minimal Urysohn.*

(b) [Ste1, Sc2]. *If a product of nonvoid spaces is Urysohn-closed, then each coordinate space is Urysohn-closed.*

(c) [Ste1, Sc2]. *The product of an H-closed, Urysohn space and a Urysohn-closed space is Urysohn-closed.*

(d) [Ste1, Sc2]. *The product of a compact space and a minimal Urysohn space is minimal Urysohn.*

Theorem 4.6.

(a) [He1]. *Every Urysohn space can be densely embedded in a Urysohn-closed space.*

(b) [Ste1]. *Every semiregular Urysohn space can be densely embedded in a semiregular, Urysohn-closed space.*

(c) [Ste1]. *Every Urysohn space can be embedded in a semiregular, Urysohn-closed space.*

Example 4.7. [He1]. This is an example of a minimal Urysohn space (X, τ) that is not compact. For any ordinal number α, let $W(\alpha)$ be the set of all ordinals strictly less than α. Let w_0 be the first infinite ordinal and w_1 the first uncountable ordinal. Let $R = [W(w_1 + 1) \times W(w_0 + 1)] - \{(w_1, w_0)\}$ and $R_n = R \times \{n\}$ where $n = 0, \pm 1, \pm 2, \ldots$ Denote the elements of R_n by (x, y, n). Identify (w_1, y, n) with $(w_1, y, n + 1)$ if n is odd and (x, w_0, n) with $(x, w_0, n + 1)$ if n is even. Call the resulting space T. To the subspace $E = R_1 \cup R_2 \cup R_3$ of T add two points a and b, and let $X = E \cup \{a, b\}$. A set $V \subset X$ is open if and only if

(a) $V \cap E$ is open in E,

(b) $a \in V$ implies there exist $\alpha_0 < w_1$ and $\beta_0 < w_0$ such that $\{(\alpha, \beta, 1) \mid \beta_0 < \beta \leqq w_0, \alpha_0 < \alpha < w_1\} \subset V$, and

(c) $b \in V$ implies there exist $\alpha_0 < w_1$ and $\beta_0 < w_0$ such that $\{(\alpha, \beta, 3) \mid \alpha_0 < < \alpha \leqq w_1, \beta_0 < \beta < w_0\} \subset V$.

Example 4.8. [He1]. This is an example of a semiregular, Urysohn-closed space (X, τ) which is not minimal Urysohn and which cannot be embedded densely in a minimal Urysohn space. Let I be the unit interval $[0, 1]$ and let I_1, I_2, I_3 be pairwise disjoint, dense subsets of X such that $I = \bigcup[I_i \mid i = 1, 2, 3]$. U is open in I provided for each $x \in U \cap I_i$ there is an interval $S_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$ such that $S_\varepsilon(x) \cap I_i \subset U$ for $i = 1$ or 2 and $S_\varepsilon(x) \cap I \subset U$ for $i = 3$.

Definition. *An open filter base $\mathfrak{F}$ in (X, τ) is completely Hausdorff if and only if for each $p \notin$ adherence of $\mathfrak{F}$, there exist an open set U containing p, $V \in \mathfrak{F}$, and a real-valued continuous function f on (X, τ) such that $f(U) = \{1\}$ and $f(V) = = \{0\}$. An open filter base $\mathfrak{F}$ in (X, τ) is completely regular if and only if for each $U \in \mathfrak{F}$, there exist $V \in \mathfrak{F}$ and a real-valued continuous function f on (X, τ) such that $f(V) = \{0\}$ and $f(X - U) \subset \{1\}$.*

Definition. *Let $\mathscr{V}$ and $\mathscr{U}$ be covers of a space (X, τ). $\mathscr{V}$ is a continuous refinement of $\mathscr{U}$ if and only if for each $V \in \mathscr{V}$, there is $U \in \mathscr{U}$ and real-valued continuous function f on (X, τ) such that $f(V) \subset \{0\}$ and $f(X - U) \subset \{1\}$. An open cover is completely Hausdorff if and only if it has a continuous refinement. An open cover $\mathscr{U}$ is co-completely regular if and only if $\mathscr{U}$ if a continuous refinement of itself.*

Theorem 4.9. *Let (X, τ) be a completely Hausdorff space. The following are equivalent*:

(a) *Every completely regular filter base in (X, τ) is fixed.*

(b) [Ste1]. *(X, τ) is completely Hausdorff-closed.*

(c) [Ste1]. *Every co-completely regular cover of (X, τ) has a finite subcover.*

(d) [Ba2]. *(X, τ_w) is compact.*

(e) [Ba2]. *(X, τ) has the Stone-Weierstrass property.*

(f) [Ha]. *Every completely Hausdorff filter base of (X, τ) is fixed.*

(g) [Ha]. *Each completely Hausdorff cover has a finite proximate subcover.*

(h) [Ste1]. *If σ is a completely regular topology on X such that $\alpha \subset \tau$, then (X, σ) is compact.*

(i) [Ste1]. *There is a unique completely regular topology σ on X such that $\sigma \subset \tau$.*

(j) [Ste1]. *For every completely Hausdorff space (Y, σ) and continuous mapping f of (X, τ) into (Y, σ), $f(X)$ is a completely Hausdorff-closed space.*

Theorem 4.10. [SSe]. *A minimal completely Hausdorff space is compact.*

Theorem 4.11. [Ste1, 2].

(a) *If the product of nonvoid spaces is completely Hausdorff-closed, then each coordinate space is completely Hausdorff-closed.*

(b) *Let* $\{X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ *be a family of completely Hausdorff-closed spaces.* $(\Pi X_\alpha, \Pi\tau_\alpha)$ *is completely Hausdorff-closed if and only if* $(\Pi\tau_\alpha)_w = \Pi((\tau_\alpha)_w)$.

(c) *Every completely Hausdorff space can be densely embedded in a completely Hausdorff-closed space.*

(d) *If every closed subset of a space is completely Hausdorff-closed, then the space is compact.*

(e) *A closed and open subset of a completely Hausdorff-closed space is completely Hausdorff-closed.*

Theorem 4.12. [Ste4]. *If* X *and* Y*are completely Hausdorffclosed spaces and* X *is a k-space, then* $X \times Y$ *is a completely Hausdorff-closed space.*

Example 4.13. [Ste1]. This is an example of a completely Hausdorff-closed space (X, τ) on which there exists a free regular filter base and which is not of second category. Let $X = [0, 1]$ with σ denoting the usual topology, $\{X(n) \mid n \in N\}$ be a family of pairwise disjoint, dense subsets of X such that $X(2n - 1)$ is countable for $n \in N$. Let τ be the topology generated by $\sigma \cup \{X(2n - 1) \mid n \in N\} \cup X(2n - 1) \cup \cup X(2n) \cup X(2n + 1) \mid n \in N\}$.

Definition. [Ba1, BS]. *An open filter base* $\mathfrak{F}$ *in* (X, τ) *is a regular filter base if and only if for each* $U \in \mathfrak{F}$, *there exists* $V \in \mathfrak{F}$ *such that* $\overline{V} \subset U$.

Definition. *Let* $\mathscr{U}$ *and* $\mathscr{V}$ *be covers of space* (X, τ). $\mathscr{V}$ *is a regular refinement of* $\mathscr{U}$ *if and only if* $\mathscr{V}$ *refines* $\mathscr{U}$ *and* $\mathscr{V}$ *is a shrinkable refinement of itself. An open cover is regular if and only if it has an open regular refinement.*

Theorem 4.14. *Let* (X, τ) *be a regular space. The following are equivalent*:

(a) (X, τ) *is regular-closed.*

(b) [Ba1, He1]. *Every regular filter base in* (X, τ) *is fixed.*

(c) [He1, Sc1]. *Every regular cover has a finite subcover.*

Theorem 4.15. *Let* (X, τ) *be a regular space. The following are equivalent*:

(a) (X, τ) *is minimal regular.*

(b) [Ba1, BS]. *Every regular filter base with at most one adherent point converges.*

(c) [Vi2]. *Every regular co-topology* $\tau(\mathscr{B}) = \tau$.

Theorem 4.16.

(a) [Ba1, BS]. *A minimal regular space is regular-closed.*

(b) [He1]. *Regular-closed is preserved by continuous functions onto a regular space.*

(c) [Ba1]. *If every closed set in a space is regular-closed, then the space is minimal regular.*

(d) [He1]. *A completely Hausdorff, minimal regular space is compact.*

(e) [He1, BS]. *A completely regular, regular-closed space is compact.*

(f) [He1]. *The set of rational numbers with the usual topology is not Katětov regular.*

Theorem 4.17.

(a) [Sc1]. *If a product of nonvoid spaces is regular-closed, then each coordinate space is regular-closed.*

(b) [Sc1, SSo]. *The product of a compact space and a regular-closed space is regular-closed.*

(c) [Sc1]. *If a product of nonvoid spaces is minimal regular, then each coordinate space is minimal regular.*

(d) [Ik, Sc1, SSo]. *The product of a compact space and minimal regular space is minimal regular.*

(e) [Ste1]. *A completely Hausdorff, regular-closed space is completely Hausdorff-closed.*

(f) [Ste1]. *A regular-closed space is of second category.*

(g) [He1]. *A Lindelöf, regular-closed space is compact.*

Example 4.18. [BS]. This is an example of a minimal regular space (X, τ) which is not compact. Let T be the space defined in example 4.7, and let $X = T \bigcup \{p, q\}$. Let $Q_n = \{(x, y, n) \in R_n \mid x < w_1, y < w_0\}$. A subset V belongs to τ provided

(a) $V \cap T$ is open in T.

(b) $p \in V$ implies there is $n \in N$ such that $Q_n \cup \bigcup [R_i \mid i > n] \subset V$, and

(c) $q \in V$ implies there is $n \in N$ such that $Q_{-n} \cup \bigcup [R_{-i} \mid i > n] \subset V$.

Example 4.19. [He1]. This is an example of a completely Hausdorff-closed, regular-closed space (Y, σ) which is not minimal regular. Let (Y, σ) be the subspace of the space (X, τ) in example 4.18 defined by $Y = \{p\} \cup \bigcup [R_k \mid k \in N]$.

Example 4.20. [He1]. This is an example of regular space (X, τ) which can not be densely embedded in a regular-closed space. Let $R(i)$ be the subspace $R_1 \cup$

$\cup \ldots \cup R_i$ of the space T in example 4.7. Let Y be the topological sum of $R(i)$ where $i \in N$ and denote elements of $R(i)$ in Y by (α, β, n, i) where $1 \leqq n \leqq i$. Let $X = Y \cup \{p\}$ where p is an additional point. A subset V belongs to τ provided $V \cap Y$ is open in Y and $p \in V$ means there is an $n \in N$ such that $\{(\alpha, \beta, h, i) \in R(i) \mid h \geqq n, i \geqq n\} \subset \subset V$.

Remark 4.21. In [Har], Douglas Harris has characterized those regular spaces which can be densely embedded in regular-closed spaces.

5. P = Completely Regular, Normal, Paracompact, Metric, Completely Normal, Locally Compact, Zero-dimensional, and Perfectly Normal. As mentioned in the introduction section, all the above properties are assumed to include the Hausdorff property. We observe that if a completely regular topological space is P-closed where P is a property common to all compact spaces, then this space is also compact.

Theorem 5.1. *For P = completely regular, normal, paracompact, metric, completely normal, locally compact, or zero-dimensional, if (X, τ) is a P-space, then the following are equivalent*:

(a) (X, τ) *is minimal* P.
(b) (X, τ) *is* P*-closed.*
(c) (X, τ) *is compact.*

For a proof of P = completely regular, locally compact, and zero-dimensional, see [Ba1, Be2]. A proof of P = normal is located in [Be2] and [SW]. A proof of P = paracompact, metric, and completely normal is located in [SSe].

Theorem 5.2. [Ste3]. *Let (X, τ) be perfectly normal. The following are equivalent*:

(a) (X, τ) *is minimal perfectly normal.*
(b) (X, τ) *is perfectly normal-closed.*
(c) (X, τ) *is countably compact.*

6. $P = T_1$ and First Countable.

Definition. *A space (X, τ) has the cofinite topology if and only if $\tau = \{\emptyset, X\} \cup \cup \{A \subset X \mid X - A$ is finite$\}$.*

Theorem 6.1. [Be2].

(a) *A T_1 space (X, τ) is minimal T_1 if and only if (X, τ) has the cofinite topology.*

(b) *Every T_1 space is Katětov T_1.*

(c) *Minimal T_1 is hereditary.*

(d) *A minimal T_1 space is Hausdorff if and only if it is finite.*

(e) *A product of non-empty spaces is minimal T_1 if and only if each coordinate space is minimal T_1, and either*

(i) *there is only one coordinate space with more than one point or*

(ii) *each coordinate space is finite and all but a finite number of coordinate spaces are singletons.*

We note that an infinite T_1 space is not T_1-closed since a one-point extension can be constructed by defining the neighborhoods of the point to be cofinite sets containing the point. So, we easily conclude that a space is T_1-closed if and only if it is finite and T_1. This gives us an example of a topological property P for which a P-closed space is minimal P without the converse holding in general.

Definition. *If P is a topological property, then $P(1)$ space will mean a space which is first countable and has property P; thus, a space is Hausdorff (1) provided it is Hausdorff and first countable.*

Definition. *A space is feebly compact if and only if every locally finite system of open sets is finite (or equivalently, every countable open filter base is fixed).*

Theorem 6.2. [Ste3]. *Let (X, τ) be a Hausdorff (1) space.*

(a) *The following are equivalent:*

(i) *(X, τ) is minimal Hausdorff (1).*

(ii) *(X, τ) is semiregular and feebly compact.*

(iii) *Every countable open filter base on (X, τ) with a unique adherent point is convergent.*

(b) *The following are equivalent:*

(i) *(X, τ) is Hausdorff (1)-closed.*

(ii) *(X, τ) is feebly compact.*

(iii) *(X, τ_s) is minimal Hausdorff (1).*

Theorem 6.3. [Ste3]. *Let (X, τ) be a Urysohn (1) space.*

(a) *(X, τ) is minimal Urysohn (1) if and only if every countable Urysohn filter base with a unique adherent point is convergent.*

(b) *(X, τ) is Urysohn (1)-closed if and only if every countable Urysohn filter base is fixed.*

Theorem 6.4. [Ste3]. *For P = regular and completely regular, let (X, τ) be a $P(1)$ space. The following are equivalent*:

(a) (X, τ) *is minimal* $P(1)$.

(b) *Every countable P filter base with at most one adherent point is convergent.*

(c) *Every countable P filter base is fixed.*

(d) (X, τ) *is* $P(1)$-*closed.*

(e) (X, τ) *is feebly compact.*

(f) (X, τ) *is minimal Hausdorff* (1).

Theorem 6.5. [Ste3]. *Let (X, τ) be a zero-dimensional* (1) *space. The following are equivalent*:

(a) (X, τ) *is minimal zero-dimensional* (1).

(b) (X, τ) *is zero-dimensional* (1)-*closed.*

(c) (X, τ) *is pseudocompact.*

Theorem 6.6. [Ste3]. *For P = paracompact, weakly normal, normal, completely normal, and perfectly normal, let (X, τ) be a $P(1)$ space.*
The following are equivalent:

(a) (X, τ) *is minimal* $P(1)$.

(b) (X, τ) *is* $P(1)$-*closed.*

(c) (X, τ) *is pseudocompact.*

(d) (X, τ) *is countably compact.*

Corollary 6.7. [Ste3]. *A paracompact* (1) *space is minimal paracompact* (1) *if and only if it is compact.*

Theorem 6.8. [Ste3].

(a) *A countable product of nonempty spaces is Hausdorff* (1)-*closed if and only if each coordinate space is Hausdorff* (1). *closed,*

(b) *For P = Hausdorff, regular, completely regular, zero-dimensional, weakly normal, a countable product of nonempty spaces is minimal $P(1)$ if and only if each coordinate space is minimal $P(1)$.*

(c) *If a countable product of nonempty spaces is minimal Urysohn* (1), *then each coordinate space is minimal Urysohn* (1).

(d) *If a countable product of nonempty spaces is Urysohn* (1)-*closed, then each coordinate space is Urysohn* (1)-*closed.*

(e) *If X and Y are Urysohn* (1)-*closed spaces and X is absolutely closed, then $X \times Y$ is Urysohn* (1)-*closed.*

(f) *The product of a compact* (1) *space and a minimal Urysohn* (1) *space is a minimal Urysohn* (1) *space.*

Theorem 6.9. [Ste3].

(a) *The closure of an open set in a Hausdorff* (1)*-closed space is Hausdorff* (1)*-closed.*

(b) *For* $P =$ *regular, completely regular, and zero-dimensional, the closure of an open set in a minimal* $P(1)$ *space is minimal* $P(1)$.

(c) *If every countable closed set in a Hausdorff space is feebly compact, then the space is countably compact.*

(d) *Every Hausdorff* (1) *space can be densely embedded in a Hausdorff* (1)*-closed space and can be embedded in a minimal Hausdorff* (1) *space.*

Theorem 6.10. [Ste3]. *For* $P =$ *regular, completely regular, zero-dimensional, weakly normal, completely normal, and perfectly normal, a minimal* $P(1)$ *space is of second category.*

The space in example 3.13 is Hausdorff (1)-closed and Urysohn (1)-closed but not minimal Hausdorff (1) or minimal Urysohn (1). The space in example 3.14 is a minimal Hausdorff (1) space which is neither countably compact nor Urysohn. The space in example 4.8 is Urysohn (1)-closed, but not minimal Urysohn (1) or feebly compact. The space in example 3.12 is minimal Hausdorff (1) but not of second category.

Example 6.11. [Ste3]. Let τ be the order topology on the set X of all ordinal numbers less than the first uncountable ordinal. (X, τ) is a minimal $P(1)$ space for $P =$ Hausdorff, Urysohn, regular, completely regular, weakly normal, normal, completely normal and zero-dimensional. (X, τ) is not compact.

7. Unsolved Problems. A Urysohn, H-closed space is Urysohn-closed and completely Hausdorff-closed; and a completely Hausdorff, regular-closed space is completely Hausdorff-closed. This leads quite naturally to the following problem posed by Stephenson.

Problem 1. [Ste1]. *Is a regular, completely Hausdorff-closed space necessarily regular-closed?*

Problem 2. *Prove or disprove that the product of Urysohn-closed spaces is Urysohn-closed.*

Problem 3. *Prove or disprove that the product of minimal Urysohn spaces is minimal Urysohn.*

Problem 4. *Prove or disprove that the product of completely Hausdorff-closed spaces is completely Hausdorff-closed.*

Problem 5. *Prove or disprove that the product of regular-closed spaces is regular-closed.*

Problem 6. [Be2]. *Prove or disprove that the product of minimal regular spaces is minimal regular.*

The embedding problem of finding necessary and sufficient conditions for a P-space to be embeddable (densely embeddable) in a P-closed space or in a minimal P space has been solved for $P =$ Hausdorff and completely Hausdorff and partially solved for Urysohn and regular. Some remaining problems are listed below.

Problem 7. *Find a necessary and sufficient condition that a Urysohn space can be embedded (or densely embedded) in a minimal Urysohn space.*

Problem 8. *Find a necessary and sufficient condition that a regular space can be embedded in a regular-closed space.*

Problem 9. *Find a necessary and sufficient condition that a regular space can be embedded (or densely embedded) in a minimal regular space.*

Since each compact Hausdorff space is of second category, it is natural to ask which P-closed and minimal P spaces are of second category. The space of example 3.12 is a minimal Hausdorff space which is not of second category. If, in the space of example 4.8, I_1 and I_2 are selected to be countable, then the space is a semi-regular, Urysohn-closed space which is not of second category. By 4.17 (d), a regular-closed space is of second category. The space in example 4.13 is completely Hausdorff-closed but not of second category. This leads to the next three problems.

Problem 10. *Is a minimal Urysohn space necessarily of second category?*

Problem 11. [Ste3]. *Is a minimal Urysohn*(1) *space necessarily of second category?*

Problem 12. [Ste1]. *Is a regular, completely Hausdorff-closed space necessarily of second category?*

Problem 13. *Does there exist a noncompact minimal perfectly normal space?*

Clearly, an affirmative answer to problem 1 would imply an affirmative answer to probem 12.

For P = Hausdorff and completely Hausdorff, if every closed subset of a space is P-closed, then the space is compact. By 4.16 if every closed set in a space is regular-closed, then the space is minimal regular. This leads to the next question posed by Banaschewski.

Problem 14. [Ba1]. *Is a space in which each closed set is regular-closed necessarily compact?*

We observe that a space in which every closed set is Urysohn-closed is minimal Urysohn.

Problem 15. *Is a space in which each closed set is Urysohn-closed necessarily compact?*

In [K1], Katětov proved that an H-closed space is not only Katětov Hausdorff but also that there exists only one minimal Hausdorff topology coarser than an H-closed topology. This motivates us to pose the following problems.

Problem 16.

(a) *Is a Urysohn-closed space necessarily Katětov Urysohn?*

(b) *Is there only one minimal Urysohn topology coarser than a Urysohn-closed, Katětov Urysohn topology?*

Problem 17.

(a) *Is a regular-closed space necessarily Katětov regular?*

(b) *Is there only one minimal regular topology coarser than a regular-closed, Katětov regular topology?*

Note: Problems 1 and 2 have been solved by Herrlich; problems 3, 10, 11, and 13 have been solved by Stephenso; problem 16 has been solved by Porter. The solutions will appear.

The authors have made every effort to include all the known references in the research on minimal topologies and give them their due credit. Undoubtedly, some errors and omissions have been made for which the authors now wish to express their apologies.

Bibliography

[AdeG] *J. M. Aarts* and *J. deGroot,* Colloquium Co-topologie, Math. Centrum (Amsterdam), Syllabus ZWA, 1964 (2).

[A1] *P. Alexandroff,* Some results in the theory of topological spaces, obtained within the last twenty-five years, Uspehi Mat. Nauk *15* (1960), 25—97; Russian Math. Surveys *15* (1960), 23—83.

[AU] *P. Alexandroff* and *P. Urysohn*, Zur Theorie der topologischen Räume, Math. Ann. *92* (1924), 258—266.

[Ba1] *B. Banaschewski*, Über zwei Extremaleigenschaften topologischen Räume, Math. Nachr. *13* (1955), 141—150.

[Ba2] *B. Banaschewski*, On the Weierstrass-Stone approximation theorem, Fund. Math. *44* (1957), 249—252.

[Ba3] *B. Banaschewski*, On the Katětov and Stone-Čech extensions, Canad. Math. Bull. *2* (1959), 1—4.

[Ba4] *B. Banaschewski*, Über Hausdorffsche-minimale Erweiterungen von Räumen, Arch. Math. *12* (1961), 355—356.

[Ba5] *B. Banaschewski*, Extensions of topological spaces, Canad. Math. Bull. *7* (1965), 1—22.

[Be1] *M. P. Berri*, Minimal topological spaces, Ph. D. dissertation, University of California at Los Angeles, 1961.

[Be2] *M. P. Berri*, Minimal topological spaces, Trans. Amer. Math. Soc. *108* (1963), 97—105.

[Be3] *M. P. Berri*, Categories of certain minimal topological spaces, J. Austral. Math. Soc. *3* (1964), 78—82.

[BS] *M. P. Berri* and *R. H. Sorgenfrey*, Minimal regular spaces, Proc. Amer. Math. Soc. *14* (1963), 454—458.

[Bo1] *N. Bourbaki*, Espaces minimaux et espaces complètement séparés, C. R. Acad. Sci. Paris *212* (1941), 215—218.

[Bo2] *M. Bourbaki*, Topologie générale, 3rd. ed., Actualités Sci. Ind. No. 1142, Hermann, Paris 1961.

[CF] *C. Chevally* and *O. Frink*, Bicompactness of Cartesian Products, Bull. Amer. Soc. *47* (1941), 612—614.

[D] *J. Dugundji*, Topology, Allyn & Bacon, Boston, 1966.

[Fl1] *J. Flachsmeyer*, *H*-abgeschlossene Räume als schwach-stetige Bilder kompakter Räume, Math. Z. *91* (1966), 336—343.

[Fl2] *J. Flachsmeyer*, Zur Theorie der *H*-abgeschlossenen Erweiterungen, ibid., *94* (1966), 349—381.

[Fo1] *S. Fomin*, Extensions of topological spaces, Ann. of Math. *44* (1943), 471—480.

[GJ] *L. Gillman* and *M. Jerison*, Rings of continuous functions, Van Nostrand, New York, 1960.

[Ha] *H. E. Hayes*, On a class of absolutely closed and minimal topological spaces, M. A. Thesis, The University of Kansas, 1967.

[Har] *D. Harris*, Regular-closed spaces and proximities, Pac. Journal of Math. *34* (1970), 675—686.

[He1] *H. Herrlich*, T_v-Abgeschlossenheit und T_v-Minimalität, Math. Z. *88* (1965), 285—294.

[He2] *H. Herrlich*, Nicht alle T_2-minimale Räume sind von 2 Kategorie, ibid., *91* (1966), 185.

[HS] *H. Herrlich* and *G. E. Strecker*, *H*-closed spaces and reflexive subcategories, Math. Ann. *177* (1968), 302—309.

[Ho] *W. Holsztyński*, Minimal Hausdorff spaces and T_1-bicompacta, Dokl. Akad. Nauk SSSR, *178* (1968), 24—26; Soviet Math. Dokl. *9* (1968), 18—20.

[Ik] *S. Ikenaga*, Product of minimal topological spaces, Proc. Japan Acad. *40* (1964), 329—331.

[Il] *S. Ilradis*, Characterization of spaces by means of *H*-closed extensions, Dokl. Akad. Nauk. SSSR *149* (1963), 1015—1018; Soviet Math. Dokl. *4* (1963), 501—504.

[K1] *M. Katětov*, Über *H*-abgeschlossene und bikompakt Räume, Časopis pěst. mat. *69* (1940), 36—49.

[K2] *M. Katětov*, On *H*-closed extensions of topological spaces, ibid., *72* (1947), 17—32.

[K3] *M. Katětov*, A note on semiregular and nearly regular spaces, ibid., *72* (1947), 97—99.

[K4] *M. Katětov,* On the equivalence of certain types of extension of topological spaces, ibid., *72* (1947), 101—106.

[Ka] *H. Kawashima,* On the topological product of minimal Hausdorff spaces, TRU Math. *1* (1965), 62—64.

[La] *R. E. Larson,* Minimal T_0-spaces and minimal T_D-spaces, Pac. Journal of Math. (to appear).

[Li] *C. T. Liu,* Absolutely closed spaces, Trans. Amer. Math. Soc. *130* (1968), 86—104.

[N] *R. L. Newcomb,* Topologies which are compact modulo an ideal Ph. D. Dissertation, University of California at Santa Barbara, 1967.

[O1] *F. Obreanu,* Filtre Deschise, Acad. Repub. Pop. Romîne Bul. Sti. Ser. Mat. Fiz. Chim. *2* (1950), 1—5.

[O2] *F. Obreanu,* Spatii Absolut Inchise, ibid., *2* (1950), 21—25.

[O3] *F. Obreanu,* Spatii Separate Minimale, An. Acad. Repub., Pop. Romîne, Sect. Sti. Mat. Fiz. Chem. Ser. A *3* (1950), 325—349.

[O4] *F. Obreanu,* Spatii Local Absolut Inchise, ibid., *3* (1950), 375—394.

[Pa] *A. S. Parhomenko,* Über eineindentige Stetige Abbildungen, Mat. Sbornik *5* (1939), 197—210 (In Russian; German summary).

[Pah] *Ki-Hyun Pahk,* Note on the characterizations of minimal T_0 and T_D spaces, Kyungpook Math. J. *8* (1968), 5—10.

[Po1] *J. R. Porter,* A study of minimal Hausdorff spaces, Ph. D. dissertation, New Mexico State University, 1966.

[Po2] *J. R. Porter,* On noncompact, *H*-closed spaces and semiregular spaces, Tech. Rept., Univ. of Okla. Preprint No. 41, 1966.

[Po3] *J. R. Porter,* On locally *H*-closed spaces, Proc. London Math. Soc. (3), *19* (1969) (to appear).

[Pr] *V. V. Proizvolov,* One-to-one continuous mappings of topological spaces, Mat. Sb. *68* (*110*) (1965), 417—431; Amer. Math. Soc. Transl. *71* (series 2) (1968), 77—92.

[PT] *J. Porter* and *J. Thomas,* On *H*-closed and minimal Hausdorff spaces, Trans. Amer. Math. Soc. *138* (1969), 159—170.

[R1] *A. Ramanathan,* Maximal-Hausdorff spaces, Proc. Indian. Acad. Sci. Sect. A *26* (1947), 31—42.

[R2] *A. Ramanathan,* A charaterization of maximal-Hausdorff spaces, J. Indian Math. Soc., N. S. *11* (1947), 73—80.

[Sc1] *C. T. Scarborough,* Minimal topologies on Hausdorff spaces, Ph. D. Dissertation, Tulane University, 1964.

[Sc2] *C. T. Scarborough,* Minimal Urysohn spaces, Pac. Journal of Math. *27* (1968), 611—617.

[Sc3] *C. T. Scarborough,* Closed graphs and closed projections, Proc. Amer. Math. Soc. *20* (1969), 465—470.

[SSo] *C. T. Scarborough* and *A. H. Stone,* Products of nearly compact spaces, Trans. Amer. Math. Soc. *124* (1966), 131—147.

[SSe] *C. T. Scarborough* and *R. M. Stephenson,* Jr., Minimal topologies, Colloq. Math. *19* (1968), 215—219.

[SW] *N. Smythe* and *C. A. Wilkins,* Minimal Hausdorff and maximal compact spaces, J. Austral. Math. Soc. *3* (1963), 167—171.

[Ste1] *R. M. Stephenson,* Jr., *P*-minimal and *P*-closed spaces, Ph. D. dissertation, Tulane University, 1967.

[Ste2] *R. H. Stephenson,* Jr., Spaces for which the Stone-Weierstrass theorem holds. Trans. Amer. Math. Soc. *133* (1968), 537—546.

[Ste3] *R. H. Stephenson,* Jr., Minimal first countable topologies, Trans. Amer. Math. Soc. *138* (1969), 115—127.

[Ste4] *R. H. Stephenson*, Jr., Product spaces for which the Stone-Weierstrass theorem holds, Proc. Amer. Math. Soc. *21* (1969), 284—288.

[Ste5] *R. M. Stephenson*, Jr., Two minimal first countable Hausdorff spaces, Math. Z. *108* (1969), 171—172.

[Sto] *M. H. Stone,* Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. *41* (1937), 374—481.

[Str] *G. E. Strecker,* Co-topologies and generalized compactness conditions, Ph. D. Dissertation, Tulane University, 1966.

[Stv] *G. E. Strecker* and *G. Viglino*, Co-topology and minimal Hausdorff spaces, Proc. Amer. Math. Soc. *21* (1969), 569—574.

[StW] *G. E. Strecker* and *E. Wattel,* On semi-regular and minimal Hausdorff embeddings, Proc. Roy. Neth. Acad. Sci. A70, (1967), 234—237; Ind. Math. *29* (1967), 234—237.

[T] *A. Tychonoff*, Über die topologische Erweiterung von Räumen Math. Ann. *102* (1930), 544—561.

[U] *P. Urysohn*, Über die Machtigkeit der zusammenhängenden Mengen, Math. Ann. *94* (1925), 262—295.

[Va] *R. Vaidyanathaswamy,* Treatise on set topology I, Indian Math. Soc. Madras, 1947.

[Ve1] *H. V. Velichko*, *H*-closed topological spaces, Mat. Sb. (N. S.) *70* (*112*) (1966), 98—112; Amer. Math. Soc. Transl. *78* (Series 2) (1969), 103—118.

[Ve2] *H. V. Velichko*, On the theory of *H*-closed topological spaces, Sibirsk. Mat. Ž. *8* (1967), 754—763; Siberian Math. J. *8* (1967), 569—575.

[Vi1] *G. Viglino,* The application of co-topology to minimal topological spaces, Ph. D. dissertation, Washington University, 1966.

[Vi2] *G. Viglino*, A co-topological application to minimal spaces, Pac. Journal of Math. *27* (1968), 197—200.

(1) LOUISIANA STATE UNIVERSITY, NEW ORLEANS, U.S.A.
(2) UNIVERSITY OF KANSAS, U.S.A.
(3) UNIVERSITY OF NORTH CAROLINA, U.S.A.