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ON TOPOLOGICAL BOOLEAN ALGEBRAS

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In this paper we shall examine how some non-elementary parts of point set topology can be extended to classical topological Boolean algebras (also called closure algebras) by considering the class of regular topological Boolean algebras which have a σ-locally finite bases. This class we call the M-algebras. We shall see that the class of M-algebras resembles very closely the class of metric spaces. Further we can show that the $\mathfrak{m}$-representable (in the algebraic sense) M-algebras of topological weight $\mathfrak{m}$ are weakly homeomorphic to quotient algebras of the form $S(X)/I$ where X is a metric space of topological weight $\mathfrak{m}$ and I is an $\mathfrak{m}$-additive ideal of $S(X)$.

The fundamental notions of point set topology viz: the separation axioms, compactness, connectedness etc. were studied in the case of topological Boolean algebras by G. Nöbeling [cf. 2]. Closure algebras with a countable regular open basis – called C-algebras – were studied by R. Sikorski [cf. 3 & 4]. We shall follow the methods of Prof. Sikorski and in doing so we shall have to overcome some natural difficulties in passing from the countable situation to the uncountable case.

We shall follow the terminology of G. Nöbeling and R. Sikorski.

1. Definitions and Known Results

Definition 1.1. *A classical topological Boolean algebra is by definition a Boolean algebra $\mathfrak{A}$ with an operation $\mathfrak{t}$ which associates with each element A in $\mathfrak{A}$ an element $\mathfrak{t}A$ in $\mathfrak{A}$ called the closure of A such that the following axioms are satisfied:*

(i) $\mathfrak{t}(A \vee B) = \mathfrak{t}A \vee \mathfrak{t}B$ *for any two elements A, B in $\mathfrak{A}$;*

(ii) $A < \mathfrak{t}A$;

(iii) $\mathfrak{t}\mathfrak{t}A = \mathfrak{t}A$;

(iv) $\mathfrak{t}0 = 0$ *where 0 is the zero element of the Boolean algebra $\mathfrak{A}$.*

We shall denote by $(\mathfrak{A}, \mathfrak{t})$ the Boolean algebra $\mathfrak{A}$ with the closure operation $\mathfrak{t}$.

Example (1). Let $(X, \mathfrak{t})$ be a topological space and let $S(X)$ be the set algebra of all subsets of X. Then $(S(X), \mathfrak{t})$ is a closure algebra.

Example (2). Let $(X, \mathfrak{t})$ be a topological space of topological weight $\mathfrak{m}$ ($\mathfrak{m}$ not less than $\aleph_0$) and let I be an $\mathfrak{m}$-additive ideal of $S(X)$. Then $S(X)/I$ is a closure algebra given by $\mathfrak{t}[A] = [A^*]$ where $[A]$ is the coset containing the element A modulo the ideal I and

$$A^* = \{\sum(G : G \text{ open in } X, \text{ and } G \wedge A \in I)\}' .$$

In the theory of closure algebras the elements of Boolean algebra play a role analogous to subsets of topological spaces. Given a closure algebra $(\mathfrak{A}, \mathfrak{t})$ we define the open (closed) elements, coverings, locally finite family of elements etc. exactly as in the case of topological spaces as follows:

An element A in $(\mathfrak{A}, \mathfrak{t})$ is open (closed) if $A' = \mathfrak{t}A'$ ($A = \mathfrak{t}A$). A class $\boldsymbol{G}$ of (open) elements is a (open) covering if $\sum(G\colon G \text{ in } \boldsymbol{G}) = |\mathfrak{A}|$ where $|\mathfrak{A}|$ is the unit element of $\mathfrak{A}$.

A class $\boldsymbol{K}$ of elements of $(\mathfrak{A}, \mathfrak{t})$ is said to be locally finite if there exists an open covering $\boldsymbol{G}$ such that each element G of $\boldsymbol{G}$ intersects (or has a non-zero intersection with) only a finite number of elements from $\boldsymbol{K}$.

Definition 1.2. *A class* $\boldsymbol{G}$ *of open elements of a closure algebra* $(\mathfrak{A}, \mathfrak{t})$ *is called an open basis for* $(\mathfrak{A}, \mathfrak{t})$ *if*

(i) $\mathfrak{A}$ *is* $\overline{\overline{\boldsymbol{G}}}$ *complete (where* $\overline{\overline{\boldsymbol{G}}}$ *is the cardinal of* $\boldsymbol{G}$),

(ii) *every open element of* $(\mathfrak{A}, \mathfrak{t})$ *can be expressed as the sum of a subset of elements from* $\boldsymbol{G}$.

The open basis $\boldsymbol{G}$ of definition 1.2 is said to be regular if for each open element G we have $G = \sum(U : \mathfrak{t}U < G, U \text{ in } \boldsymbol{G})$.

Definition 1.3. *The topological weight of a closure algebra* $(\mathfrak{A}, \mathfrak{t})$ *with an open basis is the least cardinal* $\mathfrak{m}$ *such that* $(\mathfrak{A}, \mathfrak{t})$ *has an open basis of cardinal* $\mathfrak{m}$.

Definition 1.4. *Let* $(\mathfrak{A}, \mathfrak{t})$ *be a closure algebra of topological weight* $\mathfrak{m}$. *The least* $\mathfrak{m}$-*complete subalgebra* $B_{\mathfrak{m}}(\mathfrak{A}, \mathfrak{t})$ *containing all the open elements of* $(\mathfrak{A}, \mathfrak{t})$ *is called the Borel field of* $(\mathfrak{A}, \mathfrak{t})$ *and the elements of* $B_{\mathfrak{m}}(\mathfrak{A}, \mathfrak{t})$ *are called Borel elements.*

Definition 1.5. *Two closure algebras* $(\mathfrak{A}_1, \mathfrak{t}_1)$, $(\mathfrak{A}_2, \mathfrak{t}_2)$ *are said to be homeomorphic if there exists a complete isomorphism i of* $\mathfrak{A}_1$ *onto* $\mathfrak{A}_2$ *such that for each element* A *in* $\mathfrak{A}_1$, $i(\mathfrak{t}_1 A) = \mathfrak{t}_2(iA)$.

Two topological Boolean algebras with the same topological weight are weakly homeomorphic if their Borel fields are homeomorphic.

Definition 1.6. *A topological Boolean algebra with a countable regular open basis is called a C-algebra.*

Representation Theorem for C-Algebras (cf. R. Sikorski [3]): *A closure algebra* $(\mathfrak{A}, \mathfrak{t})$ *is weakly homeomorphic to a closure algebra of the form* $S(X)/I$ *where* X *is a separable metric space and* I *is a* σ*-additive ideal of* $S(X)$ *if and only if* $(\mathfrak{A}, \mathfrak{t})$ *is a* C*-algebra.*

2. The M-Algebras

Definition 2.1. *A closure algebra* $(\mathfrak{A}, \mathfrak{t})$ *of topological weight* $\mathfrak{m}$ *is called an* M*-algebra if there exists an enumerable family of open coverings* α_i, $\{\alpha_i = (V_\lambda^i : \lambda \in \Lambda_i)\}$ *such that*

(i) *each* α_i *is locally finite cover of cardinal not greater than* $\mathfrak{m}$ *and*

(ii) *if* $\beta_i : i = 1, 2, \dots$ *is an enumerable family of coverings with* $\{\beta_i = (U_\lambda^i : \lambda \in \Lambda_i)\}$, $U_\lambda^i < V_\lambda^i$, $U_\lambda^i \in B_{\mathfrak{m}}(\mathfrak{A}, \mathfrak{t})$, *then for every open element* G *of* $(\mathfrak{A}, \mathfrak{t})$

$$G = \sum(U : U \text{ in } \bigcup\beta_i,\ \mathfrak{t}U < G).$$

It is clear from the definition that $\bigcup\alpha_i$ is a regular open basis. We call $\bigcup\alpha_i$ an M-basis for the M-algebra $(\mathfrak{A}, \mathfrak{t})$. It is also evident that a closure algebra with an open basis is an M-algebra if and only if its Borel field is an M-algebra. Again if A is an element of an M-algebra $(\mathfrak{A}, \mathfrak{t})$ and if $A\mathfrak{A}$ is the principal ideal of all elements less than A, then with the relative topology $A\mathfrak{A}$ is an M-algebra.

We observe that the C-algebras of Sikorski are M-algebras of topological weight $\aleph_0$.

Propositions 2.1 and 2.2 below follow from the Nagata-Smirnov metrizability criterion.

Proposition 2.1. *A* T_1*-topological space* $(X, \mathfrak{T})$ *is metrizable if and only if* $(S(X), \mathfrak{T})$ *is an* M*-algebra.*

Proposition 2.2. *Let* $(X, \mathfrak{t})$ *be a metric space of topological weight* $\mathfrak{m}$ *and let* I *be an* $\mathfrak{m}$*-additive ideal of* $S(X)$. *Then* $S(X)/I$ *is an* M*-algebra.*

In the next proposition we find that some important properties of metric spaces can be extended to M-algebras:

Proposition 2.3. *Every* M*-algebra is*

(i) *perfectly normal;*

(ii) *completely normal and*

(iii) *hereditarily paracompact.*

3. Representation of M-Algebras

Definition 3.1. *We call an M-algebra $(\mathfrak{A}, \mathrm{t})$ of topological weight $\mathfrak{m}$ an M-field of sets if the Boolean algebra $\mathfrak{A}$ is an $\mathfrak{m}$-additive field of sets.*

Representation Theorem 1. *Every M-field of sets is weakly homeomorphic to a metric space.*

Proof. Let $(\mathfrak{A}, \mathrm{t})$ be an M-field of sets of topological weight $\mathfrak{m}$ with the Boolean algebra $\mathfrak{A}$ an $\mathfrak{m}$-additive field of subsets of some set X.

For each x in X, let $c(x)$ denote the intersection of all closed elements of $(\mathfrak{A}, \mathrm{t})$ containing x. The Borel field $B_{\mathfrak{m}}(\mathfrak{A}, \mathrm{t})$ is an $\mathfrak{m}$-additive field of sets generated by $\mathfrak{m}$ elements and is therefore atomic. The atoms of $B_{\mathfrak{m}}(\mathfrak{A}, \mathrm{t})$ are precisely the elements $(c(x)\colon x \text{ in } X)$.

Let Y be the set $(c(x)\colon x \text{ in } X)$. The Boolean algebra $B_{\mathfrak{m}}(\mathfrak{A}, \mathrm{t})$ is isomorphic to an $\mathfrak{m}$-additive field $\mathfrak{B}$ of subsets of Y. The field $\mathfrak{B}$ with the closure operation induced by the algebraic isomorphism $B_{\mathfrak{m}}(\mathfrak{A}, \mathrm{t}) \leftrightarrow \mathfrak{B}$ is an M-field of sets. Define for each A in $S(Y)$, $\mathrm{t}A =$ the intersection of all closed elements of $(\mathfrak{B}, \mathrm{t})$ which contain A. Then clearly $\mathrm{t}A$ belongs to $\mathfrak{B}$. It can be easily checked that (Y, t) is a metric space with $(\mathfrak{B}, \mathrm{t})$ as the Borel field.

This completes the proof of representation theorem 1.

As an immediate corollary we get

Proposition 3.1. *A closure algebra is weakly homeomorphic to a metric space if and only if it is an M-field of sets.*

Next we proceed to consider M-algebras $(\mathfrak{A}, \mathrm{t})$ of topological weight $\mathfrak{m}$ in the case when the Boolean algebra $\mathfrak{A}$ is $\mathfrak{m}$-representable, i.e. $\mathfrak{A}$ is isomorphic to a quotient algebra of the form $\mathfrak{Q}/I$ where $\mathfrak{Q}$ is an $\mathfrak{m}$-additive field of sets and I is an $\mathfrak{m}$-additive ideal of $\mathfrak{Q}$. We call such M-algebras quotient M-algebras.

Representation Theorem 2. *Every quotient M-algebra of topological weight $\mathfrak{m}$ is weakly homeomorphic to a closure algebra of the form $S(X)/I$ where X is a metric space of topological weight $\mathfrak{m}$ and I is an $\mathfrak{m}$-additive ideal of $S(X)$.*

To prove this theorem we need the following lemma:

Lemma. *Let $(\mathfrak{A}, \mathrm{t})$ be an M-algebra of topological weight $\mathfrak{m}$ with the Boolean algebra $\mathfrak{A}$, $\mathfrak{m}$-representable as $\mathfrak{Q}/I$ where $\mathfrak{Q}$ is an $\mathfrak{m}$-additive field of sets and I is an $\mathfrak{m}$-additive ideal of $\mathfrak{Q}$. Then we can define a closure operation $\mathfrak{T}$ for $\mathfrak{Q}$ such that*

(1) *$(\mathfrak{Q}, \mathfrak{T})$ is an M-algebra of topological weight $\mathfrak{m}$;*

(2) *the closure algebra $(\mathfrak{A}, \mathrm{t})$ is homeomorphic to the quotient algebra $(\mathfrak{Q}/I, \mathfrak{T})$.*

Proof of the Lemma. Let $\bigcup \alpha_i$ where $\alpha_i = (R^i_\lambda : \lambda \in \lambda_i)$ be an M-basis for $(\mathfrak{A}, \mathfrak{t})$ of cardinality $\mathfrak{m}$. Let $R^i_\lambda = [B^i_\lambda]$ = the class containing the element B^i_λ, B^i_λ in $\mathfrak{Q}$. We shall proceed to find an element B in I such that

(i) the σ-family $\beta_i = (B, B' \wedge B^i_\lambda : \lambda \in \lambda_i)$ will form an open basis for a topology $\mathfrak{T}$ for $\mathfrak{Q}$;

(ii) each β_i is a locally finite open covering for $(\mathfrak{Q}, \mathfrak{T})$ and

(iii) $(\mathfrak{Q}, \mathfrak{T})$ is an M-algebra with $\bigcup \beta_i$ as an M-basis.

We may assume without loss of generality that $B^1_1 = 0$ and $\sum(B^i_\lambda : \lambda \in \Lambda_i) = |\mathfrak{Q}|$. Let $N(\lambda_i, \mu_j)$ denote the set of all (ν, k) such that $R^k_\nu < R^i_\lambda \wedge R^i_\mu$. Let $N^0_{\lambda i}$ denote the set of all (μ, j) such that $R^i_\lambda \wedge R^j_\mu = 0$. Again let N^k_ν denote the set of all (λ, i) such that $\sum(R^j_\mu : (\mu, j) \in N^0_{\lambda i})' < R^k_\nu$.

Since $\bigcup \alpha_i$ is a regular open basis for $(\mathfrak{A}, \mathfrak{t})$ we have

$$R^i_\lambda \wedge R^j_\mu = \sum\{R^k_\nu : (\nu, k) \in N(\lambda_i, \mu_j)\}$$

and

$$R^k_\nu = \sum(R^i_\lambda : (\lambda, i) \in N^k_\nu)\,.$$

Therefore the elements $L(\lambda_i, \mu_j)$ and $M(\nu, k)$ defined below belong to the ideal I:

$$L(\lambda_i, \mu_j) = (B^i_\lambda \wedge B^j_\mu) \oplus \{\sum(B^k_\nu : (\nu, k \in N(\lambda_i, \mu_j))\}$$

and

$$M(\nu, k) = B^k_\nu \oplus \{\sum(B^i_\lambda : (\lambda, i) \in N^k_\nu)\}\,.$$

Here the symbol $A \oplus B$ stands for the symmetric difference $(A \wedge B') \vee (A' \wedge B)$.

Since I is $\mathfrak{m}$-additive and $\overline{\overline{\Lambda}} \leqq \mathfrak{m}$ for all i, the elements L, M given by $L = \sum_{i,j}(\sum_{\Lambda_i, \Lambda_j} L(\lambda_i, \mu_j))$ and $M = \sum_k(\sum_{\Lambda_k} M(\nu, k))$ belong to I. Since each α_i is locally finite in $(\mathfrak{A}, \mathfrak{t})$ there exist open coverings $\boldsymbol{G}_i$ for each i such that every element of $\boldsymbol{G}_i$ intersects only a finite number of elements from α_i. Let $\boldsymbol{G}_i = \{[G^i_\delta] : \delta \in \Omega_i\}$ and $I(\delta, i) = (\lambda : B^i_\lambda \wedge G^i_\delta \notin I, \lambda \in \Lambda_i)$. Further let $K^i_\delta = \sum\{(G^i_\delta \wedge B^i_\lambda) : \lambda \notin I(\delta, i)\}$. Then each K^i_δ belongs to I and the element $K = \sum_i(\sum_{\delta \in \Omega_i} K^i_\delta)$ belongs to I.

Now each $\boldsymbol{G}_i$ is a covering of $(\mathfrak{A}, \mathfrak{t})$ implies that the elements $P = \{\sum(G^i_\delta : \delta \in \Omega_i)\}'$ belong to I and $P = \sum P_i$ belongs to I. Finally let $B = K \vee L \vee M \vee P$. Then clearly B belongs to I. Let $S^i_\lambda = B^i_\lambda \wedge B'$ and let $\beta_i = (B, S^i_\lambda : \lambda \in \Lambda_i)$. Then each β_i is a covering of $\mathfrak{Q}$ and

$$S^i_\lambda \wedge S^j_\mu = \sum(S^k_\nu : (\nu, k) \text{ in } N(\lambda_i, \mu_j))\,.$$

Therefore we can define a topology $\mathfrak{T}$ for $\mathfrak{Q}$ having $\bigcup \beta_i$ as an open basis. To show that $\bigcup \beta_i$ is regular we have only to observe that

$$S^k_\nu = \sum(S^i_\lambda : (\lambda, i) \in N^k_\nu) = \sum(\mathfrak{T} S^i_\lambda : (\lambda, i) \in N^k_\nu)\,.$$

Next we shall proceed to show that each β_i is a locally finite family. Since each $\boldsymbol{G}_i$ is an open covering of $(\mathfrak{A}, \mathfrak{t})$, we can assume without loss of generality that each G^i_δ can be expressed as a sum of elements from $(B^i_\lambda : \lambda \in \Lambda_i,\ i = 1, 2, \ldots)$. Therefore each $(B' \wedge G^i_\delta)$ can be expressed as a sum of elements from $(S^i_\lambda : \lambda \in \Lambda_i,\ i = 1, 2, \ldots)$. Therefore $\boldsymbol{H}_i = (B, B' \wedge G^i_\delta : \delta \in \Omega_i)$ is an open covering of $(\mathfrak{Q}, \mathfrak{T})$ for each i. Further each element of $\boldsymbol{H}_i$ intersects only a finite number of elements from β_i.

Now $(\mathfrak{Q}, \mathfrak{T})$ is a closure algebra of topological weight $\mathfrak{m}$ and $\bigcup \beta_i$ is a regular open basis for $(\mathfrak{Q}, \mathfrak{T})$ such that each β_i is a locally finite open covering of $(\mathfrak{Q}, \mathfrak{T})$. Also the Boolean algebra $\mathfrak{Q}$ is an $\mathfrak{m}$-additive field of sets. Therefore as in the proof of Representation theorem 1 we can show that $(\mathfrak{Q}, \mathfrak{T})$ is weakly homeomorphic to a closure algebra $S(X)$ where X is a metric space. Therefore $(\mathfrak{Q}, \mathfrak{T})$ is an M-algebra.

Finally since $[S^i_\lambda] = R^i_\lambda$ and $\bigcup_{i=1}^{\infty} (B, S^i_\lambda : \lambda \in \Lambda_i)$ is an open basis for $(\mathfrak{Q}, \mathfrak{T})$, $(\mathfrak{A}, \mathfrak{t})$ is identical with the quotient algebra $(\mathfrak{Q}/I, \mathfrak{T})$.

The proof of Representation theorem 2 follows easily from Representation theorem 1 and the Lemma.

Now we are ready to state the main theorem of this section:

Theorem 3. *A closure algebra $(\mathfrak{A}, \mathfrak{t})$ of topological weight $\mathfrak{m}$ is weakly homeomorphic to a quotient algebra of the form $(S(X)/I, \mathfrak{T})$ where $(X, \mathfrak{T})$ is a metric space of topological weight $\mathfrak{m}$ and I is an $\mathfrak{m}$-additive ideal of $S(X)$ if and only if $(\mathfrak{A}, \mathfrak{t})$ is a quotient M-algebra.*

4. Examples

We shall give a method of constructing examples of M-algebras of arbitrarily large topological weight which are not M-field of sets.

Let E_n denote the Euclidean space of dimension n and let $N(\Omega)$ be the generalized Baire space i.e. space of enumerable sequences whose elements are taken from the discrete set Ω. It is known that $N(\Omega)$ is a zero dimensional metric space of topological weight $\overline{\overline{\Omega}}$. Select Ω such that $\overline{\overline{\Omega}} = \mathfrak{m}$ with $\mathfrak{m}^{\aleph_0} > \mathfrak{m} > \mathfrak{c}$. (Here $\mathfrak{c}$ denotes the cardinal of the set of reals.) For the existence of such $\mathfrak{m}$ refer to Bachmann [1]. Consider the closure algebra $\mathfrak{A}_n = S(E_n \times N(\Omega))/J$ where J is the ideal $J = (p \times A : p$ is a fixed point of E_n and A is a subset of $N(\Omega)$ of cardinal not greater than $\mathfrak{m})$. We can show that $B_m(S(E_n \times N(\Omega))) \cap J$ is a non-semi principal $\mathfrak{m}$-ideal of $B_m(S(E_n \times N(\Omega)))$ and therefore $B_m(\mathfrak{A}_n)$ is not $\mathfrak{m}$-isomorphic to $\mathfrak{m}$-additive field of sets. Finally we can show that the topological weight of $\mathfrak{A}_n$ is exactly $\mathfrak{m}$.

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