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OPTIMAL LYAPUNOV INEQUALITIES AND APPLICATIONS TO NONLINEAR PROBLEMS*

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Abstract. This work is devoted to the study of L_p Lyapunov-type inequalities ($1 \leq p \leq \infty$) for linear partial differential equations. More precisely, we treat the case of Neumann boundary conditions on bounded and regular domains in $\mathbb{R}^N$. It is proved that the relation between the quantities p and $N/2$ plays a crucial role. This fact shows a deep difference with respect to the ordinary case. The linear study is combined with Schauder fixed point theorem to provide new conditions about the existence and uniqueness of solutions for resonant nonlinear problems.

Key words. Linear boundary problems, Lyapunov inequalities, ordinary differential equations, partial differential equations, resonant problems

AMS subject classifications. 34B05, 34B15, 35J25, 35J65

1. Introduction.

Let us consider the linear problem

$$u''(x) + a(x)u(x) = 0, \qquad x \in (0, L),\ u'(0) = u'(L) = 0 \tag{1.1}$$

where $a \in \Lambda_0$ and Λ_0 is defined by

$$\Lambda_0 = \{a \in L^1(0, L) \setminus \{0\} : \int_0^L a(x)\ \mathrm{d}x \geq 0 \ \text{ and (1.1) has nontrivial solutions } \} \tag{1.2}$$

The well-known Lyapunov inequality states that if $a \in \Lambda_0$, then $\int_0^L a^+(x)\ \mathrm{d}x > 4/L$. Moreover, the constant $4/L$ is optimal (see [3], [4], [6], [9] and [10]). An analogous result is true for Dirichlet boundary conditions. In fact, the original results were proved for this kind of boundary conditions ([9], [11], [12], [15]).
In this paper we review some more general recent results contained in [4], [5] and [6]. We consider for each p with $1 \leq p \leq \infty$, the quantity

$$\beta_p \equiv \inf_{a \in \Lambda_0} I_p(a) \tag{1.3}$$

where

$$\begin{aligned} I_p(a) = \|a^+\|_p = \left(\int_0^L |a^+(x)|^p \ \mathrm{d}x \right)^{1/p} \qquad & \forall\ a \in \Lambda_0,\ 1 \leq p < \infty, \\ I_\infty(a) = \sup \text{ess}\ a^+, \qquad & \forall\ a \in \Lambda_0 \end{aligned} \tag{1.4}$$

obtaining an explicit expression for β_p as a function of p and L.

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In the PDE case, we consider the linear problem

$$\left.\begin{aligned} -\Delta u(x) &= a(x)u(x) && x \in \Omega \\ \frac{\partial u}{\partial n}(x) &= 0 && x \in \partial\Omega \end{aligned}\right\} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded and regular domain, $\dfrac{\partial}{\partial n}$ is the outer normal derivative on $\partial\Omega$ and the function $a : \Omega \to \mathbb{R}$ belongs to the set Λ defined as

$$\Lambda = \{a \in L^\infty(\Omega) \setminus \{0\} : \int_\Omega a(x)\ \mathrm{d}x \geq 0 \text{ and (1.5) has nontrivial solutions}\} \tag{1.6}$$

Obviously, the quantity

$$\beta_p \equiv \inf_{a \in \Lambda}\ \|a^+\|_p\ ,\ 1 \leq p \leq \infty \tag{1.7}$$

is well defined and it is a nonnegative real number. The first novelty with respect to the ordinary case is that $\beta_1 = 0$ for each $N \geq 2$. Moreover, if $N = 2$, then $\beta_p > 0$, $\forall\, p \in (1, \infty]$ and if $N \geq 3$, then $\beta_p > 0$ if and only if $p \geq N/2$. Also, for each $N \geq 2$, β_p is attained if $p > N/2$. It seems difficult to obtain explicit expressions for β_p, as a function of p, Ω and N, at least for general domains (see [4], [7] and [8] for the ordinary case). The paper finishes with an application to nonlinear resonant boundary value problems.

2. Ordinary differential equations. In this section we will consider the linear boundary value problem

$$u''(x) + a(x)u(x) = 0, \qquad x \in (0, L), u'(0) = u'(L) = 0,\ a \in \Lambda_0 \tag{2.1}$$

where

$$\Lambda_0 = \{a \in L^1(0, L) \setminus \{0\} : \int_0^L a(x)\ \mathrm{d}x \geq 0 \text{ and (2.1) has nontrivial solutions}\}$$

Here $u \in H^1(0, L)$, the usual Sobolev space. For each p with $1 \leq p \leq \infty$, we can define the quantity

$$\beta_p \equiv \inf_{a \in \Lambda_0}\ \|a\|_p, \tag{2.2}$$

The main result of this section is the following.

THEOREM 2.1 ([4]). *The following statements hold:*

(1) $\beta_1 = \dfrac{4}{L}$, $\beta_\infty = \dfrac{\pi^2}{L^2}$. *The mapping* $[1, \infty) \to \mathbb{R}$, $p \to \beta_p$, *is continuous and* $\lim_{p\to\infty}\ \beta_p = \beta_\infty$. *Moreover, the mapping* $\gamma : [1, \infty) \to \mathbb{R}$, $p \to L^{-1/p}\beta_p$ *is strictly increasing.*

(2) *If* $1 < p < \infty$,

$$\beta_p = \frac{4(p-1)^{1+\frac{1}{p}}}{L^{2-\frac{1}{p}}p(2p-1)^{1/p}} \left(\int_0^{\pi/2} (\sin x)^{-1/p}\ \mathrm{d}x\right)^2 \tag{2.3}$$

(3) β_p *is attained if and only if* $1 < p \leq \infty$. *In this case,* β_p *is attained in a unique element* $a_p \in \Lambda_0$ *which is not a constant function if* $1 < p < \infty$.

Main ideas of the proof for the case $1 < p < \infty$. If $a \in \Lambda_0$ and $u \in H^1(0,L)$ is a nontrivial solution of

$$u''(x) + a(x)u(x) = 0, \qquad x \in (0,L),\ u'(0) = u'(L) = 0, \tag{2.4}$$

then

$$\int_0^L u'v' = \int_0^L auv, \quad \forall\, v \in H^1(0,L).$$

In particular, choosing $v \equiv u$ and $v \equiv 1$, we have respectively

$$\int_0^L u'^2 = \int_0^L au^2, \qquad \int_0^L au = 0 \tag{2.5}$$

Therefore, for each $k \in \mathbb{R}$, we have

$$\begin{aligned}\int_0^L (u+k)'^2 &= \int_0^L u'^2 = \int_0^L au^2 \le \int_0^L au^2 + k^2 \int_0^L a \\ &= \int_0^L au^2 + \int_0^L k^2 a + 2k\int_0^L au = \int_0^L a(u+k)^2\end{aligned}$$

It follows from Hölder inequality

$$\int_0^L (u+k)'^2 \le \|a\|_p \|(u+k)^2\|_{\frac{p}{p-1}}$$

Also, since u is a nonconstant solution of (2.4), $u+k$ is a nontrivial function. Consequently

$$\|a\|_p \ge \frac{\displaystyle\int_0^L (u+k)'^2}{\|(u+k)^2\|_{\frac{p}{p-1}}}, \qquad \forall\, a \in \Lambda_0 \tag{2.6}$$

Previous reasoning motivates the study of an special minimization problem given in the following lemma.

Lemma 2.2. *Assume* $1 < p < \infty$ *and let*

$$X_p = \left\{ u \in H^1(0,L) : \int_0^L |u|^{\frac{2}{p-1}} u = 0 \right\}$$

If $J_p : X_p \setminus \{0\} \to \mathbb{R}$ *is defined by*

$$J_p(u) = \frac{\displaystyle\int_0^L u'^2}{\left(\displaystyle\int_0^L |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}} \tag{2.7}$$

and $m_p \equiv \inf_{X_p \setminus \{0\}} J_p$, m_p *is attained. Moreover, if* $u_p \in X_p \setminus \{0\}$ *is a minimizer, then* u_p *satisfies the problem*

$$u_p''(x) + A_p(u_p)|u_p(x)|^{\frac{2}{p-1}} u_p(x) = 0, \qquad u_p'(0) = u_p'(L) = 0, \tag{2.8}$$

$$\text{where} \quad A_p(u_p) = m_p \left(\int_0^L |u_p|^{\frac{2p}{p-1}} \right)^{\frac{-1}{p}}. \tag{2.9}$$

Sketch of the proof of the lemma. If $\{u_n\} \subset X_p \backslash \{0\}$ is a minimizing sequence, the sequence $\{k_n u_n\},\ k_n \neq 0$, is also a minimizing sequence. Therefore, we can assume without loos of generality that $\int_0^L |u_n|^{\frac{2p}{p-1}} = 1$. Then $\{\int_0^L |u_n'^2|\}$ is also bounded. Moreover, for each u_n there is $x_n \in (0, L)$ such that $u_n(x_n) = 0$. Therefore, $\{u_n\}$ is bounded in $H^1(0, L)$. So, we can suppose, up to a subsequence, that $u_n \rightharpoonup u_0$ in $H^1(0, L)$ and $u_n \to u_0$ in $C[0, L]$ (with the uniform norm). The strong convergence in $C[0, L]$ gives us $\int_0^L |u_0|^{\frac{2p}{p-1}} = 1$ and $u_0 \in X_p \setminus \{0\}$. The weak convergence in $H^1(0, L)$ implies $J_p(u_0) \leq \liminf\ J_p(u_n) = m_p$. Then u_0 is a minimizer. Lagrange multiplier Theorem implies (2.8)

Now, we continue with the proof of the Theorem for the case $1 < p < \infty$. Remember that we have obtained that if $a \in \Lambda_0$ and $u \in H^1(0, L)$ is a nontrivial solution of (2.4) then we have (2.6). Then, if for each $a \in \Lambda_0$ and each u, nontrivial solution of previous problem, we choose $k_0 \in \mathbb{R}$ satisfying $u + k_0 \in X_p$ we deduce $\beta_p \geq m_p$.

Reciprocally, if $u_p \in X_p \setminus \{0\}$ is any minimizer of J_p, then u_p satisfies (2.8) and (2.9). Therefore, $A_p(u_p)|u_p|^{\frac{2}{p-1}} \in \Lambda_0$. Also,

$$\| \ A_p(u_p)|u_p|^{\frac{2}{p-1}} \ \|_p = m_p.$$

Then $\beta_p \leq m_p$. The conclusion is that $\beta_p = m_p$.

The calculus of m_p is a very delicate matter and it is based on some explicit formulas of the solutions of (2.8).

The details for the proof of the last part of the Theorem may be seen in [4] and also the cases $p = 1$ and $p = \infty$.

As an application of Theorem 2.1 to the linear problem

$$u''(x) + a(x)u(x) = f(x), \qquad x \in (0, L),\ u'(0) = u'(L) = 0, \tag{2.10}$$

we have the following corollary, which clearly generalizes in [10, Theorem 3].

Corollary 2.3. *Let $a \in L^\infty(0, L) \setminus \{0\},\ 0 \leq \displaystyle\int_0^L a(x)$, satisfying one of the following conditions:*

(1) $\|a\|_1 \leq \beta_1$,

(2) *There is some $p \in (1, \infty)$ such that $\|a\|_p < \beta_p$ or $\|a\|_p = \beta_p$ and $a \neq a_p$.*

(3) $\|a\|_\infty < \beta_\infty$ *or* $\|a\|_\infty = \beta_\infty$ *and* $a \neq a_\infty$.

Then for each $f \in L^\infty(0, L)$, the boundary value problem (2.10) *has a unique solution.*

3. Partial Differential Equations. This section will be concerned with the linear boundary value problem

$$\left.\begin{aligned} \Delta u(x) + a(x)u(x) = \quad & 0, \qquad x \in \Omega, \\ \frac{\partial u(x)}{\partial n} = \quad & 0, \qquad x \in \partial\Omega \end{aligned}\right\} \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded and regular domain, $\dfrac{\partial}{\partial n}$ is the outer normal derivative on $\partial\Omega$ and $a \in \Lambda$, where

$$\Lambda = \{\ a \in L^\infty(\Omega) \setminus \{0\} : \int_\Omega a(x)\ \ dx \geq 0 \text{ and (3.1) has nontrivial solutions}\}$$

As in the ordinary case, the positive eigenvalues of the eigenvalue problem

$$\left.\begin{aligned} \Delta u(x) + \lambda u(x) &= 0 \qquad x \in \Omega \\ \frac{\partial u(x)}{\partial n} &= 0 \qquad x \in \partial\Omega \end{aligned}\right\} \tag{3.2}$$

belong to Λ. Therefore, the quantity

$$\beta_p \equiv \inf_{a \in \Lambda} \|a\|_p \ , \ 1 \leq p \leq \infty$$

is well defined. The main result is the next theorem.

Theorem 3.1 ([5]). *The following statements hold:*

(1) $\beta_1 = 0$, *and* $\beta_\infty = \lambda_1$, $\forall\ N \geq 2$. *Here* λ_1 *is the first positive eigenvalue of the eigenvalue problem (3.2).*

(2) *If* $N = 2$, $\beta_p > 0$, $\forall p \in (1, \infty]$.
If $N \geq 3$, $\beta_p > 0 \Leftrightarrow p \in [\frac{N}{2}, \infty]$
If $N \geq 2$ *and* $\frac{N}{2} < p \leq \infty$ *then* β_p *is attained.*

(3) *The mapping* $(\frac{N}{2}, \infty) \to \mathbb{R}$, $p \mapsto \beta_p$, *is continuous and the mapping* $[\frac{N}{2}, \infty] \to \mathbb{R}$, $p \mapsto |\Omega|^{-1/p}\beta_p$, *is strictly increasing.*

Proof. The main ideas are the following:

1. If $N \geq 3$ and $\frac{N}{2} < p < \infty$, the ideas are the same as in the ordinary case. In fact, since $\frac{N}{2} < p$, then $\dfrac{2p}{p-1} < \dfrac{2N}{N-2}$ and consequently, the imbedding of the Sobolev space $H^1(\Omega)$ into $L^{2p/p-1}(\Omega)$ is compact.

2. If $N = 2$, the imbedding $H^1(\Omega) \subset L^q(\Omega)$ is compact $\forall\ q \in [1, \infty)$ and therefore, if $1 < p < \infty$, the ideas are the same as in the ordinary case.

3. If $N \geq 3$ and $1 \leq p < \frac{N}{2}$, we prove that $\beta_p = 0$ by finding appropriate minimizing sequences. Roughly speaking, the main idea is to take first a function u and to calculate the corresponding function a for which u is a solution of

$$\Delta u(x) + a(x)u(x) = 0, x \in \Omega; \qquad \frac{\partial u(x)}{\partial n} = 0, \quad x \in \partial\Omega$$

Obviously, if u is smooth enough, then we must impose two conditions: i) $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, ii) The zeros of u are also zeros of Δu. For instance, if $\Omega = B(0,1)$ we can take radial functions $u(x) = f(|x|)$ of the form $f(r) = \alpha r^{-a} - \beta r^{-b}$, $(a > 0,\ b > 0, 0 < r < 1)$.
If $N = 2$ and $p = 1$, we use the fundamental solution $\ln|x|$ to find appropriate minimizing sequences.

4. If $N \geq 3$ and $p = \frac{N}{2}$, then $\frac{2p}{p-1} = \frac{2N}{N-2}$ and the imbedding $H^1(\Omega) \subset L^{2N/N-2}(\Omega)$ is continuous but not compact. This implies that the infimum $\beta_p > 0$, but we do not know if β_p is a minimum.

□

Remark 1. It is possible to obtain an improvement of previous theorems by considering the positive part a^+ of a function $a \in \Lambda$. Specifically, if we define

$$\beta_p^+ \equiv \inf_{a \in \Lambda} \|a^+\|_p,\ 1 \leq p \leq \infty \tag{3.3}$$

it is easily seen that $\beta_p^+ = \beta_p$.

Remark 2. In the definition of the set Λ we have imposed $\int_\Omega a \geq 0$. This is not a technical but a natural assumption for Neumann boundary conditions. Otherwise, the corresponding infimum will be always zero. To see this, note that if $u \in H^1(\Omega)$ is a positive nonconstant solution of (1.5) and we consider $v = \frac{1}{u}$ as test function in the weak formulation, we obtain

$$\int_\Omega \nabla u \cdot \nabla \left(\frac{1}{u}\right) = \int_\Omega a\, u \frac{1}{u},$$

which implies

$$\int_\Omega a = -\int_\Omega \frac{|\nabla u|^2}{u^2} < 0.$$

With this in mind, if we take a nonconstant $u_0 \in C^2(\overline{\Omega})$ such that $\frac{\partial u_0}{\partial n}(x) = 0$, $\forall x \in \partial\Omega$ then, for large $n \in \mathbb{N}$, we have that $u_n = u_0 + n$ is a positive nonconstant solution of (1.5), with $a_n = \frac{-\Delta u_0}{u_0 + n}$. Clearly $\|a_n\|_p \to 0$ as $n \to \infty$ for every $1 \le p \le \infty$ and, as we have seen before, $\int_\Omega a_n < 0$.

Remark 3. We have considered Neumann boundary conditions. In the case of Dirichlet conditions it is possible to obtain analogous results in an easier way. To be more precise, consider the linear problem

$$\left.\begin{array}{ll} -\Delta u(x) = a(x)u(x) & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{array}\right\} \tag{3.4}$$

where $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded and regular domain and the function $a : \Omega \to \mathbb{R}$ belongs to the set Λ_D defined as

$$\Lambda_D = \{a \in L^\infty(\Omega) \text{ s. t. } (3.4) \text{ has nontrivial solutions}\}$$

Then, we can define the value $\beta_p^D \equiv \inf_{a \in \Lambda_D} \|a\|_p$, $1 \le p \le \infty$ and it is possible to prove that all the assertions of Theorem 3.1 remain true if we replaced β_p by β_p^D and Neumann boundary conditions of (3.1) by Dirichlet conditions.

In fact, as the Neumann case, it is possible to obtain a variational characterization of β_p^D for $N/2 < p < \infty$:

$$\beta_p^D = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\left(\displaystyle\int_\Omega |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}$$

If Ω is, moreover, a radial domain, previous minimization problem is related to a more general one which involves Rayleigh quotient

$$\frac{\displaystyle\int_\Omega |\nabla u|^2}{\left(\displaystyle\int_\Omega \rho(x)|u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}$$

where $\rho \in L^q(\Omega)$, $q = N(p-1)/(2p-N)$, is a positive function. This has been used in the study of the existence of nonsymmetric ground states of symmetric problems for nonlinear PDE's (see [1], [2] and [14]).

4. Nonlinear resonant problems. In this section we apply the previous results (on linear problems) to nonlinear boundary value problems of the form

$$\left.\begin{aligned} -\Delta u(x) &= f(x,u(x)) && x\in\Omega \\ \frac{\partial u}{\partial n}(x) &= 0, && x\in\partial\Omega \end{aligned}\right\} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded and regular domain and the function $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$, satisfies the condition

$$f,\ f_u \text{ are Caratheodory functions and } 0 \leq f_u(x,u) \text{ in } \overline{\Omega}\times\mathbb{R}. \tag{\textbf{H}}$$

The existence of a solution of (4.1) implies

$$\int_\Omega f(x,s_0)\ \mathrm{d}x = 0 \tag{4.2}$$

for some $s_0 \in \mathbb{R}$ (see [13]). Trivially, conditions **(H)** and (4.2) are not sufficient for the existence of solutions of (4.1). Indeed, consider the problem

$$\left.\begin{aligned} -\Delta u(x) &= \lambda_1 u(x) + \varphi_1(x) && x\in\Omega \\ \frac{\partial u}{\partial n}(x) &= 0, && x\in\partial\Omega \end{aligned}\right\} \tag{4.3}$$

where φ_1 is a nontrivial eigenfunction associated to λ_1. Here λ_1 is the first positive eigenvalue of the eigenvalue problem (3.2). The function $f(x,u) = \lambda_1 u + \varphi_1(x)$ satisfies **(H)** and (4.2), but the Fredholm alternative theorem shows that there is no solution of (4.3).

If, moreover of **(H)** and (4.2), f satisfies a non-uniform non-resonance condition of the type

$$\begin{gathered} f_u(x,u) \leq \beta(x) \text{ in } \overline{\Omega}\times\mathbb{R} \text{ with } \beta(x) \leq \lambda_1 \text{in } \Omega \text{ and } \beta(x) < \lambda_1 \\ \text{in a subset of } \Omega \text{ of positive measure,} \end{gathered} \tag{\textbf{h1}}$$

then it has been proved in [13] that (4.1) has solution. Let us observe that supplementary condition **(h1)** is given in terms of $\|\beta\|_\infty$. In the next result, we provide new supplementary conditions in terms of $\|\beta\|_p$, where $N/2 < p \leq \infty$, obtaining a generalization of [13, Theorem 2]. In the proof, the basic idea is to combine the results obtained in the previous section with the Schauder's fixed point theorem.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded and regular domain and $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$, satisfying:*

(1) *f, f_u are Caratheodory functions and $f(\cdot,0) \in L^\infty(\Omega)$*

(2) *There exists a function $\beta \in L^\infty(\Omega)$, satisfying*

$$0 \leq f_u(x,u) \leq \beta(x) \ \text{ in } \overline{\Omega}\times\mathbb{R} \tag{4.4}$$

and such that for some p, $N/2 < p \leq \infty$, we have $\|\beta\|_p < \beta_p$ (or $\|\beta\|_p = \beta_p$ and $\beta(x)$ is not a minimizer of the L_p-norm in Λ), where β_p is given by THEOREM 3.1.

(3)

$$\exists s_0 \in \mathbb{R} \ s.t. \int_\Omega f(x,s_0)\ \mathrm{d}x = 0, \quad \text{and} \quad f_u(x,u(x)) \not\equiv 0, \quad \forall u \in C(\overline{\Omega}). \tag{4.5}$$

Then problem (4.1) *has a unique solution.*

Proof. We first prove uniqueness. Let u_1 and u_2 be two solutions of (4.1). Then, the function $u = u_1 - u_2$ is a solution of the problem

$$-\Delta u(x) = a(x)u(x), \quad x \in \Omega, \ \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega \tag{4.6}$$

where $a(x) = \displaystyle\int_0^1 f_u(x, u_2(x) + \theta u(x))\, d\theta$. Hence $0 \leq a(x) \leq \beta(x)$ and we deduce $a(x) \in \Lambda$ and $\|a\|_p \leq \|\beta\|_p$. Applying THEOREM 3.1, we obtain $u \equiv 0$.

Next we prove existence. First, we write (4.1) in the equivalent form

$$\left.\begin{array}{ll} -\Delta u(x) = b(x, u(x))u(x) + f(x, 0), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{array}\right\} \tag{4.7}$$

where the function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by $b(x,z) = \int_0^1 f_u(x, \theta z) \ \mathrm{d}\theta$. If $X = C(\overline{\Omega})$, our hypotheses allow to define an operator $T : X \to X$, by $Ty = u_y$, being u_y the unique solution of the linear problem

$$\left.\begin{array}{ll} -\Delta u(x) = b(x, y(x))u(x) + f(x, 0), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{array}\right\} \tag{4.8}$$

where $X = C(\overline{\Omega})$ with the uniform norm.

If in X we consider the uniform norm, we will show that T is completely continuous and that $T(X)$ is bounded. The Schauder's fixed point theorem provides a fixed point for T which is a solution of (4.1).

The fact that T is completely continuous is a consequence of the compact embedding $W^{2,p}(\Omega) \subset C(\overline{\Omega})$. It remains to prove that $T(X)$ is bounded. Suppose, contrary to our claim, that $T(X)$ is not bounded. In this case, there would exist a sequence $\{y_n\} \subset X$ such that $\|u_{y_n}\|_X \to \infty$. Passing to a subsequence if necessary, we may assume that the sequence of functions $\{b(\cdot, y_n(\cdot))\}$ is weakly convergent in $L^p(\Omega)$ to a function a_0 satisfying $0 \leq a_0(x) \leq \beta(x)$, a.e. in Ω. If $z_n \equiv \dfrac{u_{y_n}}{\|u_{y_n}\|_X}$, passing to a subsequence if necessary, we may assume that $z_n \to z_0$ strongly in X (we have used again the compact embedding $W^{2,p}(\Omega) \subset C(\overline{\Omega})$), where z_0 is a nonzero function satisfying

$$\left.\begin{array}{ll} -\Delta z_0(x) = a_0(x) z_0(x), & \text{in } \Omega, \\ \dfrac{\partial z_0}{\partial n} = 0, & \text{on } \partial\Omega \end{array}\right\} \tag{4.9}$$

Trivially, there is no loss of generality if we suppose $s_0 = 0$. (Otherwise, we can do the change of variables $u(x) = v(x) + s_0$ and obtain a similar problem with the same original hypothesis). Then for every $n \in \mathbb{N}$,

$$\int_\Omega b(x, y_n(x)) u_{y_n}(x) \ \mathrm{d}x = -\int_\Omega f(x, 0) \ \mathrm{d}x = 0.$$

Therefore, for each $n \in \mathbb{N}$, the function u_{y_n} has a zero in $\overline{\Omega}$ and hence so does z_0. Thus, $a_0 \not\equiv 0$, $a_0 \in \Lambda$ and we obtain a contradiction with THEOREM 3.1 □

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