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SADDLE POINT THEOREM AND FREDHOLM ALTERNATIVE*

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Abstract. Let the operator $A : H \to H$ be linear, compact, symmetric and positive on the separable Hilbert space H. In this paper we prove that the Fredholm alternative for such an operator is a consequence of the Saddle Point Theorem.

Key words. Fredholm alternative, Saddle Point Theorem

AMS subject classifications. 35A05, 35A15, 49R50

1. Motivation. This article was inspired by a result in [4] where we deal with the nonlinear problem

$$\begin{gathered} u''(x) + m^2 u(x) + g(x, u(x)) = \tilde{f}(x), \qquad x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.1}$$

where $\tilde{f} \in L^1(0, \pi)$, $m \in \mathbb{N}$ and $g : (0, \pi) \times \mathbb{R} \to \mathbb{R}$ is a bounded, Caratheodory type function, i.e. $g(\cdot, s)$ is measurable for all $s \in \mathbb{R}$ and $g(x, \cdot)$ is continuous for a.e. $x \in (0, \pi)$. We define

$$G(x, s) = \int_0^s g(x, t)\ \mathrm{d}t$$

and

$$G_+(x) = \liminf_{s \to +\infty} \frac{G(x, s)}{s}, \quad G_-(x) = \limsup_{s \to -\infty} \frac{G(x, s)}{s}.$$

If we assume that the following potential Landesman-Lazer type condition holds

$$\begin{aligned} &\int_0^\pi \left[G_-(x)(\sin mx)^+ G_+(x)(\sin mx)^- \right]\ \mathrm{d}x \\ &\qquad < \int_0^\pi \tilde{f}(x) \sin mx\ \mathrm{d}x < \int_0^\pi \left[G_+(x)(\sin mx)^+ - G_-(x)(\sin mx)^- \right]\ \mathrm{d}x \end{aligned} \tag{1.2}$$

then using the Saddle Point Theorem we have proved in [4] that the problem (1.1) has at least one solution.

But for $g \equiv 0$ there is no function $\tilde{f}$ satisfying conditions (1.2). In this linear case

$$\begin{gathered} u''(x) + m^2 u(x) = \tilde{f}(x), \qquad x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.3}$$

we usually use a Fredholm alternative (see below) to prove the existence of solution of the problem (1.3). In this paper we obtain the existence result for the equation (1.3) using the Saddle Point Theorem and we prove that the Fredholm alternative is a consequence of the Saddle Point Theorem.

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2. Preliminaries. We set $H = W_0^{1,2}(0,\pi)$ is the Sobolev space of absolutely continuous functions $u\colon (0,\pi) \to \mathbb{R}$ such that $u' \in L^2(0,\pi)$, $u(0) = u(\pi) = 0$. We denote $(.,.)$ a scalar product on the Hilbert space H and for $u, v \in H$ we define an operator $A : H \to H$ and an element $f \in H$ by

$$(u,v) = \int_0^\pi u'v' \,\mathrm{d}x\,, \quad (Au,v) = \int_0^\pi uv \,\mathrm{d}x\,, \quad (f,v) = -\tfrac{1}{m^2}\int_0^\pi \tilde{f}v \,\mathrm{d}x\,.$$

We remark that the operator A is a linear, symmetric, positive and compact operator (with respect to the compact imbedding H into $L^2(0,\pi)$).

Now we can write a weak formulation of the equation (1.3), i.e.

$$\int_0^\pi u'v' \,\mathrm{d}x - \int_0^\pi m^2 uv \,\mathrm{d}x = -\int_0^\pi \tilde{f}v \,\mathrm{d}x \qquad \forall v \in H$$

in the form

$$\frac{1}{m^2}(u,v) - (Au,v) = (f,v)\,.$$

We put $\lambda = \frac{1}{m^2}$ and we have the equation

$$\lambda u - Au = f \tag{2.1}$$

in the Hilbert space H, and for $f \equiv 0$

$$\lambda u - Au = 0\,. \tag{2.2}$$

We denote $H_\lambda = \{u \in H \ :\ u \text{ is a solution of (2.2)}\}$.

A number λ is called an *eigenvalue* of the operator $A : H \to H$ if there exists $\varphi_\lambda \neq 0$ such that $\lambda\varphi_\lambda - A\varphi_\lambda = 0$. Such an element $\varphi_\lambda \in H$ is called an *eigenvector* of A.

We formulate the Fredholm alternative for equation (2.1) which is proved in Zeidler [5].

Theorem 2.1. **(Fredholm alternative)**
Let the operator $A\colon H \to H$ be linear, compact and self-adjoint on the separable Hilbert space H. Let $f \in H$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then the problem (2.1) has a solution iff

$$(f\,,\,u) = 0 \qquad \textit{for all} \quad u \in H_\lambda\,. \tag{2.3}$$

To use the Saddle Point Theorem instead of the Fredholm alternative we define a functional $F : H \to H$

$$F(u) = \tfrac{1}{2}\big(\lambda(u,u) - (Au,u)\big) - (f,u)$$

and we find a critical point u_0 of the functional F, i.e. $F'(u_0) = 0$ (here F' is a Frechet derivative of F). We have for all $v \in H$

$$(F'(u_0),v) = \lambda(u_0,v) - (Au_0,v) - (f,v) = 0 \ \Rightarrow \lambda(u_0,v) - (Au_0,v) = (f,v)\,.$$

We see that the critical point u_0 is also a weak solution of (1.3) and vice versa.

We say that F satisfies the *Palais-Smale condition* (PS) if every sequence $\{u_n\}$ for which $F(u_n)$ is bounded in H and $F'(u_n) \to 0$ (as $n \to \infty$) possesses a convergent subsequence.

We use the Saddle Point Theorem which is proved in Rabinowitz [3].

THEOREM 2.2. **(Saddle Point Theorem)**
Let $H = \widehat{H} \oplus \widetilde{H}$, $\dim \widehat{H} < \infty$ and $\dim \widetilde{H} = \infty$. Let $F : H \to \mathbb{R}$ be a functional such that $F \in C^1(H, \mathbb{R})$ and

(a) *there exists a bounded neighborhood D of 0 in $\widehat{H}$ and a constant α such that $F/\partial D \le \alpha$,*
(b) *there is a constant $\beta > \alpha$ such that $F/\widetilde{H} \ge \beta$.*
(c) *F satisfies the Palais-Smale Condition (PS).*

Then the functional F has a critical point in H.

The following theorem is proved in Dunford-Schwartz [1].

THEOREM 2.3. **(Courant-Fisher Principle)**
Let the operator $A : H \to H$ be linear, compact, self-adjoint and positive operator on a real separable Hilbert space H. Then all eigenvalues of A are positive reals and there exists an orthonormal basis of H which consists of eigenvectors of A. Moreover, if

$$\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_n \ge \cdots > 0$$

denote the eigenvalues of A repeated according to (finite) multiplicity and $\varphi_1, \varphi_2, \ldots, \ldots, \varphi_n, \ldots$ corresponding eigenvectors, then

$$\lambda_1 = \max_{u \in H} \frac{(Au\,,\,u)}{\|u\|^2}\,,$$

$$\lambda_{n+1} = \max_{u \in H} \left\{ \frac{(Au\,,\,u)}{\|u\|^2} \;:\; (u, \varphi_1) = \cdots = (u, \varphi_n) = 0 \right\}$$

where $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the previously obtained eigenvectors and $\lim\limits_{n\to\infty} \lambda_n = 0$.

3. Main result.

THEOREM 3.1. *Let the operator $A : H \to H$ be a linear, compact, symmetric and positive on the separable Hilbert space H and let $\lambda \in \mathbb{R}$, $\lambda_1 > \lambda > 0$. Then the Fredholm alternative for the operator A and the equation* (2.1) *is a consequence of the Saddle Point Theorem.*

Proof. Let φ_n be an eigenvector corresponding to the eigenvalue λ_n of A. We know from THEOREM 2.3 that the sequence $\{\varphi_n\}$ creates an orthonormal basis of H. We denote by $\widehat{H}$ the subspace of H spanned by the eigenvectors $\varphi_1, \ldots, \varphi_m$ corresponding to all eigenvalues $\lambda_1 \ge \ldots \ge \lambda_m > \lambda$ and by $\widetilde{H}$ the subspace of H spanned by the eigenvectors φ_n corresponding to all eigenvalues $\lambda_n < \lambda$. Then $H = \widehat{H} \oplus H_\lambda \oplus \widetilde{H}$ and $\dim \widehat{H} < \infty$, $\dim \widetilde{H} = \infty$. Let $\lambda_m = \min\{\lambda_n : \lambda_n > \lambda\}$ and $\lambda_M = \max\{\lambda_n : \lambda_n < \lambda\}$ ($\lambda_m > 0, \lambda_M > 0$). Then it follows from THEOREM 2.3

$$(Au, u) \ge \lambda_m \|u\|^2 \qquad \text{for} \quad u \in \widehat{H}\,, \tag{3.1}$$

$$(Au, u) \le \lambda_M \|u\|^2 \qquad \text{for} \quad u \in \widetilde{H}\,. \tag{3.2}$$

We will verify the assumptions of the Saddle Point THEOREM 2.2 on the space $H_\lambda^\perp = \widehat{H} \oplus \widetilde{H}$. We suppose that $f \in H_\lambda^\perp$ (i.e. $(f, u) = 0$ for all $u \in H_\lambda$ and f satisfies condition (2.3)).

(a) We take an arbitrary α and prove that there exists a bounded neighborhood D of 0 in $\widehat{H}$ such that $F/\partial D \leq \alpha$.
The inequality (3.1) yields

$$F(u) = \frac{1}{2}\big(\lambda(u,u) - (Au,u)\big) - (f,u) \leq \frac{1}{2}(\lambda - \lambda_m)\|u\|^2 + \|f\|\|u\| \leq \alpha$$

provided $\|u\|$ is sufficiently large since $\lambda < \lambda_m$.

(b) For $u \in \widetilde{H}$ we use the inequality (3.2) and we obtain

$$\begin{aligned} F(u) &= \frac{1}{2}\Big(\lambda\|u\|^2 - (Au,u)\Big) - (f,u) \\ &\geq \frac{1}{2}(\lambda - \lambda_M)\Big(\|u\|^2 - \frac{2}{(\lambda-\lambda_M)}\,\|f\|\,\|u\|\Big) \geq -\frac{\|f\|^2}{2\,(\lambda-\lambda_M)}\,. \end{aligned}$$

Hence it follows that there are constants $\alpha,\ \beta,\ \beta > \alpha$ and a bounded neighborhood D of 0 such that $F/\widetilde{H} \geq \beta$ and $F/\partial D \leq \alpha$.

(c) We prove that F satisfies the Palais-Smale Condition (PS).
We take a sequence $(u_n) \subset H_\lambda^\perp$ and suppose that there exists a constant c_1 such that

$$\big|F(u_n)\big| \leq c_1 \tag{3.3}$$

and

$$\lim_{n\to\infty} \|F'(u_n)\| = 0\,. \tag{3.4}$$

Because $\lambda \notin \sigma(A|_{H_\lambda^\perp})$ (λ is not an eigenvalue of A on $H_\lambda^\perp$) then there exists a constant $c_2 > 0$ such that

$$\|\lambda u - Au\| \geq c_2\|u\| \qquad \forall\, u \in H_\lambda^\perp \tag{3.5}$$

Suppose by contradiction that such c does not exist, i.e. there are $u_n \in H_\lambda^\perp$ such that

$$\|u_n\| = 1 \qquad \text{and} \qquad \|\lambda u_n - Au_n\| < \frac{1}{n}\|u_n\|\,.$$

Then one can find a subsequence (u_{n_k}) for which Au_{n_k} converges to some y (A is the compact operator).
Since $\lambda u_{n_k} - Au_{n_k} \to o$, $\lambda u_{n_k} \to y \in H_\lambda^\perp$ and $\|y\| = \lambda > 0$, there

$$(\lambda I - A)\lambda u_{n_k} \to (\lambda I - A)y\,, \text{ i.e. } y \in H_\lambda \cap H_\lambda^\perp\,, \text{ and thus } y = 0\,.$$

This is a contradiction because $\|y\| \neq 0$.

We use (3.4), (3.5) and we have

$$0 \leftarrow \|F'(u_n)\| = \|\lambda u_n - Au_n - f\| \geq \|\lambda u_n - Au_n\| - \|f\| \geq c_2\|u_n\| - \|f\|.$$

This implies that the sequence $(u_n) \subset H_\lambda^\perp$ is bounded. Then there exists $u_0 \in H_\lambda^\perp$ such that $u_n \rightharpoonup u_0$ in $H_\lambda^\perp$ (taking a subsequence if it is necessary) and

$$0 \leftarrow \lambda u_n - Au_n - f \rightharpoonup \lambda u_0 - Au_0 - f\,.$$

It yields

$$\begin{aligned} &\lambda u_n - Au_n - f \to \lambda u_0 - Au_0 - f \\ \Longrightarrow\quad &(\lambda I - A)(u_n - u_0) \to 0 \quad \overset{(3.5)}{\Longrightarrow} \quad u_n \to u_0 \in H_\lambda^\perp . \end{aligned}$$

This shows that F satisfies the Palais-Smale condition and the assumptions of the Saddle Point THEOREM 2.2 on $H_\lambda^\perp$.

It implies that there is a critical point $u_0 \in H_\lambda^\perp$ of F, i.e. $F'(u_0) = 0$.

Now we prove that u_0 is also a solution of (2.1). We can write an arbitrary $v \in H$ in the form $v = w + \varphi_\lambda$, where $w \in H_\lambda^\perp$, $\varphi_\lambda \in H_\lambda$. We use that A is self-adjoint and for all $v \in H$ we have

$$\begin{aligned} (F'(u_0), v) &= (F'(u_0), w) + (\lambda u_0 - Au_0, \varphi_\lambda) + (f, \varphi_\lambda) \\ &= (u_0, \lambda\varphi_\lambda - A\varphi_\lambda) + (f, \varphi_\lambda) = (f, \varphi_\lambda) \end{aligned} \tag{3.6}$$

We see that u_0 is a critical point of F and also a solution of (2.1) iff $(f, \varphi_\lambda) = 0$ for all solutions φ_λ of (2.2), i.e. we prove the Fredholm alternative for operator A. □

4. Fredholm alternative for nonlinear equation. A weak formulation of the equation (1.1) is

$$\int_0^\pi u'v' \,\mathrm{d}x - \int_0^\pi m^2 uv \,\mathrm{d}x - \int_0^\pi g(x,u)v \,\mathrm{d}x = -\int_0^\pi \tilde{f} v \,\mathrm{d}x \quad \forall v \in H$$

and the corresponding functional $F:\ H \to H$ is

$$F(u) = \frac{1}{2}\int_0^\pi [(u')^2 - m^2u^2] \,\mathrm{d}x - \int_0^\pi [G(x,u) - \tilde{f} v] \,\mathrm{d}x .$$

We set $(S(u), v) = \frac{1}{m^2}\int_0^\pi g(x,u)v \,\mathrm{d}x$ then $S: H \to H$ is a continuous compact operator and we investigate the equation

$$F'(u) = \lambda u - Au - S(u) - f = 0 \tag{4.1}$$

We add an assumptions to nonlinear operator S.
We will suppose that there is $\alpha \in (0,1)$ such that

$$|g(x,s)| \le c_3 |s|^\alpha \quad \text{for a.e. } x \in (0,\pi) \text{ and for all } s \in \mathbb{R} \tag{4.2}$$

with $c_3 > 0$. Then it holds

$$\|S(u)\| \le c_4 \|u\|^\alpha \quad \text{with} \quad c_4 > 0 \tag{4.3}$$

and

$$\lim_{|s|\to\infty} \frac{G(x,s)}{s^2} \to 0 . \tag{4.4}$$

We denote $Tu = \lambda u - Au$ and we formulate the nonlinear Fredholm alternative (see [2])

THEOREM 4.1. *Let the equation $Tu = o$ have only a trivial solution and suppose $g(x,s)$ satisfies (4.2). Then for each f the equation $Tu - S(u) = f$ has at least one solution.*

Proof. We verify that the functional F satisfies the assumptions of the Saddle Point THEOREM 2.2.

(a) For $u \in \widehat{H}$ we use the inequalities (3.1). We take an arbitrary α and the assumptions (4.2), (4.4) yield

$$\begin{aligned} F(u) &= \frac{1}{2}\big(\lambda(u,u) - (Au,u)\big) - \int_0^\pi G(x,u)\,\mathrm{d}x - (f,u) \\ &\le \frac{1}{2}(\lambda - \lambda_m)\|u\|^2 + \int_0^\pi \frac{G(x,u)}{u^2}\frac{u^2}{\|u\|^2}\,\mathrm{d}x + \|f\|\|u\| \le \alpha \end{aligned}$$

provided $\|u\|$ is sufficiently large since $\lambda < \lambda_m$.
We have proved that there exists a bounded neighborhood D of 0 in $\widehat{H}$ such that $F/\partial D \le \alpha$.

(b) For $u \in \widetilde{H}$ we use the inequalities (3.2) and we have

$$\begin{aligned} F(u) &= \frac{1}{2}\Big(\lambda\|u\|^2 - (Au,u)\Big) - \int_0^\pi G(x,u)\,\mathrm{d}x - (f,u) \\ &\ge \|u\|^2\Big(\frac{1}{2}(\lambda - \lambda_M) - \int_0^\pi \frac{G(x,u)}{u^2}\frac{u^2}{\|u\|^2}\,\mathrm{d}x - \frac{\|f\|}{\|u\|}\Big). \end{aligned}$$

Together with (4.2), (4.4) this implies that $F(u) \ge 0$ for sufficiently large $\|u\|$ since $\lambda > \lambda_M$. Hence F is bounded from below.
Then there are constants α, β, $\beta > \alpha$ and a bounded neighborhood D of 0 such that $F/\widetilde{H} \ge \beta$ and $F/\partial D \le \alpha$.

(c) We use (3.4), (3.5), (4.3) and we have

$$\begin{aligned} 0 \leftarrow \|F'(u_n)\| &= \|\lambda u_n - Au_n - S(u_n) - f\| \\ &\ge \|\lambda u_n - Au_n\| - \|S(u_n)\| - \|f\| \ge c_2\|u_n\| - c_4\|u_n\|^\alpha - \|f\|. \end{aligned}$$

Since $\alpha \in (0,1)$ this implies that the sequence $(u_n) \subset H_\lambda^\perp$ is bounded. Then there exists $u_0 \in H_\lambda^\perp$ such that $u_n \rightharpoonup u_0$, $S(u_n) \to S(u_0)$ in $H_\lambda^\perp$ (taking a subsequence if it is necessary) and

$$0 \leftarrow \lambda u_n - Au_n - S(u_n) - f \rightharpoonup \lambda u_0 - Au_0 - S(u_0) - f.$$

It yields

$$(\lambda I - A)(u_n - u_0) \to 0 \quad \overset{(3.5)}{\Longrightarrow} \quad u_n \to u_0 \in H_\lambda^\perp.$$

This shows that F satisfies the Palais-Smale condition and the assumptions of the Saddle Point THEOREM 2.2 and u_0 is a critical point of F on $H_\lambda^\perp$.

We have that if H_λ is trivial (i. e. $H_\lambda^\perp = H$) then $\lambda u - Au - S(u) = f$ has a solution for each $f \in H$. □

REFERENCES

[1] N. Dunford and J. J. Schwartz, *Linear Operators Part II*, First edition. John Wiley and Sons, New York, 1963.
[2] J. Nečas, *Sur l'alternative de Fredholm pour les oprateurs non-linaires avec applications aux limites*, Ann. Scuola Norm. Sup. Pisa **23** (1969), 331–345.
[3] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. in Math. no 65, Amer. Math. Soc. Providence, RI., (1986).
[4] P. Tomiczek, *The generalization of the Landesman-Lazer conditon*, EJDE, **2001**(04) (2001), 1–11.
[5] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Applied Mathematical Science **108**, Springer Verlag, New York, 1995.