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EXISTENCE AND ASYMPTOTICS FOR SELFDUAL PERIODIC VORTICES OF TOPOLOGICAL-TYPE*

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Abstract. We consider vortices of topological-type for a class of selfdual gauge models, with periodic boundary conditions and as the ratio of the vortex core size to the separation distance between vortex points (the scaling parameter) tends to zero. We use a gluing technique (shadowing lemma) for solutions to the corresponding semilinear elliptic equation on the plane, where the vortex points are periodically arranged. This approach is particularly convenient and natural for the study of the asymptotics as the scaling parameter tends to zero. In particular, we prove a factorization ansatz for multivortex solutions, up to an error which is exponentially small.

Key words. Selfdual vortex theories, elliptic equation, shadowing lemma

AMS subject classifications. 35J60, 58E15, 81T13

1. Introduction.

This paper presents some recent results about the existence of the solutions of topological-type to the following singular elliptic problem

$$-\Delta u=\delta^{-2}q(\mathrm{e}^{u})-4\pi\sum_{j=1}^{s}m_j\delta_{p_j}\qquad \text{in } \Omega \quad \text{and } u \text{ doubly periodic on } \partial\Omega \tag{1.1}$$

where $\Omega=(0,a)\times(0,b)$, the points $p_j\in\Omega$ are called vortex points, $m_j\in\mathbb{N}$ is the multiplicity of the vortex point p_j, δ_{p_j} is the Dirac measure at p_j and $q:[0,+\infty)\to\mathbb{R}$ is smooth and satisfies:

$$q(1)=0,\qquad q'(1)<0; \tag{q1}$$

$$q(t)>0 \qquad \text{for all } t\in(0,1) \tag{q2}$$

$$q(t)<0 \qquad \text{for all } t>1. \tag{q3}$$

The equation (1.1) arises from several self-dual gauge theories as considered, e.g. in the monographs [7], [17] and more recently in [14].

The periodic boundary conditions are justified by certain more general gauge invariant conditions introduced by ‘t Hooft [6]. Such conditions force the magnetic flux through a lattice cell to be a “quantized” value proportional to the number of vortices confined. Namely, the ‘t Hooft boundary conditions imply a topological constraint on the solutions of (1.1), exactly as for finite energy solutions on $\mathbb{R}^2$.

Let us recall some known results on problem (1.1). For $q(t)=1-t$, the equation (1.1) reduces to the equation describing self-dual Abelian Higgs vortices, see [15] for existence and uniqueness of finite energy solutions on $\mathbb{R}^2$ and [16] for the periodic case. In particular, for the self-dual Abelian Higgs vortices we refer to [8], where the asymptotic properties as $\delta\to 0^+$ are analyzed for both the planar and the periodic case.

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For $q(t) = t(1-t)$, the equation (1.1) reduces to the equation describing selfdual Chern-Simons vortices, see [11] for existence of finite energy solutions on $\mathbb{R}^2$ and [2], [12] for the periodic case.

The equation (1.1) for the general nonlinearity q satisfying (q1)–(q2)–(q3) has been considered on $\mathbb{R}^2$ by Han [5] and, by Chen, Hastings, McLeod and Yang [3], for a slightly more general nonlinearity in a radial setting, namely only for a single vortex point. In particular, Han in [5] proved the existence of topological-type solutions on $\mathbb{R}^2$, namely solutions with exponential decay at infinity and satisfying $\delta^{-2}\int_{\mathbb{R}^2} q(\mathrm{e}^u) = 4\pi N$. In the present paper we study the equation (1.1) with periodic boundary conditions and we look for solutions of topological-type, namely $u_\delta \to 0$ as $\delta \to 0^+$ in $\Omega \setminus \{p_j\}_j$.

In general, assuming only (q1)–(q2)–(q3) on the nonlinearity, the equation (1.1) may admit more than a solution with a different asymptotic behavior at infinity for the planar case and as $\delta \to 0^+$ for the periodic case, see for instance [12] for a multiplicity results for the Chern-Simons equation (topological and nontopological type solutions). Moreover, it is still an open problem if there is uniqueness of topological-type solutions assuming only (q1)–(q2)–(q3). Indeed, a uniqueness result is given by Han ([5]) for the planar case under the condition of strict monotonicity on the nonlinearity q, and for a non-monotone nonlinearity, a uniqueness result was given recently by Tarantello ([13]) for periodic solutions of topological-type for the Chern-Simon equations, taking advantage of their variational characterization.

In a non-variational setting we give in a forthcoming paper ([9]) a uniqueness result, for both the planar and the periodic case, for solutions of (1.1) assuming (q1)–(q2)–(q3), and the additional condition that $\delta^{-2}(1-\mathrm{e}^u)$ is bounded in L^1 uniformly w.r.t. to δ.

2. Main result and outline of the proof. In order to state more precisely our result, let us denote by U_N the unique (radial) solution for the single-vortex problem on $\mathbb{R}^2$ (see [5] and [3])

$$-\Delta U_N = q(\mathrm{e}^{U_N}) - 4\pi N\delta_0 \qquad \text{and } U_N \to 0 \qquad \text{as } |x| \to +\infty \tag{2.1}$$

We have the following result:

THEOREM 2.1. *Let us assume* (q1)–(q2)–(q3). *Then there exists $\delta_1 > 0$ such that for every $\delta \in (0,\delta_1)$ there exists a solution u_δ for* (1.1). *Furthermore, u_δ satisfies the approximate superposition rule*

$$u_\delta(x) = \sum_{j=1}^{s} U_{m_j}\left(\frac{|x-p_j|}{\delta}\right) + \omega_\delta, \tag{2.2}$$

where $\|\omega_\delta\|_\infty \le Ce^{-c/\delta}$, for some $C, c > 0$ independent of $\delta > 0$.

Moreover, u_δ satisfies the following properties

(i) $\mathrm{e}^{u_\delta} < 1$ *on Ω and vanishes exactly at p_j with multiplicity $2m_j$, $j = 1,\dots,s$;*

(ii) *For every compact subset K of $\Omega \setminus \cup_{j=1,\dots,s}\{p_j\}$ there exist C, $c > 0$ such that $1-\mathrm{e}^{u_\delta} \le C\mathrm{e}^{-c/\delta}$ as $\delta \to 0^+$;*

(iii) $\delta^{-2}q(\mathrm{e}^{u_\delta}) \to 4\pi\sum_{j=1}^{s} m_j\delta_{p_j}$ *in the sense of distributions, as $\delta \to 0^+$.*

Our starting point in proving THEOREM 2.1 is to consider $\delta > 0$ as a scaling parameter. Setting $\hat{u}(x) = u(\delta x)$, we have that $\hat{u}$ satisfies:

$$-\Delta\hat{u} = q(\mathrm{e}^{\hat{u}}) - 4\pi\sum_{k\in\mathcal{P}} m_k\delta_{\hat{p}_k} \qquad \text{in } \mathbb{R}^2, \tag{2.3}$$

where $\hat{p}_k = p_k/\delta$. Here $\mathcal{P}$ is a countable set, and

$$\{p_k\}_{k\in\mathcal{P}} = \{p_j + ma\underline{e}_1 + nb\underline{e}_2 \ : \ j = 1, \dots, s;\ m, n \in \mathbb{Z}\} \tag{2.4}$$

where $\underline{e}_1, \underline{e}_2$ are the unit vectors in $\mathbb{R}^2$ defining the periodic cell domain Ω.

Note that the vortex points $\hat{p}_k$ "separate" as $\delta \to 0^+$.

We use a "Shadowing-type Lemma" as introduced by Angenent in [1] (see also [10] and [8]) to prove the existence of a solution with the following form

$$\hat{u} = \sum_{j\in\mathcal{P}} \hat{\varphi}_j U_{m_j}(x - \hat{p}_j) + z$$

where $\hat{\varphi}_j$ is a suitable cut-off function around the point $\hat{p}_j$ and z is a fixed point of a suitable contractive map F_δ. Finally, we prove that the solution $u_\delta(x) = \hat{u}(\frac{x}{\delta})$ of (1.1) is in fact a periodic solution with periodic cell domain Ω and satisfies all properties stated in Theorem 2.1.

3. Properties of the single vortex. We collect in the following lemma some properties of the solution U_N of problem (2.1).

Lemma 3.1. *There exists a unique solution U_N of problem* (2.1)*. Moreover U_N is radially symmetric about the origin and satisfies the following properties:*

(i) $\mathrm{e}^{U_N(x)} < 1$ *for any* $x \in \mathbb{R}^2$.

(ii) *For every $r > 0$ there exist constants $C > 0$ and $\alpha > 0$ depending on r and N such that*

$$|1 - \mathrm{e}^{U_N(x)}| + |\nabla U_N(x)| + |U_N(x)| \le C\mathrm{e}^{-\alpha|x|},$$

for all $x \in \mathbb{R}^2 \setminus B_r(0)$.

(iii) $U_N(x) = 2N \ln|x| + O(1)$ *as* $|x| \to 0$.

(iv) *The bounded linear operator $L_N : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by*

$$L_N := -\Delta - q'(e^{U_N})e^{U_N}.$$

is invertible with a bounded inverse.

The existence and the uniqueness of the radial solution of problem (2.1) is proved in [3]. Then, Han in [5] proved that every solution of problem (2.1) is radially symmetric about the origin and consequently it is unique. Moreover in [5] the exponential decay estimate of U_N is given. The complete estimate (ii) follows by the standard elliptic theory (see e.g. [4]). (iii) follows noting that by (2.1) $v \equiv U_N - 2N\ln|x| \in H^2(\mathbb{R}^2)$, and by elliptic regularity $v \in C^\infty(\mathbb{R}^2)$.

Concerning (iv) note that the operator L_N for the Abelian Higgs model is given by $L_N := -\Delta + \mathrm{e}^{U_N}$, hence defines a strictly positive quadratic form which represents the second differential of the Euler functional associated to the variational formulation of the problem. Indeed, whenever the nonlinearity $q(t)$ is strictly monotone, namely $q'(t) < 0$, for $t \in [0, 1]$, it is easy to check that the operator is strictly positive and hence injective. However in general, the term $q'(\mathrm{e}^{U_N})$ may change sign, as for example for the Chern-Simons model. In [13] is proved that for Chern-Simons equation, U_N is a minimum of the associated Euler functional and that the second differential form, namely the quadratic form defined by the operator L_N, is strictly positive, showing that $\lambda_1 = 0$ cannot be the first eigenvalue of L_N. We generalize the proof in [13], for a general nonlinearity q, hence without any variational characterization of U_N, showing that the first eigenvalue of L_N is always strictly positive.

Proof. [LEMMA 3.1(iv)] The operator L_N is of Fredholm-type. Hence, to conclude it is enough to prove that L_N is injective. Indeed, suppose that L_N is not injective, then denoting by λ_1 the first eigenvalue of L_N, we have $\lambda_1 \leq 0$. Let $\varphi \in H^2(\mathbb{R}^2)$ be the first eigenfunction of L_N, then the function φ is positive, radially symmetric and uniquely defined by $\varphi(x, y) = \phi(\sqrt{x^2 + y^2})$ with ϕ satisfying

$$-\phi'' - \frac{1}{r}\phi' - q'(\mathrm{e}^{U_N})\,\mathrm{e}^{U_N}\,\phi = \lambda_1\phi \qquad \text{in } (0, +\infty).$$

We set $\psi(t) = \phi(\mathrm{e}^t) = \phi(r)$ and $V(t) = U_N(\mathrm{e}^t) < 0$. Then $\psi > 0$ satisfies:

$$-\ddot{\psi} = \mathrm{e}^{2t}\,q'(\mathrm{e}^V)\,\mathrm{e}^V\,\psi + \lambda_1\,\mathrm{e}^{2t}\,\psi \qquad \text{in } (-\infty, +\infty) \tag{3.1}$$

and V satisfies

$$-\ddot{V} = \mathrm{e}^{2t}\,q(\mathrm{e}^V). \tag{3.2}$$

By the properties of U_N and (3.1) easily follows

$$\lim_{t\to-\infty} \psi(t) = \phi(0) \qquad \lim_{t\to+\infty} \psi(t) = 0 \qquad \lim_{t\to\pm\infty} \dot{\psi}(t) = 0$$

and by LEMMA 3.1(ii)–(iii) and (3.2) we have

$$\lim_{t\to+\infty} \dot{V}(t) = 0, \qquad \lim_{t\to-\infty} \dot{V}(t) = 2N.$$

Now, in view of the fact that $\lim_{t\to\pm\infty} \dot{\psi}\dot{V} = 0$, multiplying (3.1) by $\dot{V}$ and integrating, we get

$$\int_{-\infty}^{+\infty} \mathrm{e}^{2t}\,\psi(2q(\mathrm{e}^V) - \lambda_1\dot{V})\,dt = \left[\mathrm{e}^{2t}\,q(\mathrm{e}^V)\psi\right]_{-\infty}^{+\infty}. \tag{3.3}$$

For $\lambda_1 \leq 0$, the l.h.s. is finite and strictly positive and $\lim_{t\to\pm\infty} \mathrm{e}^{2t}\,q(\mathrm{e}^V)\psi = 0$, a contradiction. Therefore, we may conclude, in particular, that $L_N\varphi = 0$ implies necessary $\varphi \equiv 0$. ◻

4. The shadowing Lemma. We need a suitable partition of unity. We can use the same introduced in [8], so that we only report the main aspects and we refer to [8] for more details.

There exists a locally finite open covering of $\mathbb{R}^2$, $\{P_j, Q_k\}_{(j,k)\in\mathcal{P}\times\mathcal{Q}}$ ($\mathcal{Q}$ is a countable set of indices) with the property that $P_j \cap P_{j'} = \emptyset$ for every $j' \neq j$. Moreover there exists a partition of unity $\{\varphi_j, \psi_k\}$ associated to $\{P_j, Q_k\}$, such that $\varphi_j(p_j) = 1$, $\operatorname{supp}\varphi_j \subset P_j$ and $\operatorname{supp}\psi_k \subset Q_k$. We also define the rescaled covering: $\hat{P}_j = P_j/\delta, \hat{Q}_k = Q_k/\delta$. Then $\{\hat{\varphi}_j, \hat{\psi}_k\}$ defined by $\hat{\varphi}_j(x) = \varphi_j(\delta x), \hat{\psi}_k(x) = \psi_k(\delta x)$ is a partition of unity associated to $\{\hat{P}_j, \hat{Q}_k\}$. It will also be convenient to define the sets

$$\hat{C}_j = \{x \in \hat{P}_j \,:\, \hat{\varphi}_j(x) = 1\} \qquad j \in \mathcal{P}.$$

Note that $\operatorname{supp}\{\nabla\hat{\varphi}_j, D^2\hat{\varphi}_j\} \subset \hat{P}_j \setminus \hat{C}_j$ and

$$\sup_{(j,k)}\{\|\nabla\hat{\varphi}_j\|_\infty + \|\nabla\hat{\psi}_k\|_\infty\} \leq C\delta, \qquad \sup_{(j,k)}\{\|D^2\hat{\varphi}_j\|_\infty + \|D^2\hat{\psi}_k\|_\infty\} \leq C\delta^2. \tag{4.1}$$

We shall use the following Banach spaces:

$$\hat{X}_\delta = \left\{ u \in H^2_{\mathrm{loc}}(\mathbb{R}^2) \;:\; \|u\|_{\hat{X}_\delta} \equiv \sup_{(j,k)\in\mathcal{P}\times\mathcal{Q}} \left\{ \|\hat{\varphi}_j u\|_{H^2}, \|\hat{\psi}_k u\|_{H^2} \right\} < +\infty \right\},$$
$$\hat{Y}_\delta = \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}^2) \;:\; \|f\|_{\hat{Y}_\delta} \equiv \sup_{(j,k)\in\mathcal{P}\times\mathcal{Q}} \left\{ \|\hat{\varphi}_j f\|_{L^2}, \|\hat{\psi}_k f\|_{L^2} \right\} < +\infty \right\}.$$

For any $j \in \mathcal{P}$, we define

$$\hat{U}_j(x) = U_{m_j}(x - \hat{p}_j)$$

and, for any $R > 0$, $\mathcal{B}_R = \{u \in \hat{X}_\delta \,:\, \|u\|_{\hat{X}_\delta} \le R\}$.

We make the following ansatz for the solutions $\hat{u}$ to equation (2.3):

$$\hat{u} = \sum_{j\in\mathcal{P}} \hat{\varphi}_j \hat{U}_j + z. \tag{4.2}$$

Our goal is to prove:

Proposition 4.1. *There exists $\delta_1 > 0$ such that for all $\delta \in (0, \delta_1)$ there exists unique $z_\delta \in \hat{X}_\delta$ such that $\hat{u}_\delta = \sum_{j\in\mathcal{P}} \hat{\varphi}_j \hat{U}_j + z_\delta$ is a solution to* (2.3)*. Moreover, $\|z_\delta\|_{\hat{X}_\delta} \le C\mathrm{e}^{-c/\delta}$.*

We note that the functional $F_\delta : \hat{X}_\delta \to \hat{Y}_\delta$ given by

$$F_\delta(z) = -\Delta z + \sum_{j\in\mathcal{P}} \hat{\varphi}_j q(\mathrm{e}^{\hat{U}_j}) - q(\mathrm{e}^{\sum_{j\in\mathcal{P}} \hat{\varphi}_j \hat{U}_j + z}) - \sum_{j\in\mathcal{P}} [\hat{\varphi}_j, \Delta]\hat{U}_j$$

is well-defined and C^1. Here $[\Delta, \hat{\varphi}_j] = \Delta\hat{\varphi}_j + 2\nabla\hat{\varphi}_j\nabla$. Moreover, if $z \in \hat{X}_\delta$ satisfies $F_\delta(z) = 0$, then $\hat{u}$ defined by (4.2) is a solution to (2.3).

Lemma 4.2. *For $\delta > 0$ sufficiently small, we have*

$$\|F_\delta(0)\|_{\hat{Y}_\delta} \le C\mathrm{e}^{-c/\delta} \qquad \text{as } \delta \to 0^+$$

for some constants $C, c > 0$ independent of δ.

Proof. Let

$$\mathcal{R} = \sum_{j\in\mathcal{P}} \hat{\varphi}_j q(\mathrm{e}^{\hat{U}_j}) - q(\mathrm{e}^{\sum_{j\in\mathcal{P}} \hat{\varphi}_j \hat{U}_j}), \qquad \mathcal{C} = \sum_{j\in\mathcal{P}} [\hat{\varphi}_j, \Delta]\hat{U}_j.$$

Note that $\{\operatorname{supp}\mathcal{R}, \operatorname{supp}\mathcal{C}\} \subset \cup_{j\in\mathcal{P}} \hat{P}_j \setminus \hat{C}_j$. Then, by Lemma 3.1(ii) we have

$$\|q(\mathrm{e}^{\hat{U}_j})\|_{L^\infty(\mathbb{R}^2\setminus\hat{C}_j)} \le C\|1 - \mathrm{e}^{\hat{U}_j}\|_{L^\infty(\mathbb{R}^2\setminus\hat{C}_j)} \le C\,\mathrm{e}^{-c/\delta}, \tag{4.3}$$

$$\|q(\mathrm{e}^{\hat{\varphi}_j\hat{U}_j})\|_{L^\infty(\mathbb{R}^2\setminus\hat{C}_j)} \le C\|1 - \mathrm{e}^{\hat{U}_j}\|_{L^\infty(\mathbb{R}^2\setminus\hat{C}_j)} \le C\,\mathrm{e}^{-c/\delta}\,. \tag{4.4}$$

Therefore, fixed $x \in \cup_{j\in\mathcal{P}} \hat{P}_j \setminus \hat{C}_j$, from (4.3) and (4.4) we have

$$|\mathcal{R}(x)| \le \sup_{j\in\mathcal{P}} \|\hat{\varphi}_j q(\mathrm{e}^{\hat{U}_j})\|_{L^\infty(\hat{P}_j\setminus\hat{C}_j)} + \sup_{j\in\mathcal{P}} \|q(\mathrm{e}^{\hat{\varphi}_j\hat{U}_j})\|_{L^\infty(\hat{P}_j\setminus\hat{C}_j)} \le C\mathrm{e}^{-c/\delta}.$$

On the other hand, in view of (4.1) and LEMMA 3.1(ii), for $x \in \cup_{j\in\mathcal{P}} \hat{P}_j \setminus \hat{C}_j$, we have

$$|\mathcal{C}(x)| \leq \sup_{j\in\mathcal{P}} \| [\Delta, \hat{\varphi}_j]\hat{U}_j\|_{L^\infty(\hat{P}_j\setminus\hat{C}_j)} \leq C\mathrm{e}^{-c/\delta}.$$

Hence, we conclude that, as $\delta \to 0^+$:

$$\|F_\delta(0)\|_{\hat{Y}_\delta} \leq C \sup_{j\in\mathcal{P}} \Big(\|\mathcal{R}\|_{L^2(\hat{P}_j)} + \|\mathcal{C}\|_{L^2(\hat{P}_j)} \Big) \leq C\mathrm{e}^{-c/\delta}$$

for some constants $C, c > 0$ independent of $\delta > 0$. □

Now, we consider the operator $L_\delta \equiv DF_\delta(0) : \hat{X}_\delta \to \hat{Y}_\delta$ given by

$$L_\delta = -\Delta - q'(\mathrm{e}^{\sum_{j\in\mathcal{P}} \hat{\varphi}_j\hat{U}_j})\mathrm{e}^{\sum_{j\in\mathcal{P}} \hat{\varphi}_j\hat{U}_j}.$$

and the operators $\hat{L}_0 = -\Delta - q'(1)$ and $\hat{L}_j = -\Delta - q'(\mathrm{e}^{\hat{U}_j})\mathrm{e}^{\hat{U}_j}$ for every $j \in \mathcal{P}$.

The following holds:

LEMMA 4.3. *There exist $C, c > 0$ such that for any $u \in \hat{X}_\delta$ we have*

$$\|(L_\delta - \hat{L}_j)\hat{\varphi}_j u\|_{L^2} \leq C\mathrm{e}^{-c/\delta}\|\hat{\varphi}_j u\|_{L^2}, \qquad j \in \mathcal{P},$$
$$\|(L_\delta - \hat{L}_0)\hat{\psi}_k u\|_{L^2} \leq C\mathrm{e}^{-c/\delta}\|\hat{\psi}_k u\|_{L^2}, \qquad k \in \mathcal{Q}.$$

Proof. For any $j \in \mathcal{P}$, by LEMMA 3.1(ii) follows that

$$\|q'(\mathrm{e}^{\hat{U}_j})\mathrm{e}^{\hat{U}_j} - q'(1)\|_{L^\infty(\mathbb{R}^2\setminus\hat{C}_j)} \leq C\|1 - \mathrm{e}^{\hat{U}_j}\|_{L^\infty(\mathbb{R}^2\setminus\hat{C}_j)} \leq Ce^{-c/\delta},$$
$$\|q'(\mathrm{e}^{\hat{\varphi}_j\hat{U}_j})\mathrm{e}^{\hat{\varphi}_j\hat{U}_j} - q'(1)\|_{L^\infty(\mathbb{R}^2\setminus\hat{C}_j)} \leq C\|1 - \mathrm{e}^{\hat{U}_j}\|_{L^\infty(\mathbb{R}^2\setminus\hat{C}_j)} \leq Ce^{-c/\delta}.$$

Therefore we have, as $\delta \to 0^+$

$$\begin{aligned}\|(L_\delta - \hat{L}_j)\hat{\varphi}_j u\|_{L^2} =& \|(q'(\mathrm{e}^{\hat{\varphi}_j\hat{U}_j})\mathrm{e}^{\hat{\varphi}_j\hat{U}_j} - q'(\mathrm{e}^{\hat{U}_j})\mathrm{e}^{\hat{U}_j})\hat{\varphi}_j u\|_{L^2} \\ &\leq C\,\mathrm{e}^{-c/\delta}\,\|\hat{\varphi}_j u\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}\|(L_\delta - \hat{L}_0)\hat{\psi}_k u\|_{L^2} \leq& C\sup_{j\in\mathcal{P}} \|q'(1) - q'(\mathrm{e}^{\hat{\varphi}_j\hat{U}_j})\mathrm{e}^{\hat{\varphi}_j\hat{U}_j}\|_{L^\infty(\hat{P}_j\setminus\hat{C}_j)}\|\hat{\psi}_k u\|_{L^2} \\ \leq& C\,\mathrm{e}^{-c/\delta}\|\hat{\psi}_k u\|_{L^2}.\end{aligned}$$

□

The following lemma provides an essential non-degeneracy property of L_δ:

LEMMA 4.4. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, the operator L_δ is invertible. Moreover, $L_\delta^{-1} : \hat{Y}_\delta \to \hat{X}_\delta$ is uniformly bounded with respect to $\delta \in (0, \delta_0)$.*

Proof. Once established LEMMA 3.1 and LEMMA 4.3, the proof is as in [8, Lemma 5.3] and then is omitted. □

Now we can provide the

Proof. [Proof of Proposition 4.1] We use the Banach fixed point argument. For any $\delta \in (0, \delta_0)$, with $\delta_0 > 0$ given by LEMMA 4.4, we introduce the nonlinear map $G_\delta \in C^1(\hat{X}_\delta, \hat{X}_\delta)$ defined by

$$G_\delta(z) = z - L_\delta^{-1} F_\delta(z).$$

Then, fixed points of G_δ correspond to solutions of the functional equation $F_\delta(z) = 0$. First, note that $DG_\delta(0) = 0$ and that

$$DF_\delta(z) = -\Delta - q'(\mathrm{e}^{\sum_{j\in\mathcal{P}} \hat{\varphi}_j \hat{U}_j + z})\mathrm{e}^{\sum_{j\in\mathcal{P}} \hat{\varphi}_j \hat{U}_j + z}.$$

Then, by LEMMA 4.4, for any $z \in \hat{X}_\delta$ and $u \in \hat{X}_\delta$ we have

$$\begin{aligned} \|DG_\delta(z)u\|_{\hat{X}_\delta} =& \ \|(DG_\delta(z) - DG_\delta(0))u\|_{\hat{X}_\delta} = \|L_\delta^{-1}(DF_\delta(z) - L_\delta)u\|_{\hat{X}_\delta} \\ \leq& \ C\|(-q'(\mathrm{e}^{\sum_{j\in\mathcal{P}} \hat{\varphi}_j \hat{U}_j + z})\mathrm{e}^z + q'(\mathrm{e}^{\sum_{j\in\mathcal{P}} \hat{\varphi}_j \hat{U}_j}))u\|_{\hat{Y}_\delta} \\ \leq& \ C(\|q'\|_{L^\infty(I_z)} + \|q''\|_{L^\infty(I_z)})\|(e^z - 1)u\|_{\hat{Y}_\delta}. \end{aligned}$$

where I_z is the interval $I_z = [0, e^{\|z\|_{L^\infty}}]$.

By the elementary inequality $\mathrm{e}^t - 1 \leq t\mathrm{e}^t$, for all $t > 0$, and Sobolev inequality, we have

$$\|DG_\delta(z)u\|_{\hat{X}_\delta} \leq C(\|q'\|_{L^\infty(I_z)} + \|q''\|_{L^\infty(I_z)})\|z\|_{\hat{X}_\delta} \mathrm{e}^{c\|z\|_{\hat{X}_\delta}} \|u\|_{\hat{X}_\delta}.$$

Consequently, there exists $R_1 > 0$ such that for every $R \in (0, R_1)$ we have

$$\|DG_\delta(z)\| \leq \frac{1}{2}, \qquad \forall z \in \mathcal{B}_R$$

for all $\delta \in (0, \delta_0)$. Now, for every $R \in (0, R_1)$

$$\|G_\delta(z)\|_{\hat{X}_\delta} \leq \|G_\delta(z) - G_\delta(0)\|_{\hat{X}_\delta} + \|G_\delta(0)\|_{\hat{X}_\delta} \leq \frac{1}{2}\|z\|_{\hat{X}_\delta} + \|L_\delta^{-1} F_\delta(0)\|_{\hat{X}_\delta}.$$

Since, there exist $C, c > 0$ independent of $\delta \in (0, \delta_0)$ such that

$$\|L_\delta^{-1} F_\delta(0)\|_{\hat{X}_\delta} \leq C\|F_\delta(0)\|_{\hat{Y}_\delta} \leq C\mathrm{e}^{-c/\delta}, \tag{4.5}$$

there exists $\delta_1 > 0$ such that for any $\delta \in (0, \delta_1)$ we obtain that $G_\delta(\mathcal{B}_R) \subset \mathcal{B}_R$. Hence, G_δ is a strict contraction in $\mathcal{B}_R$, for any $\delta \in (0, \delta_1)$. By the Banach fixed-point theorem, for any $\delta \in (0, \delta_1)$, there exists a unique $z_\delta \in \mathcal{B}_R$, such that $F_\delta(z_\delta) = 0$.

Moreover, in view of (4.5), we may choose $R = R_\delta = 2C\mathrm{e}^{-c/\delta}$ and we get $\|z_\delta\|_{\hat{X}_\delta} \leq R_\delta = 2C\mathrm{e}^{-c/\delta}$. □

5. Proof of Theorem 2.1. As a consequence of PROPOSITION 4.1, there exists $\delta_1 > 0$ such that for every $\delta \in (0, \delta_1)$ the function u_δ defined by

$$u_\delta(x) = \hat{u}_\delta\left(\frac{x}{\delta}\right) = \sum_{j\in\mathcal{P}} \varphi_j(x) U_{m_j}\left(\frac{x - p_j}{\delta}\right) + z_\delta\left(\frac{x}{\delta}\right) \tag{5.1}$$

is a solution to (1.1). Then arguing as in [8], lemma 6.1 we have that the solution u_δ defined in (5.1) satisfies the approximate superposition rule (2.2). Then points (i)–(ii) follow by maximum principle and by the properties of the solution, see [8, Lemma 6.2] for more details. To show that (iii) holds, let $\varphi \in C_c^\infty(\mathbb{R}^2)$. Then,

$$-\int_{\mathbb{R}^2} u_\delta \Delta\varphi = \delta^{-2}\int_{\mathbb{R}^2} q(\mathrm{e}^{u_\delta})\varphi - 4\pi \sum_{j\in\mathcal{P}} m_j \varphi(p_j).$$

It can be proved, as in [8], lemma 6.2, that $\int_{\mathbb{R}^2} u_\delta \Delta\varphi \to 0$ as $\delta \to 0^+$, and (iii) is established.

Finally, we want to prove that if the p_j's are doubly periodically arranged in $\mathbb{R}^2$ (see (2.4)), then u_δ is in fact a doubly periodic solution to (1.1). We define $\underline{\hat{e}}_l = \underline{e}_l/\delta$, $l = 1, 2$. Equivalently, we show that

$$\hat{u}_\delta(x + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2) = \hat{u}_\delta(x) \qquad \text{for any } x \in \mathbb{R}^2,\ m, n \in \mathbb{Z}.$$

Indeed, we may assume that $\hat{\varphi}_j(x - (ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2)) = \hat{\varphi}_{j'}(x)$, where

$$p_{j'} = p_j + ma\underline{e}_1 + nb\underline{e}_2 \quad \text{and} \quad \hat{\psi}_k(x - (ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2)) = \hat{\psi}_{k'}(x)$$

for a suitable $k' \in \mathcal{Q}$. Then,

$$\hat{u}_\delta(x + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2) = \sum_{j\in\mathcal{P}} \hat{\varphi}_j(x)\hat{U}_j(x) + z_\delta(x + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2).$$

Hence, it is sufficient to prove that $z_\delta(x + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2) = z_\delta(x)$, for every $x \in \mathbb{R}^2$, $m, n \in \mathbb{Z}$.

First, we claim that $z_\delta(\,\cdot\, + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2) \in \mathcal{B}_R$. Indeed, for every $(j, k) \in \mathcal{P} \times \mathcal{Q}$ we have

$$\|\hat{\varphi}_j z_\delta(\,\cdot\, + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2)\|_{H^2} = \|\hat{\varphi}_{j'} z_\delta\|_{H^2} \qquad \text{and}$$

$$\|\hat{\psi}_k z_\delta(\,\cdot\, + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2)\|_{H^2} = \|\hat{\psi}_{k'} z_\delta\|_{H^2}.$$

Hence, we obtain

$$\|z_\delta(\,\cdot\, + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2)\|_{\hat{X}_\delta} = \|z_\delta\|_{\hat{X}_\delta} \le R.$$

Moreover, if $F_\delta(z_\delta) = 0$ we also have $F_\delta(z_\delta(\,\cdot + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2)) = 0$. Therefore, $z_\delta(\,\cdot + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2)$ is a fixed point of G_δ in $\mathcal{B}_R$. By uniqueness, we conclude that $z_\delta(\,\cdot + ma\underline{\hat{e}}_1 + nb\underline{\hat{e}}_2) = z_\delta$, as asserted.

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