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SYSTEMS FOR PHASE TRANSITIONS WITH HYSTERESIS EFFECT

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Abstract. The paper deals with a system of nonlinear PDEs which describes a phase transition model with vector hysteresis and diffusion effect. Existence of solutions for the system under consideration is obtained by the method of Yosida approximation, L^∞-energy method and energy type inequalities in L^2. Uniqueness result has been obtained in the case when the coefficient of the interfacial energy in the kinetics equation of the order parameter is zero. Results on existence and uniqueness are given also for the ODE system analogue.

Key words. Nonlinear PDEs, existence of solutions, subdifferential, Yosida approximation, hysteresis, L^∞-energy method, method of L^1-semigroups, ordinary differential systems.

AMS subject classifications. 35R70, 35K50, 74N30, 34C55, 34A60

1. Introduction. The present paper deals with a system of nonlinear PDEs which is a model of a class of phase transitions where the hysteresis and diffusive effects are taken into account:

$$a\mathbf{w}_t - \kappa\Delta\mathbf{w} + \partial\mathbf{I}_{K(u)}(\mathbf{w}) \ni \mathbf{F}(\mathbf{w}, u) \quad \text{in } Q, \tag{1.1}$$

$$\mathbf{c}\cdot\mathbf{w}_t + du_t - \Delta u = h(\mathbf{w}, u) \quad \text{in } Q. \tag{1.2}$$

Here N, m are positive integers, $\mathbf{w} = (w_1, \ldots, w_m)$, $T > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $Q = (0,T)\times\Omega$; a, κ, $\mathbf{c} = (c_1, \ldots, c_m)$, d are given constants; $\mathbf{F} : \mathbb{R}^m \times \mathbb{R} \to R^m$, $h : \mathbb{R}^m \times \mathbb{R} \to R$, $f_{i*}, {f_i}^* : \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, m)$ are given functions. We assume that $f_{i*}, {f_i}^* \in C^2(\mathbb{R})$, $f_{i*} \le {f_i}^*$ on $\mathbb{R}$ and there exist constants $k_i > 0$ such that $f_{i*} = {f_i}^*$ on $(-\infty, -k_i] \cup [k_i, \infty)$, $i = 1, \ldots, m$.

For each $u \in \mathbb{R}$ we denote by $\partial I_u^{(i)}(\cdot)$ the subdifferential of the indicator function $I_u^{(i)}(\cdot)$ of the interval $[f_{i*}(u), f_i^*(u)]$, $(i = 1, \ldots, m)$, namely,

$$I_u^{(i)}(w_i) = \begin{cases} 0 & \text{if } f_{i*}(u) \le w_i \le f_i^*(u) \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\partial I_u^{(i)}(w_i) = \begin{cases} \emptyset & \text{if } \ w_i > f_i^*(u) \ \text{ or } \ w_i < f_{i*}(u) \\ [0, +\infty) & \text{if } \ w_i = f_i^*(u) > f_{i*}(u) \\ \{0\} & \text{if } \ f_{i*}(u) < w_i < f_i^*(u) \\ (-\infty, 0] & \text{if } \ w_i = f_{i*}(u) < f_i^*(u) \\ R & \text{if } \ w_i = f_i^*(u) = f_{i*}(u). \end{cases}$$

Define $K(u) = \{\mathbf{w} \in \mathbb{R}^m : \ f_{i*}(u) \le w_i \le {f_i}^*(u), \ i = 1, \ldots, m\}$.

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We denote by $I_{K(u)}(\cdot)$ the indicator function of the set $K(u)$ and $\partial\mathbf{I}_{K(u)}(\cdot)$ denotes the subdifferential of $I_{K(u)}(\cdot)$. The subdifferential $\partial\mathbf{I}_{K(u)}(\mathbf{w})$ is a set - valued mapping and in our statement of the problem $\partial\mathbf{I}_{K(u)}(\mathbf{w}) = \{0\}$ if $\mathbf{w} \in \mathrm{int}K$, and $\partial\mathbf{I}_{K(u)}(\mathbf{w})$ coincides with the cone of normals to K at the point $\mathbf{w}$ if $\mathbf{w} \in \partial K$. In our statement of the problem it is easy to see that

$$\partial\mathbf{I}_{K(u)}(\mathbf{w}) = (\partial I_u^{(1)}(w_1), \dots, \partial I_u^{(m)}(w_m)).$$

In this paper we study the system (1.1), (1.2) together with the following boundary and initial conditions

$$\frac{\partial \mathbf{w}}{\partial \nu} = 0, \qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Sigma = (0,T) \times \partial\Omega, \tag{1.3}$$

$$\mathbf{w}(0,x) = \mathbf{w}_0(x), \qquad u(0,x) = u_0(x) \qquad \text{in } \Omega, \tag{1.4}$$

where ν is the unit outward normal vector on $\partial\Omega$, $\mathbf{w}_0$, u_0 are given initial data.

The system (1.1), (1.2) is a model for solid-liquid phase transition of a multi – component substance where we take into account the hysteresis effect in the evolution of the interface. Equations (1.1) and (1.2) correspond respectively to the kinetics of the vector order parameter $\mathbf{w}$ and the balance of the internal energy; u is the relative temperature of the physical system under consideration. The right hand sides of the equations of system (1.1), (1.2) describe possible nonlinearities respectively in the kinetics of the order parameter and the external energy suply.

The hysteresis effect is described by the term $\partial\mathbf{I}_{K(u)}(\mathbf{w})$ in differential inclusion (1.1). It is known that some types of hysteresis operators can be represented by ordinary (or partial) differential inclusion containing subdifferential of the indicator function of a closed set (whose shape could possibly depend on the unknown variables). Let us note that this characterization of hysteresis operators was used for analysis of many nonlinear phenomena, for example, a real-time control problems (see [9]), solid-liquid phase transitions (see [7], [19]), shape memory alloys (see [1]), filtration problems (see [14]). Very recently this approach has been used to study the phenomena of hysteresis in processes in population dynamics (see [2], [23]).

Differential inclusion (1.1) decribes the relaxation dynamics of the vector order parameter. The relation assigning to a function $u(t)$ the solution $\mathbf{w}(t)$ of differential inclusion (1.1) corresponds to generalized vector play hysteresis operator which is often used to describe solid-liquid phase transitions with supercooling effect and martensite-austenite phase transitions in shape memory allows. Let us note that models with hysteresis are object of active recent investigations (see papers [8], [10], [12], [13] as well as the monographs [5], [11], [18], [24]).

Various special cases of the system (1.1),(1.2) have been already studied. In [7], P. Colli, N. Kenmochi and M. Kubo studied the following system

$$aw_t - \kappa\Delta w + \partial I_u(w) \ni F(w,u) \qquad \text{in } Q,$$

$$cw_t + du_t - \Delta u = g(x,t) \qquad \text{in } Q$$

as a model for Stefan problem with phase relaxation and temperature dependent constraint for the scalar order parameter. Later, M. Kubo in [14] studied filtration problems with hysteresis described by similar systems with convective term (we refer the reader also to the papers [8], [9], the monograph [5] as well as the references therein).

Recently, in [23], M. Ôtani studied the following nonlinear parabolic system with hysteresis effect

$$w_t - \nabla\cdot(\nabla w + \vec{\lambda}(w)) + \partial I_U(w) \ni F(w,U) \qquad \text{in } Q,$$
$$u_{it} - \nabla\cdot(\nabla u_i + \vec{\mu}_i(u_i)) = h_i(w,U) \qquad \text{in } Q, \quad i=1,\dots,m,$$

which is a model for population interaction with hysteresis effect of $1+m$ biological species with densities (w,U), $U=(u_1,\dots,u_m)$. To this end in [23], a further extention of the recently proposed L^∞-energy method is developed. It should be noted that the L^∞-energy method (proposed in [22], [23] and the references therein) was found to an be effective tool applicable to various types of parabolic equations and systems including doubly nonlinear parabolic equations, porous medium equations, strongly nonlinear parabolic equations governed by the ∞-Laplacian, complicated parabolic systems from applied sciences, etc.

In mathematical aspect the present paper has been influenced mainly by the papers [23] and [7]. Using the L^∞-energy method we will obtain results for boundedness and existence of solutions of the system (1.1)–(1.4). As concerns for uniqueness, the result presented here is based on the method of L^1-semigroups proposed in [8], and later develop in [7].

Detailed proofs of the results presented in this paper can be found in [20] and [21].

2. Preliminary Notes. Denote by H the Hilbert space $L^2(\Omega)$ with the usual scalar product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$, and by $\mathbf{H}$ the product space $H\times\dots\times H$ (m-times). Denote by V the Sobolev space $H^1(\Omega)$ equipped with the norm $|u|_V=(u,u)_V^{1/2}$, where $(u,v)_V=(u,v)_H+a(u,v)$, $a(u,v)=\int_\Omega \nabla u(x)\cdot\nabla v(x)dx$, $u,v\in V$, and by $\mathbf{V}$ the product space $V\times\dots\times V$ (m-times).

We give the definition of solutions in a weak (variational) sense for the system (1.1)–(1.4).

Definition 2.1. Let $\kappa>0$. A pair of functions $\{\mathbf{w},u\}$, ($\mathbf{w}=(w_1,\dots,w_m)$) is called a solution of the system (1.1)–(1.4) if:

(i) $w_i, u \in L^\infty(0,T;V\cap L^\infty(\Omega))\cap L^2(0,T;H^2(\Omega))\cap W^{1,2}(0,T;H)$, $i=1,\dots,m$.

(ii) $a\mathbf{w}_t-\kappa\Delta\mathbf{w}+\partial\mathbf{I}_{K(u)}(\mathbf{w})\ni\mathbf{F}(\mathbf{w},u)$ in $\mathbf{H}$, a.e. in $(0,T)$.

(iii) $\mathbf{c}\cdot\mathbf{w}_t+du_t-\Delta u=h(\mathbf{w},u)$ in H, a.e. in $0,T)$.

(iv) $\dfrac{\partial w_i}{\partial\nu}=0$, $\dfrac{\partial u}{\partial\nu}=0$ in $L^2(\partial\Omega)$, a.e. in $(0,T)$, $i=1,\dots,m$.

(v) $\mathbf{w}(0)=\mathbf{w}_0$, $u(0)=u_0$.

For simplicity of the notation we will denote in the sequel by $\mathbf{w}'$ and u' the time-derivatives $\mathbf{w}_t$ and u_t of $\mathbf{w}$ and u, respectively.

Note that the inclusion (ii) is equivalent to the following conditions:

(ii)–(a) $\mathbf{w}\in K(u)$ a.e. in Q.

(ii)–(b) $(a\mathbf{w}'(t)-\kappa\Delta\mathbf{w}(t)-\mathbf{F}(\mathbf{w}(t),u(t)),\mathbf{w}(t)-\mathbf{z})\le 0$ for all $\mathbf{z}\in\mathbf{H}$ with $\mathbf{z}\in K(u(t))$ a.e. in Ω for a.e. $t\in(0,T)$.

Throughout the paper we suppose that the following assumptions hold:

H1. $a>0$, $c_i\ne 0$, $d>0$ are given constants, $i=1,\dots,m$.

H2. $f_{i*}, {f_i}^*\in C^2(R)$ are such that $f_{i*}\le {f_i}^*$ on R and there exist constants $k_i>0$ such that $f_{i*}(u)={f_i}^*(u)=r_iu+s_i$ on $(-\infty,-k_i]$ and $f_{i*}(u)={f_i}^*(u)=p_iu+q_i$ on $[k_i,\infty)$, where r_i, s_i, p_i, q_i are given constants, $i=1,\dots,m$. Moreover, if $c_i>0$

($c_i < 0$) then f_{i*}, f_i^* are assumed to be non-decreasing (non-increasing) functions on $\mathbb{R}$, $i = 1, \dots, m$.

H3. $w_{0i}, u_0 \in L^\infty(\Omega) \cap V$, $i = 1, \dots, m$ and $\mathbf{w}_0 \in K(u_0)$ a.e. in Ω, $\mathbf{w}_0 = (w_{01}, \dots, w_{0m})$.

H4. $\mathbf{F}$ and h are locally Lipschitz continuous functions from $\mathbb{R}^m \times R$ into $\mathbb{R}^m$ and $\mathbb{R}$, respectively.

H5. There exist positive constants $C_{\mathbf{F}}$ and C_h such that $F_i(\mathbf{w}, u) w_i \le C_{\mathbf{F}}(|w_i|^2 + |u|^2 + 1)$, $i = 1, \dots, m$, and $h(\mathbf{w}, u) u \le C_h(|\mathbf{w}|^2 + |u|^2 + 1)$, $\mathbf{w} \in \mathbb{R}^m$, $u \in \mathbb{R}$.

3. Auxiliary Problems. Let $M > 0$ be a constant large enough which is to be fixed later and consider the cut-off function:

$$\chi_M(r) = \begin{cases} -M & \text{if } r < -M \\ r & \text{if } -M \le r \le M \\ M & \text{if } r > M \end{cases}$$

and define the auxiliary functions

$$\begin{aligned} \chi_M(w_i)(t,x) &= \chi_M(w_i(t,x)), && (t,x) \in Q,\ i = 1, \dots, m, \\ \chi_M(\mathbf{w})(t,x) &= (\chi_M(w_1)(t,x), \dots, \chi_M(w_m)(t,x)), && (t,x) \in Q, \\ \chi_M(u)(t,x) &= \chi_M(u(t,x)), && (t,x) \in Q. \end{aligned}$$

In this section we introduce an approximate system with approximation parameters M and $\mu > 0$. To this end for $(\mathbf{w}, u) \in \mathbb{R}^m \times \mathbb{R}$ we denote

$$\begin{aligned} \partial \mathbf{I}^\mu_{K_M(u)}(\mathbf{w}) &= \left(\partial {I^\mu_{u,M}}^{(1)}(w_1), \dots, \partial {I^\mu_{u,M}}^{(m)}(w_m)\right) \\ &= \frac{1}{\mu}\left([w_1 - f^*_{1,M}(u)]^+ - [f_{1*,M}(u) - w_1]^+,\right. \\ &\qquad \left. \dots, [w_m - f^*_{m,M}(u)]^+ - [f_{m*,M}(u) - w_m]^+\right), \end{aligned}$$

and

$$\begin{aligned} \mathbf{J}_{u,M}\mathbf{w} &= (J^{(1)}_{u,M} w_1, \dots, J^{(m)}_{u,M} w_m) \\ &= (\max\{\min\{w_1, f^*_{1,M}(u)\}, f_{1*,M}(u)\}, \dots, \max\{\min\{w_m, f^*_{m,M}(u)\}, f_{m*,M}(u)\}), \end{aligned}$$

where $f^*_{i,M}(u) = f^*_i(\chi_M(u))$, $f_{i*,M}(u) = f_{i*}(\chi_M(u))$, $i = 1, \dots, m$. Moreover, denote

$$\begin{aligned} \mathbf{J}_u\mathbf{w} &= (J^{(1)}_u w_1, \dots, J^{(m)}_u w_m) \\ &= (\max\{\min\{w_1, f^*_1(u)\}, f_{1*}(u)\}, \dots, \max\{\min\{w_m, f^*_m(u)\}, f_{m*}(u)\}). \end{aligned}$$

Note that $\partial \mathbf{I}^\mu_{K_M(u)}$ is the Yosida regularization of the subdifferential graph of the indicator function $I_{K_M(u)}$ of the set

$$K_M(u) = \{\mathbf{w} \in \mathbb{R}^m :\ f_{i*,M}(u) \le w_i \le f^*_{i,M}(u),\ i = 1, \dots, m\}.$$

Consider the following approximate system of PDEs

$$a\mathbf{w}' - \kappa\Delta\mathbf{w} + \partial\mathbf{I}^{\mu}_{K_M(u)}(\mathbf{w}) = \mathbf{F}_M(\mathbf{w}, u) \quad \text{in } Q, \tag{3.1}$$

$$\mathbf{c}\cdot(\mathbf{J}_u\mathbf{w})' + du' - \Delta u = h_{M,\mathbf{J}}(\mathbf{w}, u) \quad \text{in } Q, \tag{3.2}$$

$$\frac{\partial\mathbf{w}}{\partial\nu} = 0, \qquad \frac{\partial u}{\partial\nu} = 0 \quad \text{on } \Sigma, \tag{3.3}$$

$$\mathbf{w}(0,x) = \mathbf{w}_0(x), \quad u(0,x) = u_0(x) \quad \text{in } \Omega, \tag{3.4}$$

where

$$\mathbf{F}_M(\mathbf{w}, u) = \mathbf{F}(\chi_M(\mathbf{w}), \chi_M(u)), \qquad h_{M,\mathbf{J}}(\mathbf{w}, u) = h(\mathbf{J}_{u,M}\mathbf{w}, \chi_M(u)).$$

4. Main results.

4.1. System (1.1)–(1.4) with $\kappa > 0$.

We formulate the following:

THEOREM 4.1. *Suppose that assumptions* **H1–H5** *are satisfied. Then there exists a constant $\kappa_0 > 0$ such that for each $0 < \kappa < \kappa_0$ the system* (1.1)–(1.4) *possesses at least one solution.*

4.2. System (1.1)-(1.4) with $\kappa = 0$.

In this section consider the system (1.1)–(1.4) with $\kappa = 0$, namely the following system

$$a\mathbf{w}_t + \partial\mathbf{I}_{K(u)}(\mathbf{w}) \ni \mathbf{F}(\mathbf{w}, u) \quad \text{in } Q, \tag{4.1}$$

$$\mathbf{c}\cdot\mathbf{w}_t + du_t - \Delta u = h(\mathbf{w}, u) \quad \text{in } Q. \tag{4.2}$$

$$\frac{\partial u}{\partial\nu} = 0 \quad \text{on } \Sigma = (0,T)\times\partial\Omega, \tag{4.3}$$

$$\mathbf{w}(0,x) = \mathbf{w}_0(x), \qquad u(0,x) = u_0(x) \quad \text{in } \Omega. \tag{4.4}$$

DEFINITION 4.2. A pair of functions $\{\mathbf{w}, u\}$ is called a solution of the system (4.1)–(4.4) if:

(i) $w_i \in L^\infty(0,T;L^\infty(\Omega)) \cap W^{1,2}(0,T;H)$, $i = 1,\dots,m$.

(ii) $u \in L^\infty(0,T;V\cap L^\infty(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap W^{1,2}(0,T;H)$.

(iii) $a\mathbf{w}_t + \partial\mathbf{I}_{K(u)}(\mathbf{w}) \ni \mathbf{F}(\mathbf{w}, u)$ in $\mathbf{H}$, a.e. in $(0,T)$.

(iv) $\mathbf{c}\cdot\mathbf{w}_t + du_t - \Delta u = h(\mathbf{w}, u)$ in H, a.e. in $(0,T)$.

(v) $\frac{\partial u}{\partial\nu} = 0$ in $L^2(\partial\Omega)$, a.e. in $(0,T)$.

(vi) $\mathbf{w}(0) = \mathbf{w}_0$, $u(0) = u_0$.

THEOREM 4.3. *Suppose that assumptions* **H1–H5** *are satisfied. Then the system* (4.1)–(4.4) *possesses a unique solution. Moreover,* $\mathbf{w} \in L^\infty(0,T;\mathbf{V})$.

4.3. Local solutions.

If we suppose in Theorems 4.1 and 4.3 that the functions $\mathbf{F}$, h are locally Lipschitz continuous functions on $\mathbb{R}^m\times\mathbb{R}$ (without any growth conditions), it can be proved existence of local solutions of the respective systems, namely, the following theorems hold true:

THEOREM 4.4. *Suppose that assumptions* **H1–H4** *are satisfied. Then there exist a positive number T_0 (depending only on $|\mathbf{w}_0|_\infty$ and $|u_0|_\infty$) as well as a constant $\kappa_0 > 0$*

such that for each $0 < \kappa < \kappa_0$ *the system* (1.1)–(1.4) *possesses at least one solution on* $[0, T_0] \times \Omega$.

THEOREM 4.5. *Suppose that assumptions* **H1**–**H4** *are satisfied. Then there exists a positive number* T_0 *(depending only on* $|\mathbf{w}_0|_\infty$ *and* $|u_0|_\infty$*) such that the system* (4.1)–(4.4) *possesses a unique solution on* $[0, T_0] \times \Omega$.

5. Remarks for the ODE case. This section deals with the following class of systems of nonlinear ordinary differential equations (ODEs) with hysteresis effect:

$$a(w(t), u(t))w'(t) + \partial I_{u(t)}(w(t)) \ni g(w(t), u(t)), \qquad 0 < t < T, \tag{5.1}$$

$$c(w(t), u(t))w'(t) + d(w(t), u(t))u'(t) = h(w(t), u(t)), \quad 0 < t < T, \tag{5.2}$$

with initial conditions:

$$w(0) = w_0, \qquad u(0) = u_0, \tag{5.3}$$

where $T > 0$, $a, c, d, g, h : \mathbb{R}^2 \to R$, $f_*, f^* : \mathbb{R} \to \mathbb{R}$ are given functions such that $f_* \le f^*$ on $\mathbb{R}$ and there exists a constant $k_0 > 0$ such that $f_* = f^*$ on $(-\infty, -k_0] \cup [k_0, \infty)$.

For each $u \in R$ we denote by $\partial I_u(\cdot)$ the subdifferential of the indicator function $I_u(\cdot)$ of the interval $[f_*(u), f^*(u)]$.

The system (5.1)–(5.3) was studied in [24] in the partucular case when $a \equiv 1$, $c \equiv 1$, $d \equiv 1$, $g \equiv 0$ and $h \equiv 0$. The idea for uniqueness proof is based on the L^1-theory of nonlinear semigroups (cf. [8], [24]). Very recently the system (5.1)–(5.3) was considered in [10] in a general framework, where existence and uniqueness results as well as numerical simulation of the solutions are presented. However, the assumption of global Lipschitz continuity of the coefficient functions, the right-hand sides and the constraint functions is essential for the existence proof presented in [10]. In the present section, based on the L^∞-energy method approach presented in [23], we will formulate results for local and global existence as well as uniqueness of the solutions of the ODEs system with hysteresis effect under the relaxed assumption of local Lipschitz continuity of all functions involved in (5.1)–(5.3). However, we must note that local Lipschitz continuity will imply existence of local solutons, while to get global solutions, certain growth assumptions on the right-hand sides are to be supposed (cf. [23]).

We give the definition of solutions of the system (5.1)–(5.3).

DEFINITION 5.1. A pair of functions $\{w, u\}$ is called a solution of the system (5.1)–(5.3) if: $w, u \in W^{1,2}(0, T)$; $\{w, u\}$ satisfy (5.1), (5.2) a.e. on $[0, T]$ as well as the initial condition (5.3).

Introduce the following assumptions:

H6. f_*, f^* are nondecreasing and locally Lipschitz continuous functions on $\mathbb{R}$ such that $f_* \le f^*$ on $\mathbb{R}$, there exists a constant $k_0 > 0$ such that $f_* = f^*$ on $(-\infty, -k_0] \cup [k_0, \infty)$ and $f_*(u_0) \le w_0 \le f^*(u_0)$.

H7 The derivatives $f_*', f^{*\prime}$ are locally Lipschitz continuous functions on $\mathbb{R}$ and there exists a positive constant μ_0 such that

$$d(f_*(u), u) + c(f_*(u), u)f_*'(u) \ge \mu_0,$$

$$d(f^*(u), u) + c(f^*(u), u)f^{*\prime}(u) \ge \mu_0$$

for any $u \in \mathbb{R}$.

H8. a, c, d are locally Lipschitz continuous functions on $\mathbb{R}^2$ and there exists a positive constant c_0 such that $a \geq c_0$ and $d \geq c_0$ on $\mathbb{R}^2$.

H9. g, h are locally Lipschitz continuous functions on $\mathbb{R}^2$ such that $\frac{g(w,u)w}{a(w,u)} \leq C_{g,a}(|w|^2 + |u|^2 + 1)$, $h(w,u)u \leq C_h(|w|^2 + |u|^2 + 1)$, where $C_{g,a}$, C_h are positive constants, $w \in \mathbb{R}$, $u \in \mathbb{R}$.

H10. $|f^*(u)| \leq C_f(|u| + 1)$ on $(-\infty, -k_0] \cup [k_0, \infty)$, where C_f is a positive constant.

Let $M > 0$ be a constant large enough which is to be fixed later and consider the cut-off function:

$$\chi_M(r) = \begin{cases} -M & \text{if } r < -M \\ r & \text{if } -M \leq r \leq M \\ M & \text{if } r > M \end{cases}$$

and define the auxiliary functions

$$\chi_M(w)(t) = \chi_M(w(t)), \qquad 0 < t < T,$$
$$\chi_M(u)(t) = \chi_M(u(t)), \qquad 0 < t < T.$$

We introduce an approximate system with approximation parameters M and $\mu > 0$. For $w \in \mathbb{R}$, $u \in \mathbb{R}$ we denote:

$$\partial I^{\mu}_{u,M}(w) = \frac{1}{\mu}[w - f^*_M(u)]^+ - \frac{1}{\mu}[f_{*,M}(u) - w]^+,$$

and

$$J_{u,M}w = \max\{\min\{w, f^*_M(u)\}, f_{*,M}(u)\},$$

where $f^*_M(u) = f^*(\chi_M(u))$, $f_{*,M}(u) = f_*(\chi_M(u))$. Note that $\partial I^{\mu}_{u,M}$ is the Yosida regularization of the subdifferential graph of the indicator function of the interval $[f_{*,M}(u), f^*_M(u)]$.

Consider the following approximate system of ODEs

$$w'(t) + \partial I^{\mu}_{u,M}(w(t)) = \frac{g_M(w(t), u(t))}{a_M(w(t), u(t))}, \qquad 0 < t < T, \quad (5.4)$$

$$c_{M,J}(w(t), u(t))(J_{u,M}w)'(t) + d_{M,J}(w(t), u(t))u'(t) = h_{M,J}(w(t), u(t)), 0 < t < T, \quad (5.5)$$

$$w(0) = w_0, \qquad u(0) = u_0, \quad (5.6)$$

where

$$a_M(w,u) = a(\chi_M(w), \chi_M(u)),$$
$$c_{M,J}(w,u) = c(J_{u,M}w, \chi_M(u)), \qquad d_{M,J}(w,u) = d(J_{u,M}w, \chi_M(u)),$$
$$g_M(w,u) = g(\chi_M(w), \chi_M(u)), \qquad h_{M,J}(w,u) = h(J_{u,M}w, \chi_M(u)).$$

THEOREM 5.2. *Suppose that assumptions* **H6–H10** *are satisfied. Then:*

1. *The system* (5.1)–(5.3) *possesses at least one solution.*
2. *The system* (5.1)–(5.3) *possesses a unique solution if* $c \geq c_0$ *on* $\{(w,u) \in \mathbb{R}^2 : f_*(u) \leq w \leq f^*(u)\}$.

3. *The system* (5.1)–(5.3) *possesses a unique solution if* $c \equiv 0$ *on* $\{(w,u) \in \mathbb{R}^2 : f_*(u) \le w \le f^*(u)\}$.

REMARK. Assumptions **H9, H10** in Theorem 5.2 can be replaced by the following assumption:

H11. g, h are locally Lipschitz continuous functions on $\mathbb{R}^2$ such that $\frac{g(w,u)w}{a(w,u)} \le C_{g,a}(|w|^2+1)$, $h(w,u)u \le C_h(|u|^2+1)$, where $C_{g,a}$, C_h are positive constants, $w \in \mathbb{R}$, $u \in \mathbb{R}$.

THEOREM 5.3. *Suppose that assumptions* **H6–H8, H11** *are satisfied. Then:*

1. *The system* (5.1)–(5.3) *possesses at least one solution.*
2. *The system* (5.1)–(5.3) *possesses a unique solution if* $c \ge c_0$ *on* $\{(w,u) \in \mathbb{R}^2 : f_*(u) \le w \le f^*(u)\}$.
3. *The system* (5.1)–(5.3) *possesses a unique solution if* $c \equiv 0$ *on* $\{(w,u) \in \mathbb{R}^2 : f_*(u) \le w \le f^*(u)\}$.

REMARK. If we suppose in Theorem 5.2 that the functions g, h are locally Lipschitz continuous functions on $\mathbb{R}^2$ (without any growth conditions), it can be proved existence of local solutions of the system (5.1)–(5.3).

THEOREM 5.4. *Suppose that assumptions* **H6–H8** *are satisfied and the functions* g, h *are locally Lipschitz continuous functions on* $\mathbb{R}^2$*. Then there exists a positive number* T_0 *(depending only on the initial data) such that:*

1. *The system* (5.1)–(5.3) *possesses at least one solution on* $[0, T_0]$.
2. *The system* (5.1)–(5.3) *possesses a unique solution on* $[0, T_0]$ *if* $c \ge c_0$ *on* $\{(w,u) \in \mathbb{R}^2 : f_*(u) \le w \le f^*(u)\}$.
3. *The system* (5.1)–(5.3) *possesses a unique solution on* $[0, T_0]$ *if* $c \equiv 0$ *on* $\{(w,u) \in \mathbb{R}^2 : f_*(u) \le w \le f^*(u)\}$.

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