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BOUNDARY VALUE PROBLEMS FOR STRONGLY NONLINEAR SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we examine nonlinear differential inclusions with Dirichlet boundary conditions and a forcing term with no growth restrictions and satisfying instead a generalized sign condition. Using techniques from multivalued analysis and theory of nonlinear operators of monotone type, we establish the existence of a solution.

Key words. Boundary value problem,Strongly nonlinear problem, sign condition, measurable multifunction, upper and lower semicontinuous multifunctions,truncation, Moreau-Yosida regularization.

AMS subject classifications. 34B15

1. Intorduction. In this paper we examine nonlinear differential inclusions

$$\left.\begin{aligned} &a(t,x(t),x'(t))' - \partial\varphi(x(t)) - F(t,x(t)) \ni h(t) - \beta x'(t) \text{ a.e on } T=[0,b] \\ &x(0)=x(b)=0. \end{aligned}\right\} \quad (1.1)$$

Here $a : T \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ and $F : T \times \mathbb{R} \to 2^{\mathbb{R}}$ are multifunctions and $\partial\varphi(x)$ is the subdifferential of the convex function $\varphi(\cdot)$. We do not impose any growth condition on the multifunction $F(t,x)$. Instead we use a generalized sign condition. Using techniques from multivalued analysis and the theory of the operators of monotone type, we prove the existence of a solution for problem (1.1).

A particular case of this problem was studied by Boccardo-Drabek-Giachetti-Kucera [1], Del Pino-Elgueta-Manasevich [3], Drabek [4], where $a(t,x,y) = \|y\|^{p-2} y, \varphi = 0, F$ is single-valued and $h = 0, \beta = 0$. In addition in all these papers, F has a restricted growth, the usual $(p-1)$-polynomial growth condition and satisfies also a nonresonance condition. The approach in those papers is different from ours and is based on degree theoretic arguments. Problems with multivalued terms were studied by Pruszko [13], Frigon-Granas [6], Frigon [5], Kandilakis-Papageorgiou [12] and Halidias-Papageorgiou [8]. In all these works $a(t,x,y) = y$ (semilinear problem) , $\varphi = 0$, but F depends also on y in a nonlinear in general way. Also F has restricted growth. It should be pointed out that the works of Pruszko [13], Kandilakis-Papageorgiou [12] and Halidias-Papageorgiou [8] deal with the vector problem and moreover, in Kandilakis-Papageorgiou [12] and Halidias-Papageorgiou [8] the authors use boundary conditions which are nonlinear and multivalued and incorporate the Dirichlet boundary conditions as a special case. The approach of the Pruszko [13] is based on degree theory, Frigon-Granas [6] and Frigon [5] employ the method of upper and lower solutions, while Kandilakis-Papageorgiou [12] and Halidias-Papageorgiou [8] use the theory of nonlinear monotone operators and the multivalued Leray-Schauder alternative theorem. Recently there have been works involving the one-dimensional p-Laplacian (i.e $a(t,x,y) = \|y\|^{p-2} y$), f single-valued but depending also nonlinearly on y and with periodic or Neumann boundary conditions. We refer to the papers of Guo [7] and Dang-Oppenheimer [2] and the references therein. They impose polynomial growth restrictions on f.

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2. Preliminaries. Our analysis of problem (1.1) is based on notions and results from multivalued analysis and the theory of nonlinear operators of monotone type. So, for easy reference, in this section we recall some basic definitions and facts from these areas. For details we refer to the book of Hu-Papageorgiou [10].

Let (Ω, Σ) be a measurable space and X a separable Banach space. We use the following notations: $P_{f(c)}(X) = \{A \subseteq X :$ nonempty closed (and convex)$\}$ and $P_{(w)k(c)}(X) = \{A \subseteq X$ nonempty, (weakly-) compact (and convex)$\}$.

Also if $A \in 2^X \setminus \{\emptyset\}, x \in X$ and $x^* \in X^*$, we set $|A| = \sup[\|a\| : a \in A]$, $d(x, A) = \inf[\|x - a\| : a \in A]$ (the distance function from A) and $\sigma(x^*, A) = \sup[(x^*, a) : a \in A]$ (the support function of A).

A multifunction $F : \Omega \to P_f(X)$ is said to be *"measurable"*, if for all $x \in X$, the $\mathbb{R}_+$-valued function $\omega \to d(x, F(\omega))$ is measurable.

A multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ is *graph measurable*, if
$GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$ with $B(X)$ being the Borel σ-field of X.

For multifunctions with values in $P_f(X)$, measurability implies graph measurability, while the converse is true if Σ is complete, i.e $\Sigma = \widehat{\Sigma}$(=the universal σ-field corresponding to Σ). Suppose μ is a finite measure on Σ. Given a multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$, by S_F^p we denote the set of all selectors of F which belong in the Lebesgue-Bochner space $L^p(\Omega, X)$, i.e $S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega)\mu -$ a.e$\}$. In general this set may be empty. However, if $F(\cdot)$ is graph measurable and there exists a function $\xi \in L^p(\Omega)$such that $\inf[\|x\| : x \in F(\omega)] \leq \xi(\omega)\mu$-a.e, then $S_F^p \neq \emptyset$.

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z \setminus \{\emptyset\}$ is *"lower semicontinuous"* (lsc for short)(resp *"upper semicontinuous"* (usc for short)), if for all $C \subseteq Z$ closed, the set $G^+(C) = \{y \in Y : G(y) \subseteq C\}$ (resp $G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$) is closed in Y. As usc multifunction G with nonempty closed values (i.e. for all $y \in Y, G(y) \in P_f(Y)$) has a closed graph
$GrG = \{(y, z) \in Y \times Z : z \in G(y)\}$. The converse is true if in addition $G(\cdot)$ is locally compact, i.e. for every $y \in Y$, there exists a neighborhood U of y such that $\overline{G(U)}$ is compact in Z.

If Z is a metric space, then in $P_f(Z)$ we can define a generalized metric, known as the Hausdorff metric, by

$$h(A, B) = \max[\sup_{a \in A} \mathrm{d}(a, B), \sup_{b \in B} \mathrm{d}(b, A)].$$

We set $h^*(A, B) = \sup_{a \in A} \mathrm{d}(a, B)$ (the excess of A from B) and $h^*(B, A) = \sup_{b \in B} \mathrm{d}(b, A)$ (the excess of B from A). If Y is a Hausdorff topological space and $G : Y \to 2^Z \setminus \{\emptyset\}$, then G is h-lower semicontinuous (h-lsc for short) (resp h-upper semicontinuous) (h-usc for short), if the function $y \to h^*(G(y), G(x)))$ (resp. the function $y \to h^*(G(x), G(y))$ is continuous at x for every $x \in X$. A multifunction G which is both h-usc and h-lsc is said to be *"h-continuous"*, i.e. G is continuous from Y into the pseudometric space $(2^Z \setminus \{\emptyset\}, h)$. In general we have that usc implies h-usc and h-lsc implies lsc. However, for multifunctions with nonempty and compact values h-usc $\Longleftrightarrow$ usc and h-lsc $\Longleftrightarrow$ lsc. Moreover, if $G : Y \to 2^Z \setminus \{\emptyset\}$ is h-usc, then for every $z \in Z, y \to \mathrm{d}(z, G(y))$ is lower semicontinuous, while G is lsc if and only if for every $z \in Z, y \to \mathrm{d}(z, G(y))$ is upper semicontinuous.

Now let us recall a few definitions and facts from the theory of nonlinear operators of monotone type. So let X be a reflexive Banach space and X^* its topological dual. A map $A : D \subseteq X \to 2^{X^*}(D = \{x \in X : A(x) \neq \emptyset\}$, the domain of A), is said to be *"monotone"*, if for all $x^* \in A(x), y^* \in A(y)$, we have $(x^* - y^*, x - y) \geq 0$ (here by $(\cdot, \cdot)$ we denote the duality brackets for the pair (X, X^*)). If $(x^* - y^*, x - y) = 0$ implies $x = y$, then we say that A is *"strictly monotone"*. The operator A is *"maximal monotone"*, if it is monotone and $y^* \in A(y)$ if $(x^*-y^*, x-y) \geq 0$ for all $x \in D$ and $x^* \in A(x)$, i.e. GrA is maximal with respect to inclusion among the graphs of all monotone operators. A maximal monotone map A has a graph which is sequentially closed in $X \times X_w^*$ and in $X_w \times X^*$, where by X_w and X_w^* we denote the spaces X and X^* with their respective weak topologies. If $A : X \to P_{fc}(X^*)$ is a monotone map and for every $x, y \in X$ $\lambda \to A(x+\lambda y)$ is usc from $[0,1]$ into X_w^*, then A is maximal monotone. Now let $A : X \to X^*$ be a single-valued map which is everywhere defined (i.e. $D = X$). We say that A is *"demicontinuous"*, if $x_n \to x$ in X, then $A(x_n) \xrightarrow{w} A(x)$ in X^*. A monotone, demicontinuous map is maximal monotone. A map $A : D \subseteq X \to 2^{X^*}$ is said to be *"coercive"*, if D is bounded or D is unbounded and $\inf\{\|x^*\| : x^* \in A(x)\} \to \infty$ as $\|x\| \to \infty$. A maximal monotone, coercive map is surjective.

An operator $A : X \to 2^{X^*}$ is said to be *" pseudomonotone"* , if the following are true:

(a) for every $x \in X, A(x) \in P_{wkc}(X^*)$;

(b) A is usc from every finite dimensional subspace Z of X into X_w^*; and

(c) if $x_n \xrightarrow{w} x, x_n^* \in A(x_n)$ and $\overline{\lim}_{n\to\infty}(x_n^*, x_n - x) \leq 0$, then for every $y \in X$, there exists $x^*(y) \in A(x)$ such that $(x^*(y), x-y) \leq \underline{\lim}_{n\to\infty}(x_n^*, x_n - y)$.

If the operator A is bounded (i.e. it maps bounded sets in X to bounded sets in X^*) and satisfies (c) above, then it satisfies also (b). An operator $A : X \to 2^{X^*}$ is said to be *"generalized pseudomonotone"*, if $x_n \xrightarrow{w} x$ in X, $x_n \xrightarrow{w} x^*$ in X^*, $x_n^* \in A(x_n)$ and $\overline{\lim}_{n\to\infty}(x_n^*, x_n - x) \leq 0$, then $x^* \in A(x)$ and $(x_n^*, x_n) \to (x^*, x)$. Every maximal monotone map is generalized pseudomonotone. Also a pseudomonotone operator is generalized pseudomonotone. The converse is true if the operator A has values in $P_{wkc}(X^*)$ and is bounded. A pseudomonotone operator which is coercive, is surjective.

Let X be a reflexive Banach space and $\varphi : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a proper, convex function (i.e. $\operatorname{dom} f = \{x \in X : \varphi(x) < +\infty\} \neq \emptyset$ and $\operatorname{epi} f = \{(x, \lambda) \in X \times \mathbb{R} : \varphi(x) \leq \lambda\}$ is convex). Given $x \in X$, we denote by $\partial\varphi(x)$ the set of all $x^* \in X^*$ such that $(x^*, y - x) \leq \varphi(y) - \phi(x)$ for all $y \in X$. Such elements $x^* \in X^*$ are called subgradients of φ at x and $\partial\varphi(x)$ is the subdifferential of φ at x. The generally multivalued operator $\partial\varphi : X \to 2^{X^*}$ is monotone and if φ is also lower semicontinuous. then $\partial\varphi$ is maximal monotone. Also $\partial\varphi(x)$ is always a closed convex set, possibly empty. If φ is Gateaux differentiable, then $\partial\varphi(x) = \{\varphi'(x)\}$ and $\varphi(x) = \inf_X \varphi$ if and only if $0 \in \partial\varphi(x)$. For every $\lambda > 0$ we define

$$\varphi_\lambda(x) = \inf\left[\varphi(y) + \frac{1}{2\lambda}\|x-y\|^2 : y \in X\right].$$

The function φ_λ is convex and everywhere finite on X and it is called the Moreau-Yosida approximation of φ. If X is reflexive and φ is proper, convex and lower semicontinuous, then φ_λ is Gateaux differentiable and $\lim_{\lambda\downarrow 0}\varphi_\lambda(x) = \varphi(x)$ for all $x \in X$. Moreover, $\partial\varphi_\lambda = (\partial\varphi)_\lambda$ (=the Yosida approximation of the maximal monotone map $\partial\varphi$).

Our hypotheses on the data of (1.1) are the following:

$\mathbf{H}(\alpha)$: $a : T \times \mathbb{R} \times \mathbb{R} \to P_{fc}(\mathbb{R})$ is a multifunction such that for almost all $t \in T$ and all $x \in \mathbb{R}$, $0 \in a(t, x, 0)$ and

(i) $(t, x, y) \to a(t, x, y)$ is graph measurable;

(ii) for almost all $t \in T$ and all $x \in \mathbb{R}, a(t, x, \cdot)$ is maximal monotone, while for almost all $t \in T$ and all $y \in \mathbb{R}, a(t, \cdot, y)$ is lsc;

(iii) for almost all $t \in T$ and all $x, y \in \mathbb{R}$

$$|a(t,x,y)| \leq \gamma(t) + c(|x|^{p-1} + |y|^{p-1})$$

with $\gamma \in L^q(T)(\frac{1}{p} + \frac{1}{q} = 1)$, $c > 0$;

(iv) for almost all $t \in T$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}$ and all $v \in a(t,x,y)$ we have

$$vy \geq c_1 |y|^p - \gamma_1(t)$$

with $\gamma_1 \in L^1(T), c_1 > 0$.

$\mathbf{H}(\varphi)$: $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and $\varphi(0) = \inf_{\mathbb{R}}\varphi$.

$\mathbf{H}(\mathbf{F})$: $F : T \times \mathbb{R} \to P_{kc}(\mathbb{R})$ is a multifunction such that

(i) $(t,x) \to F(t,x)$ is measurable;

(ii) for all $t \in T, x \to F(t,x)$ is usc;

(iii) for every $r > 0$ there exists $\xi_r \in L^1(T)$ such that for almost all $t \in T$ and all $|x| \leq r$ we have

$$|F(t,x)| \leq \xi_r(t);$$

(iv) there exists $M_1 > 0$ such that for all $t \in T$ we have $x. \min\{v \in F(t,x)\} \geq 0$ if $x \leq -M_1$ and $x. \max\{v \in F(t,x)\} \geq 0$ if $x \geq M_1$.

By virtue of hypotheses $H(a)$, for every $x \in W_0^{1,p}(T), S^q_{a(\cdot,x(\cdot),x'(\cdot))} \neq \emptyset$. Then term $a(t,x(t),x'(t))'$ in (1.1), as usual, is interpreted in the following way

$$a(t,x(t),x'(t))' = \left\{g' : g \in S^q_{a(\cdot,x(\cdot),x'(\cdot))}\right\}.$$

The derivative of g is defined in the sense of distributions. From a well-known representation theorem for the space $W^{-1,q}(T) = W_0^{1,p}(T)^*$ (see for example Hu-Papageorgiou [11], theorem A.1.25, p. 866), we see that $a(t,x(t),x'(t))' \in P_{fc}(W^{-1,q}(T))$. By a solution of (1.1) we mean a function $x \in W_0^{1,p}(T)$ such that for some $g \in S^q_{a(\cdot,x(\cdot),x'(\cdot))}, u \in S^q_{\partial\varphi(x(\cdot)}$ and $f \in S^q_{F(\cdot,x(\cdot))}$ such that

$$g'(t) - u(t) - f(t) = h(t) - \beta x'(t) \qquad \text{a.e on } T.$$

3. Auxiliary results. In this section we prove some auxiliary results which will be used in Section 4 to prove the main existence theorem concerning problem (1.1).

We start by establishing the existence of an approximate a Caratheodory selection of F which satisfies a generalized sign contition. Recall $g : T \times \mathbb{R} \to \mathbb{R}$ is Caratheodory function if it is measurable in $t \in T$ and continuous in $x \in \mathbb{R}$. Such a function is jointly measurable.

Proposition 3.1. *If $F : T \times \mathbb{R} \to P_{kc}(\mathbb{R})$ is a multifunction which satisfies hypotheses* **H(F)** *and $\varepsilon > 0$ is given, then there exists $g_\varepsilon : T \times \mathbb{R} \to \mathbb{R}$ a Caratheodory function such that*

(a) *for all $(t,x) \in T \times \mathbb{R}, g_\varepsilon(t,x) \in F(t, x + \overline{B}_\varepsilon) + \overline{B}_\varepsilon$ where $\overline{B}_\varepsilon = [-\varepsilon, \varepsilon]$;*

(b) *for every $M > 0$ there exists $m_M \in L^1(T)$ such that for almost all $t \in T$ and all $|x| \leq M$, $|g_\varepsilon(t,x)| \leq m_M(t)$ and $m_M(\cdot)$ can be chosen to be independent of $0 < \varepsilon \leq 1$;*

(c) *for all $t \in T$ and all $|x| \geq M_1 + 1$, we have $xg_\varepsilon(t,x) \geq 0$.*

Proof. Let $\vartheta_1(t,x)$ and $\vartheta_2(t,x)$ be the two measurable functions introduced in the remark of Section 2. We define

$$\widehat{\vartheta}_1(t,x)=\begin{cases}\theta_1(t,x) & \text{if } x\geq M_1\\ \max\{0,\vartheta_1(t,x)\} & \text{if } x<M_1\end{cases}\quad\text{and}$$

$$\widehat{\vartheta}_2(t,x)=\begin{cases}\theta_2(t,x) & \text{if } x\geq -M_1\\ \min\{0,\vartheta_2(t,x)\} & \text{if } x<-M_1\end{cases}.$$

It is easy to see that $\widehat{\vartheta}_1,\widehat{\vartheta}_2$ are both measurable and $\widehat{\vartheta}_1(t,\cdot)$ is lower semicontinuous while $\widehat{\vartheta}_2(t,\cdot)$ is upper semicontinuous. We set $\widehat{F}(t,x)=[\widehat{\vartheta}_1(t,x),\widehat{\vartheta}_2(t,x)]$. Then $\widehat{F}(\cdot,\cdot)$ is measurable and for all $t\in T,\widehat{F}(t,\cdot)$ is usc. Moreover, since $\theta_1\leq\widehat{\vartheta}_1\leq\widehat{\vartheta}_2\leq\vartheta_2$, we have that $\widehat{F}(t,x)\subseteq F(t,x)$ for all $(t,x)\in T\times\mathbb{R}$.

Fix $t\in T$. Invoking Theorem I.4.42, p. 107, of Hu-Papageorgiou [10], we know that given $\delta>0$, we can find $r_\delta:\mathbb{R}\to\mathbb{R}$ a continuous map such that $r_\delta(x)\in\widehat{F}(t,x+B_\delta)+B_\delta$ for all $x\in\mathbb{R}$, with $B_\delta=(-\delta,\delta)$.

Choose $\delta\leq\min\left\{\varepsilon,\frac{1}{2}\right\}$. We have

$$\max\left\{u\in\widehat{F}(t,x+B_\delta)\right\}\leq 0\qquad\text{if } x<-\left(M_1+\frac{1}{2}\right)$$

$$\text{and}\min\left\{u\in\widehat{F}(t,x+B_\delta)\right\}\geq 0\qquad\text{if } x>M_1+\frac{1}{2}.$$

Define $\widehat{r}_\delta:\mathbb{R}\to\mathbb{R}$ by

$$\widehat{r}_\delta(x)=\begin{cases}\min\{0,r_\delta(x)\} & \text{if } x\leq-(M_1+1)\\ r_\delta(x) & \text{if } -\left(M_1+\frac{1}{2}\right)\leq x\leq M_1+\frac{1}{2}\\ \max\{0,r_\delta(x)\} & \text{if } x\geq M_1+1\end{cases}$$

and on the remaining intervals $[-(M_1+1),-(M_1+\frac{1}{2})]$ and $[M_1+\frac{1}{2},M_1+1]$, we define linear connections. Then $\widehat{r}_\delta(\cdot)$ is continuous, for $|x|\geq M_1+1$ we have $x\widehat{r}_\delta(x)\geq 0$ and

$$\widehat{r}_\delta(x)\in\widehat{F}(t,x+B_\delta)+B_\delta$$

$$\Longrightarrow\widehat{r}_\delta(x)\in F(t,x+B_\delta)+B_\delta.$$

Now we will choose such a continuous approximate selector of F, which depends measurably on $t\in T$. To this end, define the multifunction $S_\varepsilon:T\to 2^{C(\mathbb{R})}$ by

$$S_\varepsilon(t)=\{r\in C(\mathbb{R}):r(x)\in F(t,x+B_\epsilon)+B_\epsilon, xr(x)\geq 0\text{ for all } |x|\geq M_1+1\}.$$

From the first part of the proof, we know that S_ϵ has nonempty values. Let $F^*(t,x)=F(t,x+\overline{B}_\varepsilon)+\overline{B}_\epsilon$. Clearly $F^*(t,\cdot)$ is usc and has compact and convex values (i.e. for every $(t,x)\in T\times\mathbb{R},F^*(t,x)$ is a bounded, closed interval). For every $y\in\mathbb{R}$ we have

$$\sigma(y,F^*(t,x))=\sup[\sigma(y,F(t,x+z)):z\in\overline{B}_\varepsilon]+\varepsilon.$$

Invoking Proposition II.2.23, p. 161 of Hu-Papageorgiou [10], we have that $t\to\sup[\sigma(y,F(t,x+z)):z\in\overline{B}_\varepsilon]$ is measurable and then so is $t\to\sigma(y,F^*(t,x))$ from which we infer the Lebesgue measurability of $t\to F^*(t,y)$. We have

$$GrS_\epsilon=\{(t,r)\in T\times C(\mathbb{R}):\mathrm{d}(r(x),F^*(t,x))=0\text{ for all } x\in\mathbb{R},\ xr(x)\geq 0\text{ for } |x|\geq M_1+1\}.$$

Let $\{a_n\}_{n\geq 1} \subseteq \mathbb{R}$ be an enumeration of the rationals in the real line and $\{c_m\}_{m\geq 1}$ an enumeration of the rationals in $|x| \geq M_1+1$. Also note that since $F^*(t,\cdot)$ is usc, for every $r \in C(\mathbb{R}), x \to \mathrm{d}(r(x), F^*(t,x))$ is lower semicontinuous. Indeed, suppose that $x_n \to x$ in $\mathbb{R}$. For all $n \geq 1$ we have

$$\begin{aligned} & h^*(F^*(t,x_n), F^*(t,x)) \geq \mathrm{d}(r(x_n), F^*(t,x)) - \mathrm{d}(r(x_n), F^*(t,x_n)) \\ \Longrightarrow\ & \mathrm{d}(r(x_n), F^*(t,x_n)) \geq \mathrm{d}(r(x_n), F^*(t,x)) - h^*(F^*(t,x_n), F^*(t,x)) \end{aligned}$$

Since $F^*(t,\cdot)$ is usc, it is also h-usc and so $h^*(F^*(t,x_n), F^*(t,x)) \to 0$ as $n \to \infty$. Hence $\mathrm{d}(r(x), F^*(t,x)) = \lim_{n\to\infty} \mathrm{d}(r(x_n), F^*(t,x)) \leq \underline{\lim}_{n\to\infty} \mathrm{d}(r(x_n), F^*(t,x_n))$ $\Longrightarrow x \to \mathrm{d}(r(x), F^*(t,x))$ is lower semicontinuous as claimed.

Exploiting this fact, we can write

$$GrS_\varepsilon = \bigcap_{n\geq 1, m\geq 1} \{(t,r) \in T \times C(\mathbb{R}) : \mathrm{d}(r(a_n), F^*(t,a_n)) = 0, c_m r(c_m) \geq 0\}.$$

Note that for every $n \geq 1, (t,r) \to \mathrm{d}(r(a_n), F^*(t,a_n))$ is a Caratheodory function on $T \times C(\mathbb{R})$ (on $C(\mathbb{R})$ we consider the topology of uniform convegence on compacta which makes it a separable Frechet space). So the function is jointly measurable (see Hu-Papageorgiou [10], Proposition II.1.6, p. 142). Therefore $GrS_\varepsilon \in \mathcal{L}(T) \times B(C(\mathbb{R}))$, with $\mathcal{L}(T)$ being the Lebesgue σ-field of T and $B(C(\mathbb{R}))$ the Borel σ-field of the space $C(\mathbb{R})$. Apply the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [10], Theorem II.2.14, p. 158) and obtain $\widehat{g} : T \to C(\mathbb{R})$ a Lebesgue measurable map such that $\widehat{g}_\varepsilon(t) \in S_\varepsilon(t)$ for all $t \in T$. If we set $g_\varepsilon(t,x) = \widehat{g}_\varepsilon(t)(x)$, then clearly $g_\varepsilon(t,x)$ is the desired Caratheodory selector. ∎

Let $A : W_0^{1,p}(T) \to 2^{L^q(T)}$ be defined by

$$A(x) = S^q_{a(\cdot,x(\cdot),x'(\cdot))}$$

and let $\alpha : W_0^{1,p}(T) \to 2^{W^{-1,q}}(T)$ be defined by

$$\alpha(x) = \{-g' : g \in A(x)\}$$

and for every $x \in W_0^{1,p}(T)$, let $V_x : W_0^{1,p}(T) \to 2^{W^{-1,q}}(T)$ be defined by

$$V_x(v) = \left\{-w' : w \in S^q_{a(\cdot,x(\cdot),v'(\cdot))}\right\}.$$

In both these operators, the derivatives involved in their definitions are defined in the sense of distributions. We will show that α is pseudomonotone. For this we need the following intermediate lemma about the operator V_x.

Lemma 3.2. *If hypotheses $H(a)$ hold, then $V_x : W_0^{1,p}(T) \to 2^{W^{-1,q}(T)}$ is maximal monotone.*

Proof. By virtue of hypotheses $\mathbf{H}$(a)(i) and (iii) for every $v \in W^{1,p}(T), S^q_{a(\cdot,x(\cdot),v'(\cdot))} \in P_{wkc}(W^{-1,q}(T))$. So in order to prove the maximal monotonicity of $V_x(\cdot)$, it suffices to show that for all $u \in W_0^{1,p}(T), \lambda \to V_x(v+\lambda u)$ is usc from $[0,1]$ into $W^{-1,q}(T)_w$ (by $W^{-1,q}(T)_w$ we denote the space $W^{-1,q}(T)$ furnished with the weak topology). Since $V_x(\cdot)$ is bounded and $W^{-1,q}(T)$ is reflexive, because of Proposition I.2.23, p. 43, of Hu-Papageorgiou [10], we know that it suffices to show that if $\lambda_n \to \lambda, \eta_n \xrightarrow{w} \eta$ in $W^{-1,q}(T)$

and $\eta_n \in V_x(v+\lambda_n u)$, then $\eta \in V_x(v+\lambda u)$. We have $\eta_n = -g_n'$, with $g_n \in L^q(T)$ and satisfies

$$g_n(t) \in a(t, x(t), (v+\lambda_n u)'(t)) \qquad \text{a.e on } T.$$

Because of hypothesis $\mathbf{H}(a)$(iii) and since $\eta_n \xrightarrow{w} \eta$ in $W^{-1,q}(T)$, we infer that $g_n \xrightarrow{w} g$ in $L^q(T)$ with $-g' = \eta$. From Proposition VII. 3.9, p. 694, of Hu-Papageorgiou [10], we obtain

$$g(t) \in \overline{\text{conv}}\,\overline{\lim}\, a(t, x(t), (v+\lambda_n u)'(t)) \subseteq a(t, x(t), (v+\lambda u)'(t)) \quad \text{a.e on } T,$$

the last inclusion following from the convexity of the values of a and since $a(t, x(t), \cdot)$ has closed graph (since it is maximal monotone). Hence $g \in S^q_{a(\cdot, x(\cdot), (v_\lambda + u)'(\cdot))}$, from which it follows that $\eta \in V_x(v+\lambda u)$. Therefore $\lambda \to V_x(v+\lambda u)$ is usc from $[0,1]$ into $W^{-1,q}(T)_w$ and so $V_x(\cdot)$ is maximal monotone. ∎

Using this intermediate result we can now prove the pseudomonotonicity of α.

Proposition 3.3. *If hypotheses $H(a)$ hold, then $\alpha : W_0^{1,p}(T) \to 2^{W^{-1,q}(T)}$ is pseudomonotone.*

Proof. Note that because of hypotheses $\mathbf{H}(a)$(i) and (iii), $S^q_{a(\cdot, x(\cdot), x'(\cdot))} \in P_{wkc}(L^q(T))$ and so we see that α has nonempty, weakly compact and convex values. Moreover, α is bounded, So in order to show the pseudomonotonicity of α, it suffices to show that α is generalized pseudomonotone. To this end we take $a_n \in \alpha(x_n), n \geq 1$, and assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(T)$, $a_n \xrightarrow{w} a$ in $W^{-1,q}(T)$ and $\overline{\lim}_{n\to\infty} \langle a_n, x_n - x\rangle \leq 0$, with $\langle \cdot, \cdot \rangle$ being the duality brackets for the pair $(W_0^{1,p}(T), W^{-1,q}(T))$. By definition we have $a_n = -g_n'$, with $g_n \in S^q_{a(\cdot, x_n(\cdot), x_n'(\cdot))}$. From hypothesis $\mathbf{H}(a)$(iii) we have

$$\begin{aligned} |g_n(t)| \ &\leq \ \gamma(t) + c(|x_n(t)|^{p-1} + |x_n'(t)|^{p-1}) \text{ a.e on } T, \\ &\Longrightarrow \{g_n\}_{n\geq 1} \subseteq L^q(T) \text{ is bounded.} \end{aligned}$$

Passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} g$ in $L^q(T)$. For every $z \in W_0^{1,p}(T)$ we have

$$\begin{aligned} \langle -g_n', z\rangle = \int_0^b g_n z' \,\mathrm{d}t \to \int_0^b g z' \,\mathrm{d}t = \langle -g', z\rangle \\ \Longrightarrow a_n \xrightarrow{w} -g' \text{ in } W^{-1,q}(T), \quad \text{i.e.} \quad a = g'. \end{aligned}$$

First we will show that $g(t) \in a(t, x(t), x'(t))$ a.e on T. For this purpose let $v \in W_0^{1,p}(T)$. Then by virtue of hypothesis $H(a)(i), t \to a(t, x(t), v'(t))$ is measurable and so by the Yankov-von Neumann-Aumann selection theorem, we can find $w : T \to \mathbb{R}$ a measurable map such that $w(t) \in a(t, x(t), v'(t))$ for all $t \in T$, i.e. $w \in S^q_{a(\cdot, x(\cdot), v'(\cdot))} \neq \emptyset$. Also for every $n \geq 1$ let

$$\Gamma_n(t) = \{u \in a(t, x_n(t), v'(t)) : |w(t) - u| = d(w(t), a(t, x_n(t), v'(t)))\}.$$

Because $t \to a(t, x_n(t), v'(t))$ is measurable, the function $(t,u) \to \eta_n(t,u) = |w(t) - u|$ $-$ $\mathrm{d}(w(t), a(t, x_n(t), v'(t)))$ is Caratheodory, thus jointly measurable. Hence $Gr\Gamma_n \in$

$\mathcal{L}(T) \times B(\mathbb{R})$ and we can apply once again the Yankov-von Neumann-Aumann selection theorem and obtain $w_n \in S^q_{a(\cdot,x_n(\cdot),v'(\cdot))}$ $n \geq 1$, such that $w_n(t) \in \Gamma_n(t)$ a.e on T. Therefore we have

$$\begin{aligned} |w(t) - w_n(t)| &= \mathrm{d}(w(t), a(t, x_n(t), v'(t))) \\ &\leq h^*(a(t, x(t), v'(t)), a(t, x_n(t), v'(t))) \quad \text{a.e on } T \end{aligned}$$

and because $x_n \to x$ in $C(T)$ (from the compact embedding of $W_0^{1,p}(T)$ into $C(T)$) and from the lower semicontinuity of $a(t, \cdot, v'(t))$ (hypothesis $\mathbf{H}(a)$(ii)), we have that

$$\begin{aligned} &h^*(a(t, x(t), v'(t)), a(t, x_n(t), v'(t))) \to 0 \quad \text{a.e on } T, \\ \Longrightarrow\ &|w(t) - w_n(t)| \to 0 \quad \text{a.e on } T. \end{aligned}$$

So by the dominated convergence theorem, we have $w_n \to w$ in $L^q(T)$ and then $w_n' \to w'$ in $W^{-1,q}(T)$. From the monotonicity of $a(t, x_n(t), \cdot)$ we have

$$\begin{aligned} 0 &\leq \int_0^b (g_n(t) - w_n(t))(x_n'(t) - v'(t))\,\mathrm{d}t \\ &= \int_0^b g_n(t)(x_n'(t) - x'(t))\,\mathrm{d}t + \int_0^b g_n(t)(x'(t) - v'(t))\,\mathrm{d}t \\ &\quad + \int_0^b w_n(t)(v'(t) - x_n'(t))\,\mathrm{d}t \\ &= \langle a_n, x_n - x\rangle + \int_0^b g_n(t)(x - v)'(t)\,\mathrm{d}t + \int_0^b w_n(t)(v - x_n)'(t)\,\mathrm{d}t \end{aligned}$$

Passing to the limit and using the fact that $\overline{\lim}_{n\to\infty} \langle a_n, x_n - x\rangle \leq 0$, we obtain

$$\begin{aligned} &0 \leq \int_0^b g(t)(x - v)'(t)\,\mathrm{d}t + \int_0^b w(t)(v - x)'(t)\,\mathrm{d}t, \\ \Longrightarrow\ &0 \leq \langle -g' + w', v - x\rangle \quad \text{for all } (v, w) \in GrV_x. \end{aligned}$$

Invoking Lemma 3.2, we deduce that $-g' \in V_x(x')$, hence $g(t) \in a(t, x(t), x'(t))$ a.e on T. Therefore $a_n \xrightarrow{w} a = -g'$ in $W^{-1,q}(T)$ with $g \in S^q_{a(\cdot,x(\cdot),x'(\cdot))}$, hence $a \in \alpha(x)$. To finish the proof, it remains to show that $\langle a_n, x_n\rangle \to \langle a, x\rangle$. As above let $w_n \in S^q_{a(\cdot,x_n(\cdot),x'(\cdot))}$ such that $w_n \to g$ in $L^q(T)$. We have

$$\begin{aligned} &\int_0^b g_n(t)(x_n - x)'(t)\,\mathrm{d}t = \int_0^b (g_n - w_n)(t)(x_n - x)'(t)\,\mathrm{d}t \\ &\quad + \int_0^b w_n(t)(x_n - x)'(t)\,\mathrm{d}t \\ \Longrightarrow\ &\langle a_n, x_n - x\rangle \geq \int_0^b w_n(t)(x_n - x)'(t)\,\mathrm{d}t \ \ \text{(since } a(t, x_n(t), \cdot) \text{ is monotone)} \\ \Longrightarrow\ &\underline{\lim}_{n\to\infty} \langle a_n, x_n - x\rangle \geq 0. \end{aligned}$$

On the other hand by hypothesis $\overline{\lim}_{n\to\infty} \langle a_n, x_n - x\rangle \leq 0$ and so finally $\langle a_n, x_n\rangle \to \langle a, x\rangle$, which finishes the proof of the proposition. ∎

Now we consider the Caratheododory selector $g_\varepsilon(t,x)$ from PROPOSITION 3.1 and define its truncation at level $k>0$:

$$g_\varepsilon^k(t,x)=\begin{cases} g_\varepsilon(t,x) & \text{if } \ |g_\varepsilon(t,x)|\le k\\ k\,\mathrm{sgn}(g_\varepsilon(t,x)) & \text{if } \ |g_\varepsilon(t,x)|>k\end{cases}$$

In what follows by N_k^ε we denote the Nemitsky operator corresponding to the trunceted function g_ε^k,i.e. $N_k^\varepsilon(x)(\cdot)=g_\varepsilon^k(\cdot,x(\cdot))$. Evidently $N_k^\varepsilon : L^p(T)\to L^q(T)$ is continuous and $N_k^\varepsilon|_{W_0^{1,p}} : W_0^{1,p}(T)\to W_0^{-1,q}(T)$ is compact and has values in $L^\infty(T)$.

Since by hypothesis $H(\varphi),\varphi$ is $\mathbb{R}$-valued, it is continuous (in fact locally Lipschitz) and so for every $x\in W_0^{1,p}(T)\subseteq C(T),\varphi(x(\cdot))\in L^\infty(T)$. So we can define the function $\Phi : W_0^{1,p}(T)\to\mathbb{R}$ by $\Phi(x)=\int_0^b\varphi(x(t))\,\mathrm{d}t$. Evidently Φ is continuous, convex. Therefore it is subdifferentiable everywhere, i.e. $D(\partial\Phi)=\left\{x\in W_0^{1,p}(T):\partial\Phi(x)\ne\emptyset\right\}=W_0^{1,p}(T)$. For every $\lambda>0$ we consider the Moreau-Yosida approximation of Φ, which is continuous, convex and Gateaux differentiable. Moreover, $\partial\Phi_\lambda=(\partial\Phi)_\lambda$(= the Yosida approximation of the maximal monotone map $\partial\Phi$).

Finally let $K : W_0^{1,p}(T)\to W_0^{-1,q}(T)$ be defined by

$$Kx=-\beta x'$$

Exploiting the compact embedding of $L^p(T)$ into $W^{-1,q}(T)$, we see that K is a compact linear operator. Consider the following auxiliary operator inclusion:

$$\alpha(x)+K(x)+\partial\Phi_\lambda(x)+N_k^\varepsilon(x)\ni -h \tag{3.1}$$

In the next proposition we obtain a solution for (3.1)

PROPOSITION 3.4. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(\varphi)$, $\mathbf{H}(F)$ *hold and* $h\in L^1(T)$, *then the operator inclusion* (3.1) *has a solution* $x\in W_0^{1,p}(T)$.

Proof. By PROPOSITION 3.3, α is pseudotonotone. Also we already mentioned that $K\in\mathcal{L}(W_0^{1,p}(T),W^{-1,q}(T))$ is compact. Hence $\alpha_1=\alpha+K : W_0^{1,p}(T)\to W^{-1,q}(T)$ is pseudomonotone. Also $\partial\Phi_\lambda : W_0^{1,p}(T)\to W^{-1,q}(T)$ is maximal monotone and earlier we pointed out that $N_k^\varepsilon : W_0^{1,p}(T)\to W^{-1,q}(T)$ is compact. Since maximal monotone maps defined everywhere and compact maps are pseudomonotone and the sum of pseudomonotone maps is pseudomonotone (see Hu-Papageorgiou [10], pp. 365–368) it follows that $x\to U(x)=(\alpha_1+\partial\Phi_\lambda+N_k^\varepsilon)(x)$ is pseudomonotone. Also for every $x\in W_0^{1,p}(T)$ and every $a=-g'\in\alpha(x)$ with $g\in A(x)$, we have

$$\langle -g',x\rangle-\beta(x,x')_{pq}+\langle\partial\Phi_\lambda(x),x\rangle+(N_k^\varepsilon(x),x)_{pq}=\langle U(x),x\rangle$$

where by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(T),L^q(T))$. From integration by parts and using hypothesis $\mathbf{H}(a)$(iv), we have

$$\langle -g',x\rangle=\int_0^b g(t)x'(t)\,\mathrm{d}t\ge c_1\left\|x'\right\|_p^p-\left\|\gamma_1\right\|_1.$$

Also $\beta(x,x')_{pq}=\beta\int_0^b x(t)x'(t)\,\mathrm{d}t=\frac{\beta}{2}\int_0^b\frac{d}{\mathrm{d}t}x(t)^2\,\mathrm{d}t=0$. In addition since by hypothesis $H(\varphi),0\in\partial\varphi(0)$, we deduce that $\partial\Phi_\lambda(0)=0$ and so $\langle\partial\Phi_\lambda(x),x\rangle\ge 0$. Finally

$$(N_k^\varepsilon(x),x)_{pq}=\int_o^b N_k^\varepsilon(x)(t)x(t)\,\mathrm{d}t\le\left\|N_k^\varepsilon(x)\right\|_1\left\|x\right\|_\infty\le\vartheta_1(k)\left\|x\right\|$$

for some $\theta_1(k) > 0$. Therefore finally we can write that for all $u \in U(x)$

$$\langle u, x\rangle \geq c_1 \|x'\|_p^p - \vartheta_1(k) \|x\| . \tag{3.2}$$

Recalling that $\|x'\|_p^p$ is an equivalent norm on $W_0^{1,p}(T)$, from (3.2) it follows that $U(\cdot)$ is coercive. But a pseudomonotone, coercive operator is surjective. Since $h \in L^1(T) \subseteq W^{-1,q}(T)$, we see that there exists $x \in W_0^{1,p}(T)$ such that $-h \in U(x)$ ∎

4. Existence theorem. In this section, using the preparatory work of Section 3, we prove the following existence theorem for problem (1.1).

THEOREM 4.1. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(\varphi)$, $\mathbf{H}(F)$ *hold and* $h \in L^1(T)$, *then problem* (1.1) *has a solution* $x \in W_0^{1,p}(T)$.

Proof. Let $\lambda_k, \varepsilon_k \downarrow 0$ and let $x_k \in W_0^{1,p}(T)$ be solutions of

$$\alpha(x) + K(x) + \partial\Phi_{\lambda_k}(x) + N_k^{\varepsilon_k}(x) \ni -h, k \geq 1.$$

We take the duality brackets with x_k. So we have

$$\langle -g_k', x_k\rangle + (K(x_k), x_k)_{pq} + (\partial\Phi_{\lambda_k}(x_k), x_k)_{pq} + (N_k^{\varepsilon_k}(x_k), x_k)_{pq} = (-h, x_k)_{1,\infty}$$

where $g_k \in A(x_k)$. As before (proof of PROPOSITION 3.4), we have

$$\langle -g_k', x_k\rangle = \int_0^b g_k(t)x_k'(t)\,\mathrm{d}t \geq c_1 \|x_k'\|_p^p - \|\gamma_1\|_1$$
$$(K(x_k), x_k)_{pq} \geq 0 \ \text{ and } (\partial\Phi_{\lambda_k}(x_k), x_k)_{pq} \geq 0.$$

Thus we can write that

$$c_1 \|x_k'\|_p^p - \|\gamma_1\|_1 + \int_0^b N_k^{\varepsilon_k}(x_k)(t)x_k(t)\,\mathrm{d}t \leq \|h\|_1 \|x_k\|_\infty . \tag{4.1}$$

Let $T(k) = \{t \in T : |x_k(t)| \leq M_1 + 1 = M_2\}$. We have

$$\begin{aligned}\int_0^b N_k^{\varepsilon_k}(x_k)(t)x_k(t)\,\mathrm{d}t &= \int_{T(k)} N_k^{\varepsilon_k}(x_k)(t)x_k(t)\,\mathrm{d}t + \int_{T\setminus T(k)} N_k^{\varepsilon_k}(x_k)(t)x_k(t)\,\mathrm{d}t\\ &\geq \int_{T(k)} N_k^{\varepsilon_k}(x_k)(t)x_k(t)\,\mathrm{d}t \text{ (see PROPOSITION 3.1)}\\ &\geq -\vartheta_2 > 0 \qquad (4.2)\\ &\qquad \text{hypothesis } \mathbf{H}(F)\text{(iii) and PROPOSITION 3.1).}\end{aligned}$$

for some ϑ_2. Using this in (4.1) and from the Poincare inequality, we obtain

$$c_1 \|x_k'\|_p^p \leq \theta_3 \|x_k'\| + \vartheta_4 \ \text{ for some } \ \vartheta_3, \vartheta_4 > 0 \text{ independent of } k \geq 1.$$
$$\Longrightarrow \{x_k\}_{k\geq 1} \subseteq W_0^{1,p}(T) \text{ is bounded.}$$

So by passing to a subsequence if necessary, we may assume that $x_k \xrightarrow{w} x$ in $W_0^{1,p}(T)$. Now let $T' \subseteq T$ be a measurable subset and let $\mu > M_2 = M_1 + 1$. Set $T_{1k}' = \{t \in T' : |x_k(t)| \leq \mu\}$, $T_{2k}' = \{t \in T' : |x_k(t)| > \mu\}$ and $T_k = \{t \in T : |x_k(t)| > M_2\}$.

Evidently $T'_{2k} \subseteq T_k$. Since $\mu > M_2$, we have

$$\begin{aligned}\int_{T'} |N_k^{\varepsilon_k}(x_k)(t)| \,\mathrm{d}t &= \int_{T'_{1k}} |N_k^{\varepsilon_k}(x_k)(t)| \,\mathrm{d}t + \int_{T'_{2k}} |N_k^{\varepsilon_k}(x_k)(t)| \,\mathrm{d}t \\ &\leq \int_{T'} \xi_\mu(t) \,\mathrm{d}t + \int_{T'_{2k}} |N_k^{\varepsilon_k}(x_k)(t)| \,\mathrm{d}t \\ &\leq \int_{T'} \xi_\mu(t) \,\mathrm{d}t + \frac{1}{\mu} \int_{T'_{2k}} N_k^{\varepsilon_k}(x_k)(t) x_k(t) \,\mathrm{d}t \\ &\leq \int_{T'} \xi_\mu(t) \,\mathrm{d}t + \frac{1}{\mu} \int_{T_k} N_k^{\varepsilon_k}(x_k)(t) x_k(t) \,\mathrm{d}t \\ &= \int_{T'} \xi_\mu(t) \,\mathrm{d}t + \frac{1}{\mu} [\int_{T} N_k^{\varepsilon_k}(x_k)(t) x_k(t) \,\mathrm{d}t - \int_{T \setminus T_k} N_k^{\varepsilon_k}(x_k)(t) x_k(t) \,\mathrm{d}t].\end{aligned}$$

Directly from the equation and because $\{x_k\}_{k\geq 1} \subseteq W_0^{1,p}(T)$ is bounded, we can write that

$$\int_T N_k^{\varepsilon_k}(x_k)(t) x_k(t) \,\mathrm{d}t \leq \vartheta_5 \text{ for some } \vartheta_5 > 0 \text{ independent of } \; k \geq 1.$$

So we can write that

$$\int_{T'} |N_k^{\varepsilon_k}(x_k)(t)| \,\mathrm{d}t \leq \int_{T'} \xi_\mu(t) \,\mathrm{d}t + \frac{1}{\mu}[\vartheta_5 + \vartheta_2] \text{ (see(4.2))}$$

Because $\mu > M_2$ was arbitary and $\xi_\mu \in L^1(T)$, it follows that

$$\begin{gathered}\lim_{|T'| \downarrow 0} \sup_{k \geq 1} \int_{T'} |N_k^{\varepsilon_k}(x_k)(t) \,\mathrm{d}t| = 0 \\ \Longrightarrow \{N_k^{\varepsilon_k}(x_k)(\cdot)\}_{k \geq 1} \text{ is uniformly integrable.}\end{gathered}$$

Therefore by the Dunford-Pettis theorem and by passing to a subsequence if necessary, we may assume that $N_k^{\varepsilon_k}(x_k) \xrightarrow{w} v$ in $L^1(T)$ and in $W^{-1,q}(T)$, since $L^1(T)$ is embedded continuously in $W^{-1,q}(T)$

Because $g_k \in A(x_k)$, from hypothesis $H(a)(iii)$, we have that $\{g_k\}_{k\geq 1} \subseteq L^q(T)$ is bounded. Hence $\{g'_k\}_{k\geq 1} \subseteq W^{-1,q}(T)$ is bounded. Thus by passing to a subsequence if necessary, we may assume that $g_k \xrightarrow{w} g$ in $L^q(T)$ and $g'_k \xrightarrow{w} z$ in $W^{-1,q}(T)$. Clearly $z = g'$. Because

$$-g'_k - \beta x'_k + \partial\Phi_{\lambda_k}(x_k) + N_k^{\varepsilon_k}(x_k) = -h, \; k \geq 1$$

we infer that

$$\partial\Phi_{\lambda_k}(x_k) \xrightarrow{w} u \text{ in } W^{-1,q}(T)$$

(recall $x_k \xrightarrow{w} x$ in $W_0^{1,p}(T)$). Also from Hu-Papageorgiou [10] (Proposition III.2.29(c), p. 325), for all $k \geq 1$ and all $y \in W_0^{1,p}(T)$, we have

$$\|\partial\Phi_{\lambda_k}(y)\| \leq \left\|\partial\Phi^0(y)\right\| \tag{4.3}$$

$$\text{and } \partial\Phi_{\lambda_k}(y) \xrightarrow{w} \partial\Phi^0(y) \text{ in } W^{-1,q}(T) \text{ as } k \to \infty \tag{4.4}$$

Recall that $\partial\Phi^0(y) = w \in \partial\Phi(y)$ such that $\|w\| = \inf[\|\overline{w}\| : \overline{w} \in \partial\Phi(y)]$. From (4.3) with $y = x$, we have that

$$\overline{\lim}_{k\to\infty} \|\partial\Phi_{\lambda_k}(x)\| \leq \left\|\partial\Phi^0(x)\right\|, \tag{4.5}$$

while from (4.4) and the weak lower semicontinuity of the norm, we have

$$\underline{\lim}_{k\to\infty} \|\partial\Phi_{\lambda_k}(x)\| \geq \left\|\partial\Phi^0(x)\right\|. \tag{4.6}$$

From (4.5) and (4.6), we have that $\|\partial\Phi_{\lambda_k}(x)\| \to \left\|\partial\Phi^0(x)\right\|$ and since $W^{-1,q}(T)$ in uniformly convex, from the Kadec-Klee property, we have that $\partial\Phi_{\lambda_k}(x) \to \partial\Phi^0(x)$ in $W^{-1,q}(T)$.

For every $k \geq 1$ we have

$$\begin{aligned}\langle -g_k', x_k - x\rangle - \beta(x_k', x_k - x)_{pq} + \langle \partial\Phi_{\lambda_k}(x_k), x_k - x\rangle + (N_k^{\varepsilon_k}(x_k), x_k - x)_{pq} \\ = (-h, x_k - x)_{1\infty}\end{aligned} \tag{4.7}$$

Exploiting the monotonicity of $\partial\Phi_{\lambda_k}$, we have

$$\begin{aligned}\langle -g_k', x_k - x\rangle - \beta(x_k', x_k - x)_{pq} + \langle \partial\Phi_{\lambda_k}(x), x_k - x\rangle + (N_k^{\varepsilon_k}(x_k), x_k - x)_{pq} \\ \leq (-h, x_k - x)_{1\infty}.\end{aligned}$$

Observe that $(x_k', x_k - x)_{pq} \to 0, \langle \partial\Phi_{\lambda_k}(x), x_k - x\rangle \to 0, (N_k^{\varepsilon_k}(x_k), x_k - x)_{pq} \to 0$ and $(-h, x_k - x)_{1\infty} \to 0$ (in the last convergence we have used the fact that $x_k \to x$ in $C(T)$ from the compact embedding of $W_0^{1,p}(T)$ into $C(T)$). So we have

$$\overline{\lim}_{k\to\infty} \langle -g_k', x_k - x\rangle \leq 0.$$

From Proposition 3.3, we know that α is pseudomonotone. So from the above inequality, we infer that $-g' \in \alpha(x)$, hence $g(t) \in a(t, x(t), x'(t))$ a.e on T and $\langle -g_k', x_k\rangle \to \langle -g', x\rangle$. Since $\partial\Phi_{\lambda_k} = (\partial\Phi)_{\lambda_k}$, we have that

$$\lambda_k \|\partial\Phi_{\lambda_k}(x_k)\|_* = \|x_k - J_{\lambda_k}(x_k)\|$$

where $\|\cdot\|_*$ denotes the norm in $W^{-1,q}(T)$ and J_{λ_k} the resolvent operator (see Hu-Papageorgiou [10], p. 325). Since $\{\partial\Phi_{\lambda_k}(x_k)\}_{k\geq 1} \subseteq W^{-1,q}(T)$ is bounded, we deduce that $\|x_k - J_{\lambda_k}(x_k)\| \to 0$ as $k \to \infty$. Therefore $J_{\lambda_k}(x_k) \xrightarrow{w} x$ in $W_0^{1,p}(T)$. Moreover, since $\langle -g_k', x_k\rangle \to \langle -g', x\rangle$, from (4.7) we obtain that

$$\lim_{k\to\infty} \langle \partial\Phi_{\lambda_k}(x_k), x_k - x\rangle = 0$$

We have that

$$\langle \partial\Phi_{\lambda_k}(x_k), x_k - x\rangle = \langle \partial\Phi_{\lambda_k}(x_k), x_k - J_{\lambda_k}(x_k)\rangle + \langle \partial\Phi_{\lambda_k}(x_k), J_{\lambda_k}(x_k) - x\rangle.$$

Observe that $|\langle \partial\Phi_{\lambda_k}(x_k), x_k - J_{\lambda_k}(x_k)\rangle| \leq \|\partial\Phi_{\lambda_k}(x_k)\|_* \|x_k - J_{\lambda_k}(x_k)\| \to 0$. So it follows that

$$\lim_{k\to\infty} \langle \partial\Phi_{\lambda_k}(x_k), J_{\lambda_k}(x_k) - x\rangle = 0$$

Recall that $\partial\Phi_{\lambda_k}(x_k) \in \partial\Phi(J_{\lambda_k}(x_k))$ and $J_{\lambda_k}(x_k) \xrightarrow{w} x$ in $W_0^{1,p}(T)$. But $\partial\Phi$ being maximal monotone, it is generalized pseudomonotone. So because $\partial\Phi_{\lambda_k}(x_k) \xrightarrow{w} u$ in $W^{-1,q}(T)$ we have $u \in \partial\Phi(x)$. Therefore in the limit as $k \to \infty$ we obtain

$$-g' - \beta x' + u + v = -h$$

with $g \in A(x), u \in \partial\Phi(x)$. Moreover, using Proposition VII. 3.9, p. 624 of Hu-Papageorgiou [10], we can easily check that $v \in S^1_{F(\cdot,x(\cdot))}$. Thus $x \in W^{1,p}(Z)$ solves (1.1). ∎

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