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NUMERICAL APPROXIMATION OF SINGULAR BOUNDARY VALUE PROBLEMS FOR A NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this work we are concerned about singular boundary value problems for certain nonlinear second order ordinary differential equations on finite and infinite domains. An asymptotic expansion is obtained for the family of solutions satisfying the boundary condition at the origin. In the case of infinite domains, the asymptotic behavior of the solutions is also analysed for large values of the independent variable. Based on the asymptotic expansions, computational methods are introduced to approximate the solution of the problem.

Key words. nonlinear singular boundary value problem, bubble-type solution, asymptotic approximation, one-parameter family of solutions, shooting method

AMS subject classifications. 65L05

1. Introduction. In this work, we are concerned about the nonlinear second order differential equation

$$y''(x) + \tfrac{N-1}{x} y'(x) = c(x) f(y), \tag{1.1}$$

where $c(x)$ is a continuous function in $[0, +\infty[$, there exist real numbers $C_{\max}$ and $C_{\min}$ such that $0 < C_{\min} \leq c(x) \leq C_{\max}$, $\forall x \in [0, +\infty[$ and $L = \lim_{x \to +\infty} c(x) \neq 0$, f is a polynomial, such that $f(0) = 0$. We search for solutions of this equation which satisfy the boundary conditions:

$$y'(0) = 0, \quad y(M) = 0, \quad M \in \mathbb{R}^+, \tag{1.2}$$

or

$$y'(0) = 0, \quad \lim_{x \to +\infty} y(x) = 0. \tag{1.3}$$

These two boundary value problems have been studied in [3]. They arise when looking for radial solutions of the elliptic equation $\triangle y = c(|x|) f(y)$ in a ball $B(0, M) \subset \mathbb{R}^N$ or in all $\mathbb{R}^N$, respectively ($N > 1$). The case when $c(x) \equiv 1$ has also been studied in [2] and [1]. A more general class of problems has been analysed in [7], where the linear operator on the left-hand side of Eq. (1.1) has been replaced by

$$(|y'|^{m-2} y')' + \frac{N-1}{x} |y'|^{m-2} y',$$

which represents the radial part of the so-called degenerate laplacian (it coincides with the usual laplacian in the case $m = 2$). An equation with an even more general operator was analysed in [6]. In all these papers, sufficient conditions where imposed on f that

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guarantee the existence of at least one positive solution to the considered problems. A theorem about uniqueness of solution was also presented in [6].

A related problem arises in hydrodynamics when modeling the formation of microscopical bubbles in a non-homogeneous fluid. In this case the density of the fluid ρ satisfies the so-called density profile equation (see [4]). This equation was studied in [10] and [11], where it was written in the form:

$$\rho''(r) + \frac{N-1}{r}\rho'(r) = 4\lambda^2(\rho+1)\rho(\rho-\xi),\ 0 < r < \infty, \tag{1.4}$$

where $\lambda > 0$, $N > 1$. The authors investigated strictly increasing solutions of Eq. (1.4) which satisfy the boundary conditions

$$\lim_{r\to 0+} \rho'(r) = 0, \tag{1.5}$$

$$\lim_{r\to\infty} \rho(r) = \xi. \tag{1.6}$$

As we shall see, this problem can be reduced to a particular case of (1.1). The theoretical results of [7] can then be used to show that problem (1.4), (1.5), (1.6), with $0 < \xi < 1$, has at least one strictly monotone solution. With this purpose, let us set in (1.4)

$$\rho(r) = \xi - y(r).$$

Then the boundary conditions (1.5), (1.6) reduce to (1.3) and equation (1.4) reduces to (1.1), where

$$f(y) = 4\lambda^2 y(y-\xi)(y-\xi-1) \tag{1.7}$$

and $c(x) \equiv 1$. Moreover, it is easy to verify that, if $0 < \xi < 1$, the function f given by (1.7) satisfies the conditions of [7, Theorem 1].

On the other hand, it was shown in [11] that a solution of (1.4) with the considered properties exists only if $0 < \xi < 1$.

If in the considered problem we replace the boundary condition (1.6) by

$$\rho(M) = \xi \tag{1.8}$$

where $M > 0$, and $N < 4$, then, using again the substitution $\rho(r) = \xi - y(r)$, we obtain a problem of the type (1.1), (1.2). According to [3, Theorem 19 and Remark 5], if M is sufficiently large, this problem has at least one solution. Moreover, the mentioned theorem is still applicable if the right-hand side of equation (1.4) is multiplied by a function $c(r)$, such that $0 < C_{\min} \le c(r) \le C_{\max}$, $\forall r \in [0, +\infty[$.

In Section 2, we consider the mentioned above case of $f(y) = 4\lambda^2 y(y-\xi)(y-\xi-1)$ and extend the results obtained in [10] and [11] for nonconstant functions $c(r)$. In particular, we shall obtain, for this case, series expansions of the solutions of the considered boundary value problem near the singularities at zero and infinity.

In section 3, we consider $f(y) = y - y^3$. For this case, we also obtain series expansions of the solutions of (1.1) at zero and at infinity.

Numerical results for all the considered boundary value problems are presented in Section 4.

We finish this paper with some conclusions and remarks on future work.

2. The case $f(y) = 4\lambda^2 y(y-\xi)(y-\xi-1)$. As we have seen in the introduction, when the function on the right-hand side of our equation has this form, with $0 < \xi < 1$, and $c(x) \equiv 1$, the boundary value problem (1.1), (1.3) is equivalent to the problem, considered in [10] and [11]. In order to compare our results with those obtained in the cited works, we perform the variable substitution $y = \xi - \rho$. Then (1.1) can be rewritten as

$$\rho''(x) + \frac{N-1}{x}\rho'(x) = 4\lambda^2(\rho+1)\rho(\rho-\xi)c(x),\ 0 < x < \infty, \tag{2.1}$$

and the boundary conditions (1.2) and (1.3) become, respectively

$$\rho'(0) = 0, \tag{2.2}$$
$$\rho(M) = \xi \tag{2.3}$$

and

$$\rho'(0) = 0, \tag{2.4}$$
$$\lim_{x\to\infty} \rho(x) = \xi. \tag{2.5}$$

2.1. The singularity at zero. Let us consider equation (2.1) with initial conditions

$$\lim_{x\to 0^+} \rho(x) = \rho_0, \quad \lim_{x\to 0^+} x\rho'(x) = 0, \tag{2.6}$$

where ρ_0 is a real parameter. Note that this problem is singular at zero for all $N > 1$.

We shall assume that $c(x)$ is analytic in a neighborhood of 0 and can be expanded in the form

$$c(x) = \gamma_0 + \sum_{k=1}^{\infty} \gamma_k x^k, \qquad x \le \delta. \tag{2.7}$$

If we linearize equation (2.1) in the neighborhood of $x = 0$, taking (2.6) into account, we obtain an equation with a regular singularity at zero, with characteristic exponents $\lambda_1 = 0$ and $\lambda_2 = 2 - N$.

Therefore, according to [9, Theorem 4 and Theorem 5], the Cauchy problem (2.1), (2.6) for each value of ρ_0 has a unique solution that can be represented in the form of the series

$$\rho(x) = \rho_0 + \sum_{k=2}^{+\infty} \rho_k(\rho_0) x^k, \quad 0 \le x \le \delta, \quad \delta > 0. \tag{2.8}$$

The coefficients ρ_k can be determined substituting (2.8) in (2.1). For $k = 2, 3, 4$ we obtain the following formulae:

$$\begin{aligned}
\rho_2(\rho_0) &= \frac{2\lambda^2}{N}\rho_0(\rho_0+1)(\rho_0-\xi)\gamma_0, \\
\rho_3(\rho_0) &= \frac{4\lambda^2}{3N+3}\rho_0(\rho_0+1)(\rho_0-\xi)\gamma_1, \\
\rho_4(\rho_0) &= \frac{\lambda^2(-\gamma_2\xi\rho_0 + \gamma_2\rho_0^2 - \gamma_2\xi\rho_0^2 + \gamma_2\rho_0^3 - \gamma_0\xi\rho_2 + 2\gamma_0\rho_0\rho_2 - 2\gamma_0\xi\rho_0\rho_2 + 3\gamma_0\rho_0^2\rho_2)}{N+2},
\end{aligned} \tag{2.9}$$

where, for the sake of simplicity, we have replaced $\rho_2(\rho_0)$ by ρ_2. Hence, we conclude that each solution of the form (2.8) satisfies the condition (2.4) and therefore the singular Cauchy problem (2.1), (2.4) has a one parameter family of solutions.

EXAMPLE 2.1.1. Considering, for example, $c(x) = \cos\left(\frac{1}{2+x^2}\right)$, we have $c(x) = \cos(\frac{1}{2}) + \frac{1}{4}\sin(\frac{1}{2})x^2 + O(x^4)$, as $x \to 0$, and therefore $\gamma_0 = \cos(\frac{1}{2})$ and $\gamma_2 = \frac{1}{4}\sin(\frac{1}{2})$. Substituting these values in the right-hand side of formulae (2.9), we obtain the corresponding values of $\rho_k(\rho_0)$, $k = 2, 3, \ldots$.

2.2. The singularity at infinity. Let us now consider equation (2.1) with initial condition

$$\lim_{x\to+\infty}(\rho(x) - \xi) = \lim_{x\to+\infty}\rho'(x) = 0. \tag{2.10}$$

In order to analyze the asymptotic behavior of the solutions of (2.1) at infinity, where this equation has an irregular singular point, we perform the variable substitution $z = x^{\frac{N-1}{2}}(\rho - \xi)$. Then (2.1) becomes

$$z'' = 4\lambda^2\left(\frac{z}{x^{\frac{N-1}{2}}} + \xi + 1\right)\left(\frac{z}{x^{\frac{N-1}{2}}} + \xi\right)zc(x) + \frac{(N-1)(N-3)}{4x^2}z, \tag{2.11}$$

and (2.10) take the form

$$\lim_{x\to+\infty} z(x) = \lim_{x\to+\infty} z'(x) = 0. \tag{2.12}$$

If $N = 1$, equation (2.11) is an autonomous equation. When $N > 1$, this equation is asymptotically autonomous, this is, when $x \to +\infty$, we obtain an autonomous equation whose characteristic roots are

$$\tau_{1,2} = \pm\tau, \quad \tau = 2\lambda\sqrt{\xi(\xi+1)L}, \quad L = \lim_{x\to+\infty} c(x) > 0.$$

Following, as in [10] and [11], a method introduced by Lyapunov [12], the singular Cauchy problem (2.11),(2.12) has a one parameter family of solutions that can be represented by

$$z(x,b) = C_1(x)b\,\mathrm{e}^{-\tau x} + \sum_{k=2}^{+\infty} C_k(x)b^k\,\mathrm{e}^{-\tau kx}, \quad x \geq x_\infty, \tag{2.13}$$

where b is the parameter and $|b\,\mathrm{e}^{-\tau x_\infty}|$ is small. The coefficients $C_k(x)$ can be obtained substituting (2.13) in (2.11). For $k = 1$, we obtain the linear differential equation

$$C_1''(x) - 2\tau C_1'(x) = \left(\frac{(N-1)(N-3)}{4x^2} + (c(x) - L)\right)C_1(x), \tag{2.14}$$

and the initial conditions

$$\lim_{x\to+\infty} C_1(x) = 1, \quad \lim_{x\to+\infty} C_1'(x) = 0. \tag{2.15}$$

If $\int_{x_\infty}^{\infty}(c(x) - L)\,\mathrm{d}x < \infty$, according to [9, Theorem 3 and Note 3], problem (2.14), (2.15) has an unique solution, whose smoothness depends on the smoothness of $c(x)$. In particular, if $c(x)$ allows an expansion in the form

$$c(x) = L + \sum_{k=2}^{\infty}\frac{d_k}{x^k}, \qquad x > x_\infty, \tag{2.16}$$

the solution of the Cauchy problem (2.14), (2.15) may be expanded as

$$C_1(x) = 1 + \sum_{k=1}^{+\infty} \frac{b_k}{x^k}, \qquad x > x_\infty, \tag{2.17}$$

where the coefficients b_k may be determined substituting (2.17) and (2.16) in (2.14). After determining b_1 and b_2, we obtain the following expansion for C_1:

$$\begin{aligned} C_1(x) =& 1 + \frac{(N-1)(N-3) + 16 d_2 \lambda^2 \xi(\xi+1)}{8\tau x} \\ &+ \frac{((N-1)(N-3) + 16 d_2 \lambda^2 \xi(\xi+1))((N-5)(N+1)}{128 \tau^2 x^2} \\ &+ \frac{16 d_2 \lambda^2 \xi(\xi+1)) + 256 d_3 \lambda \xi (1+\xi)\sqrt{\xi(1+\xi)}}{128 \tau^2 x^2} + O\left(\frac{1}{x^3}\right), \end{aligned} \tag{2.18}$$

as $x \to +\infty$.

EXAMPLE 2.2.1. Considering, for example, $c(x) = \cos\left(\frac{1}{2+x^2}\right)$, we have $L = \lim_{x\to\infty} c(x) = 1$ and it is easy to verify that $\int_{x_\infty}^{\infty} (c(x) - 1)\,\mathrm{d}x < \infty$.

Moreover, in this case we have

$$c(x) = 1 - \frac{1}{2}\left(\frac{1}{x^4}\right) + O\left(\frac{1}{x^6}\right), \qquad x \to \infty,$$

and therefore $d_2 = 0$, $d_3 = 0$ and $d_4 = -\frac{1}{2}$. By replacing these coefficients in (2.18), we obtain the first terms of the expansion of C_1.

Concerning the higher terms of the expansion (2.13), we note that the coefficients C_k, $k = 2, 3, \ldots$ may be obtained in a similar way, by solving singular Cauchy problems, analogous to (2.14), (2.15). Finally, we can conclude that for each value of b the singular Cauchy problem (2.11), (2.12) has exactly one solution, which can be expanded in the form (2.13).

Returning now to the initial variable ρ, we conclude that the Cauchy problem (2.1), (2.10) has a one parameter family of solutions that can be represented by

$$\rho(x, b) = \xi + \frac{1}{x^{\frac{N-1}{2}}} \sum_{k=1}^{+\infty} C_k(x) b^k \,\mathrm{e}^{-\tau k x}, \quad x \geq x_\infty,$$

where b is the parameter. Note that, since the needed solution is increasing, we must have $b < 0$.

3. The case $f(y) = y - y^3$. Let us now consider the singular boundary value problems (1.1)–(1.2) and (1.1)–(1.3) with $f(y) = y - y^3$ and $N > 2$. It can be easily proved that, in this case, the conditions of [3, Theorem 19 and Theorem 20] are satisfied if $N < 4$, and therefore these problems have at least one positive solution (in the case of the first one, for a sufficient large M).

3.1. The singularity at zero. Let us consider equation (1.1) with initial conditions

$$\lim_{x\to 0^+} y(x) = y_0, \quad \lim_{x\to 0^+} x y'(x) = 0, \tag{3.1}$$

where y_0 is a real parameter.

As in subsection 2.1, we assume that c allows an expansion in the form (2.7). In this case, equation (1.1) also has a regular singularity at $x = 0$ and its characteristic exponents are the same as in subsection 2.1. Then, using again [9, Theorem 4 and Theorem 5], we can assure that the singular Cauchy problem (1.1), (3.1) has exactly one solution for each value of y_0, which can be represented in the form

$$y(x) = y_0 + \sum_{k=2}^{+\infty} y_k(y_0)x^k, \quad 0 \le x \le \delta, \quad \delta > 0. \tag{3.2}$$

The coefficients y_k can be determined substituting (3.2) in (1.1). For $k = 2, 3, 4$ we obtain:

$$\begin{aligned} y_2(y_0) &= \frac{-y_0(y_0+1)(y_0-1)\gamma_0}{2N}, \\ y_3(y_0) &= \frac{-y_0(y_0+1)(y_0-1)\gamma_1}{3N+3}, \\ y_4(y_0) &= \frac{\gamma_2 y_0 - \gamma_2 y_0^3 + \gamma_0 y_2 - 3\gamma_0 y_0^2 y_2}{4(2+N)}. \end{aligned}$$

All the solutions of the form (3.2) satisfy $y'(0) = 0$ and therefore the equation (1.1) has a one-parameter family of solutions that satisfy this condition.

Note that in this case the positive root of $F(t) = \int_0^t (s - s^3)\,\mathrm{d}s$ is $\beta = \sqrt{2}$; therefore, according to [3] and [7], we must have $y(0) > \sqrt{2}$.

3.2. The singularity at infinity. Let us now consider equation (1.1) with the initial condition

$$\lim_{x\to+\infty} y(x) = \lim_{x\to+\infty} y'(x) = 0. \tag{3.3}$$

In order to analyze the asymptotic behavior of the solutions of (1.1) at infinity, we follow the same steps as in subsection 2.2. By means of the variable substitution $z = x^{\frac{N-1}{2}} y$, we obtain the new equation

$$z'' = \left(\frac{z}{x^{\frac{N-1}{2}}} + 1\right)\left(1 - \frac{z}{x^{\frac{N-1}{2}}}\right) zc(x) + \frac{(N-1)(N-3)}{4x^2} z, \tag{3.4}$$

and the conditions (3.3) become

$$\lim_{x\to+\infty} z(x) = \lim_{x\to+\infty} z'(x) = 0. \tag{3.5}$$

Equation (3.4) is asymptotically autonomous, since, when $x \to +\infty$, we obtain the autonomous equation $z'' = Lz$, where $L = \lim_{x\to+\infty} c(x) > 0$, whose characteristic roots are

$$\tau_{1,2} = \pm\tau, \quad \tau = \sqrt{L}.$$

Using again the results of [12], the singular Cauchy problem (3.4), (3.5) has a one parameter family of solutions that can be represented in the form

$$z(x, b) = C_1(x) b\,\mathrm{e}^{-\tau x} + \sum_{k=2}^{+\infty} C_k(x) b^k\,\mathrm{e}^{-\tau k x}, \quad x \ge x_\infty, \tag{3.6}$$

where b is the parameter and $|b\,\mathrm{e}^{-\tau x_\infty}|$ is small.

The coefficients $C_k(x)$ can be obtain substituting (3.6) in (3.4). In particular, for the coefficient $C_1(x)$, we obtain the singular Cauchy problem (2.14), (2.15). Therefore, using the same arguments as in subsection 2.2, we conclude that C_1 satisfies a series expansion of the form (2.17).

The coefficients C_k, for $k = 2, 3, \ldots$ can be computed in a similar way. By substituting these coefficients in the series (3.6) we obtain, for each b, a solution of the Cauchy problem (3.4), (3.5).

Returning to the initial variable y, we conclude that the Cauchy problem (1.1),(3.3) has a one parameter family of solutions that can be represented by

$$y(x,b) = \frac{1}{x^{\frac{N-1}{2}}} \sum_{k=1}^{+\infty} C_k(x) b^k \, \mathrm{e}^{-\tau k x}, \quad x \geq x_\infty,$$

where b is the parameter. Since in this case the needed solution of (1.1) is decreasing, we must have $b > 0$.

4. Numerical Results.

4.1. The case $f(y) = 4\lambda^2 y(y-\xi)(y-\xi-1)$. Numerical results for the problem (2.1), (2.4), (2.5)(on the half-line), in the case $N = 3$, have been obtained in [10] and [11], using a shooting method, similar to the one of the present paper. More recently, in [8], collocation methods have been used with success to approximate the solution of the same problem. Here we will begin by presenting some results, concerning the problem (2.1), (2.2), (2.3) (on a finite domain). In order to solve numerically this problem, for a certain $M > 0$, we use the following method:

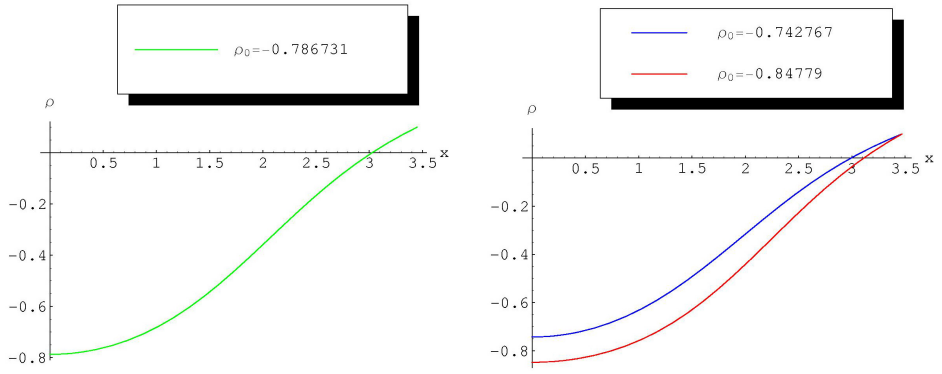
- solve equation (2.1) with initial condition $\rho(\delta, \lambda) = \rho_0 + \sum_{k=2}^{n_1} \rho_k(\rho_0)\delta^k$, where n_1 is a sufficiently large integer so that the remainder of the series is negligible (for the considered value of δ).
- shoot on the parameter ρ_0 so as to satisfy the other boundary condition $\rho(M) = \xi$.

Using this method, we have determined numerically the value M_0, such that, if $M > M_0$, then the problem (2.1), (2.2), (2.3) has at least one solution. The approximate values of M_0 in the case $N = 3$, $\lambda = 1$ and $c(x) = 1$ are given in TABLE. 4.1, for different values of ξ, with the corresponding values of ρ_0. For comparison, we give also the corresponding values of R and $\rho_0(\inf)$ (the bubble radius R is, by definition, a positive real number that satisfies $\rho(R) = 0$, and ρ_0 (inf) denotes ρ_0 for the solution in the infinite domain).

ξ	M_0	ρ_0	$\rho_0(\inf)$	R
0.1	3.45	-0.786731	-0.305	3.32
0.2	3.61	-0.831891	-0.568	2.68
0.3	3.83	-0.893815	-0.771	2.58
0.4	4.14	-0.966880	-0.903	2.72
0.5	4.61	-0.982158	-0.9711	3.07

TABLE 4.1
$N = 3$, $c(x) = 1$

For example, in the case $N = 3$, $\lambda = 1$, $c(x) = 1$ and $\xi = 0.1$, we can say that if $M < 3.45$, the problem (2.1), (2.2), (2.3) has no solution; if $M = 3.45$ this problem has one solution corresponding to $\rho_0 = -0.786731$; and if $M = 3.47$ (for example) we will have two solutions corresponding to $\rho_0 = -0.84779$ and $\rho_0 = -0.742767$, as it is shown in FIGURE 4.1.



$M = 3.45$ $M = 3.47$

FIG. 4.1. $\lambda = 1$, $c(x) = 1$ *and* $\xi = 0.1$

FIGURE 4.2 shows the graphics of some solutions and corresponding derivatives of the problem (2.1), (2.2), (2.3) in the case $c(x) = \sin\left(1 + \frac{1}{1+x^2}\right)$.

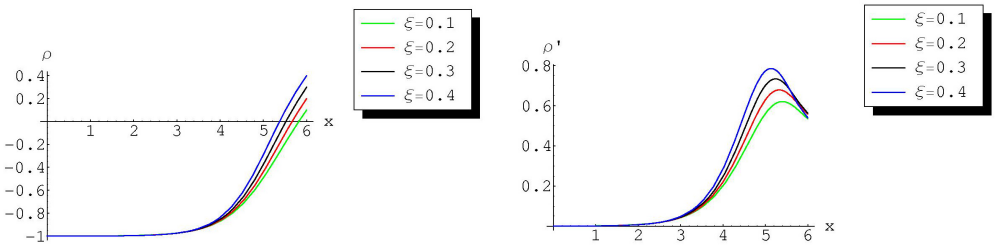


FIG. 4.2. *Graphics of solutions and their derivatives in the case* $N = 3$, $M = 6$ *and* $c(x) = \sin\left(1 + \frac{1}{1+x^2}\right)$

The boundary value problem (2.1), (2.4), (2.5) (on the half-line) was considered only for $c(x) = \text{const}$. When analysing this problem, we must take into the consideration that it is singular both at zero and at infinity. Therefore, instead of shooting from one endpoint, we must divide the original problem into two auxiliary problems: the first, on the interval $[\delta, x_0]$, and the second, on $[x_0, x_\infty]$. Here, x_0 is such that $y(x_0) = 0$. Since each of these problems has exactly one singularity, it can be solved by the shooting method in the usual way. Then we must "couple" the solutions of the two auxiliary problems in order to obtain the solution of the original problem. Let us describe this algorithm in detail:

- fix certain values of x_0, δ and x_∞ such that $x_\infty > x_0 > \delta$;
- solve the Cauchy problem with initial condition $\rho(\delta, \lambda) = \rho_0 + \sum_{k=2}^{n_1} \rho_k(\rho_0)\delta^k$ with an arbitrary λ;
- shoot on the parameter ρ_0 so as to be satisfied $\rho(x_0, \lambda) = 0$ and denote by $\rho_-(x, \lambda)$ the obtained solution in $[\delta, x_0]$;
- solve the Cauchy problem with initial condition $\rho(x_\infty, \lambda) = \xi + \frac{1}{x_\infty^{\frac{N-1}{2}}} \sum_{k=1}^{n_2} C_k(x_\infty) b^k e^{-2\lambda\sqrt{\xi(\xi+1)}Lkx_\infty}$ with the same value of λ; here n_2 is a sufficiently large integer, so that the remainder of the series is negligible, for the given r_∞;

- shoot on the parameter b so as to satisfy $\rho(x_0, \lambda) = 0$ and denote by $\rho_+(x, \lambda)$ the obtained solution in $[x_0, x_\infty]$;
- compute $\triangle(\rho_0, \lambda) = \lim_{x \to x_0^-} \rho'(x, \lambda) - \lim_{x \to x_0^+} \rho'(x, \lambda)$;
- find the value $\hat{\lambda} \in \mathbb{R}^+$ that satisfies $\triangle(x_0, \hat{\lambda}) = 0$, by the secant method;
- if $\rho(x, \hat{\lambda})$ is the solution of (2.1)–(2.4)–(2.5) for a given value $\hat{\lambda}$, then for an arbitrary value of λ the corresponding solution can be calculated by the formula $\rho(x, \lambda) = \rho(x\lambda/\hat{\lambda}, \hat{\lambda})$.

When ξ is less than a certain value ($\xi < 0.3$, in the case of $N = 3$), the described method becomes unstable. In this case, it is preferable to shoot from $x = \delta$ and replace the boundary condition at x_∞ by

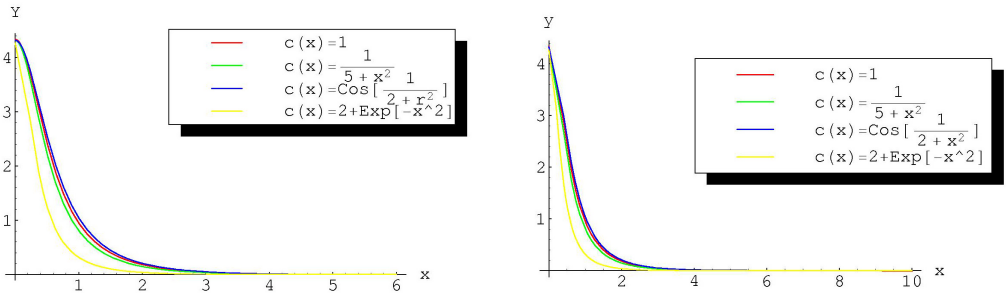
$$\rho'(x_\infty) = \left(2\lambda\sqrt{\xi(\xi+1)} + \frac{N-1}{2x_\infty} - \frac{C_1'(x_\infty)}{C_1(x_\infty)} \right) (\xi - \rho(x_\infty))$$

4.2. The case $f(y) = y - y^3$. In this case, according to the previous section, problem (1.1), (1.2) has at least one solution, for sufficiently large M, if $N < 4$. Concerning problem (1.1), (1.3) (on the halfline), the existence of at least one solution is also guaranteed, if N satisfies the same restriction.

The numerical algorithms used to solve both problems are similar to the ones, used in the previous section, in the case of a different function f. The main difference is that, when solving the problem on an infinite domain, we have to find the root of a nonlinear system of equations, and the Newton method is applied with this purpose.

The outline of the algorithm for the problem (1.1), (1.2) is as follows:

- solve the Cauchy problem with initial condition $y(\delta) = y_0 + \sum_{k=2}^{n_1} y_k(y_0)\delta^k$ in $[\delta, M]$;
- shoot on the parameter y_0 so as to satisfy the condition $y(M) = 0$.



(a) *Solutions of the problem* (1.1)–(1.2) *with* $M = 6$ (b) *Solutions of the problem* (1.1)–(1.3)

FIG. 4.3. *The case* $f(y) = y - y^3$ *with* $N = 3$

To obtain the numerical solution of the problem (1.1), (1.3) we have used the following numerical algorithm:

- fix certain values of δ and x_∞ such that $x_\infty > \delta$;
- solve the Cauchy problem with initial condition $y(\delta) = y_0 + \sum_{k=2}^{n_1} y_k(y_0)\delta^k$ in $[\delta, \frac{x_\infty}{2}]$;
- solve the Cauchy problem with initial condition $y(x_\infty) = \frac{1}{x_\infty^{\frac{N-1}{2}}} \sum_{k=1}^{n_2} C_k(x_\infty) b^k \, \mathrm{e}^{-2\lambda\sqrt{L}kx_\infty}$ in $[\frac{x_\infty}{2}, x_\infty]$;
- shoot on the parameters y_0 and b, so that the solutions and their first derivatives coincide at $\frac{x_\infty}{2}$. This gives us a nonlinear system of two equations which is solved by the Newton method.

In FIGURE 4.3(a) we plot the solutions of problem (1.1),(1.2) with $M = 6$, $N = 3$ and $f(y) = y - y^3$. In FIGURE 4.3(b), the solutions of the problem (1.1), (1.3) are displayed for the same value of N and the same function f. In both cases, we consider different forms of the function $c(x)$. The corresponding values of y_0 in the finite domain for $M = 6$ and $y_0(inf)$ in the infinite domain are given in TABLE 4.2.

$c(x)$	$y_0(M = 6)$	y_0(inf)
1	4.3376	4.33747
$1 + \frac{1}{x^2+5}$	4.32029	4.32017
$\cos\left(\frac{1}{2+x^2}\right)$	4.32125	4.32099
$2 + e^{-x^2}$	4.26728	4.26728

TABLE 4.2
Values of y_0

5. Conclusions and Future Work. In this paper we have continued the work developed in [10] and [11] on the analysis and numerical solution of nonlinear singular second-order boundary value problems on unbounded domains. The numerical methods introduced in those papers have been adapted to new classes of problems. Numerical experiments have been carried out which have confirmed the theoretical existence results of [3] and [7].

In the future, we are planning to extend the considered numerical methods to new problems, which arise when searching for radial solutions of the quasilinear equation $\Delta_m u + f(u) = 0$ in R^n, where Δ_m is the degenerate Laplace operator.

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REFERENCES

[1] F. V. Atkinson and L. A. Peletier, *Ground states of* $-\Delta u = f(u)$ *and the Emden-Fowler equation*, Archive for Rational Mechanics and Analysis, **93** (1986), 103–127.
[2] H. Berestycki and P.-L. Lions, *Nonlinear Scalar Field Equations, I Existence of a Ground State*, Archive for Rational Mechanics and Analysis, **82** (1983), 313–345.
[3] D. Bonheure, J. M. Gomes and L. Sanchez, *Positive solutions of a second-order singular ordinary differential equations*, Nonlinear Analysis, **61** (2005), 1383–1399.
[4] F. Dell'Isola, H. Gouin and G. Rotoli, *Nucleation of spherical shell-like interfaces by second gradient theory: numerical simulations*, Finite Difference Methods: theory and Applications, R.Ciegis, European Journal of Mechanics, B/Fluids, **15**(4), (1996), 545–568.
[5] G. H. Derrick,*Comments on nonlinear wave equations as models for elementary particles*, J. Math. Physics, **5** (1965), 1252–1254.
[6] B. Franchi, E. Lanconelli and J. Serrin, *Existence and uniqueness of nonnegative solutions of quasilinear equations in* $\mathbb{R}^n$, Advances in Mathematics, **118** (1996), 177–243.
[7] F. Gazzola, J. Serrin and M. Tang, *Existence of ground states and free boundary problems for quasilinear elliptic operators*, Advances in Differential Equations, **5**(1–3),(2000), 1–30.
[8] G. Kitzhofer, O. Koch, P. Lima and E. Weinmuller, *E*fficient solution of the density profile equation in hydrodynamics, to appear in *Journal of Scientific Computing*.
[9] N. B. Konyukhova, *S*ingular Cauchy problems for systems of ordinary differential equations, USSR Comput. Maths. Math. Phys., **23** (1983), 72–82.
[10] P. M. Lima, N. V. Chemetov, N. B. Konyukhova and A. I. Sukov, *Analytical-numerical approach to a singular boundary value problem*, Proceedings of CILAMCE XXIV, Ouro Preto, Brasil, 2003.

[11] P. M. Lima, N. B. Konyukhova, A. I. Sukov and N. V. Chemetov, *Analytical-Numerical Investigation of Bubble-Type Solutions of Nonlinear Singular Problems*, J. Comput. Appl. Math., **189** (2006), 260–273.

[12] A. M. Lyapunov, *Problème General de la Stabilité du Mouvement*, Ann. Fac. Sci. Univ. Toulouse, **2**(9) (1907), 203–474 or Ann. of Math. Studies, vol. **17**, Princeton University, Princeton, N. J., 1947.