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ON BLOW-UP AT SPACE INFINITY FOR SEMILINEAR HEAT EQUATIONS

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We are interested in solutions of semilinear heat equations which blow up at space infinity.

In [**7**], we considered a nonnegative blowing up solution of

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^n, \qquad t > 0$$

with initial data u_0 satisfying

$$0 \le u_0(x) \le M, \qquad u_0 \not\equiv M \quad \text{and} \quad \lim_{|x|\to\infty} u_0(x) = M,$$

where $p > 1$ and $M > 0$ is a constant. We proved in [**7**] that the solution u blows up exactly at the blow-up time for the spatially constant solution with initial data M. We moreover proved that u blows up only at the space infinity. In this paper we would like to generalize this result in the following directions.

- (i) (Initial data) We consider more general initial data u_0 which may not converge to M for all directions of x, for example $u_0 \to M$ as $|x| \to \infty$ only for x in some sector. It is convenient to introduce a notion of blow up direction at the space infinity. We are able to give necessary and sufficient condition so that a particular direction is a blow-up direction.
- (ii) (Nonlinear term) We extend the class of nonlinearities. It includes e^u and $u^p + u^q$ for p, $q > 1$.

In [**8**] we consider solutions of the initial value problem for the equation

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1}$$

The nonlinear term f is assumed to be nonnegative and locally Lipschitz in $\mathbb{R}$ with the property that

$$\liminf_{b \ge b_0,\, \delta \in (\delta_0, 1)} \frac{\delta^p f(b)}{f(\delta b)} > 0 \quad \text{for} \quad b_0 > 0, \quad \delta_0 \in (0,1), \quad p > 1. \tag{2}$$

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We take two constants M and N satisfying $M+N>0$ and

$$f(M)>0. \tag{3}$$

The initial data u_0 is assumed to be a measureable function in $\mathbb{R}^n$ satisfying

$$-N \le u_0 \le M \text{ a.e.} \qquad \text{and} \qquad u_0 \not\equiv M \text{ a.e.} \tag{4}$$

We are interested in initial data such that $u_0 \to M$ as $|x| \to \infty$ for x in some sector of $\mathbb{R}^n$. We assume that

$$\operatorname{essinf}_{x\in\tilde{B}_m}(u_0(x) - M_m(x-x_m)) \ge 0 \qquad \text{for } m=1,2,\ldots, \tag{5}$$

where

$$\tilde{B}_m = B_{r_m}(x_m) \tag{6}$$

with a sequence $\{r_m\}$ and a sequence of vectors $\{x_m\}_{m=1}^{\infty}$ and a sequence of functions $\{M_m(x)\}$ satisfying

$$\lim_{m\to\infty} r_m = \infty, \qquad M_m(x) \le M_{m+1}(x) \qquad \text{for } m \ge 1$$

$$\lim_{m\to\infty} \inf_{s\in[1,r_m]} \frac{1}{|B_s|}\int_{B_s(0)} M_m(x)\,\mathrm{d}x = M.$$

Here $B_r(x)$ denotes the closed ball of radius r centered at x. (In fact, it follows from (4) that $|x_m| \to \infty$ as $m \to \infty$.)

Problem (1) has a unique bounded solution at least locally in time. However, the solution may blow up in finite time. For a given initial value u_0 and nonlinear term f let $T^* = T^*(u_0, f)$ be the maximal existence time of the solution. If $T^* = \infty$, the solution exists globally in time. If $T^* < \infty$, we say that the solution blows up in finite time. It is well known that

$$\limsup_{t\to T^*} \|u(\cdot,t)\|_\infty = \infty, \tag{7}$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm in space variables.

In this paper, we are interested in the behavior of a blowing up solution near space infinity as well as the location of blow-up points defined below. A point $x_{BU} \in \mathbb{R}^n$ is called a *blow-up point* (with value $\pm\infty$) if there exists a sequence $\{(x_m,t_m)\}_{m=1}^{\infty}$ such that

$$t_m \uparrow T^*, \quad x_m \to x_{BU} \quad \text{and} \quad u(x_m,t_m) \to \pm\infty \quad \text{as} \quad m\to\infty.$$

If there exists a sequence $\{(x_m,t_m)\}_{m=1}^{\infty}$ such that

$$t_m \uparrow T^*, \quad |x_m| \to \infty \quad \text{and} \quad u(x_m,t_m) \to \pm\infty \quad \text{as} \quad m\to\infty,$$

then we we say that the solution blows up to $\pm\infty$ at space infinity.

A direction $\psi \in S^{n-1}$ is called a *blow-up direction for the value* $\pm\infty$ if there exists a sequence $\{(x_m,t_m)\}_{m=1}^{\infty}$ with $x_m \in \mathbb{R}^n$ and $t_m \in (0,T^*)$ such that $u(x_m,t_m) \to \pm\infty$ (as $m\to\infty$) and

$$\frac{x_m}{|x_m|} \to \psi \quad \text{as} \quad m\to\infty. \tag{8}$$

We consider the solution $v(t)$ of an ordinary differential equation

$$\begin{cases} v_t = f(v), \quad t > 0, \\ v(0) = M. \end{cases} \tag{9}$$

Let $T_v = T^*(M, f)$ be the maximal existence time of the solution of (9), i. e.,

$$T_v = \int_M^\infty \frac{\mathrm{d}s}{f(s)}.$$

We are now in position to state our main results.

Theorem 1. *Assume that f is locally Lipschitz in $\mathbb{R}$ and satisfies* (2) *and* (3). *Let u_0 be a continuous function satisfying* (4) *and* (5), *and $T_v \le T^*(-N, f)$. Then there exists a subsequence of $\{x_m\}_{m=1}^\infty$ (still denoted by $\{x_m\}$, independent of t) such that*

$$\lim_{m\to\infty} u(x_m, t) = v(t).$$

The convergence is uniform in every compact subset of $\{t : 0 \le t < T_v\}$. Moreover, the solution blows up at T_v.

Remark. Our assumption $T_v \le T^*(-N, f)$ says that the solution does not blow up to minus infinity before it blows up to plus infinity. From the condition (4), it follows that $\lim_{m\to\infty} |x_m| = \infty$.

This result in particular implies that

$$\sup_{0<t<T^*} v^{-1}(t)\|u(\cdot, t)\|_\infty < \infty. \tag{10}$$

When we set $f(u) = |u|^{p-1}u$, such a blow-up rate estimate is known for subcritical p; see e.g. [**3**], [**5**], [**6**] for general bounded initial data without assuming (4) and (5). However, for supercritical p such a blow-up rate estimate (10) may not hold in general; see e.g. [**1**], [**9**]. If one considers only radial solutions of (1) for supercritical p less than $1 + 4/(n - 4 - 2(n - 1)^{1/2})$ or $n \le 10$, then the estimate (10) holds [**11**]. We would like to emphasize that Theorem 1 does not require any restriction on p.

Our second main result is on the location of blow-up points.

Theorem 2. *Assume the same hypotheses as in Theorem 1. Then the solution of* (1) *has no blow-up points with $+\infty$ in $\mathbb{R}^n$. (It blows up only at space infinity.)*

There is a huge literature on location of blow-up points since the work of Weissler [**13**] and Friedman-McLeod [**2**]. (We do not intend to list references exhaustively in this paper.) However, most results consider either bounded domains or solutions decaying at space infinity; such a solution does not blow up at space infinity [**4**].

As far as the authors know, before the result of [**7**] the only paper discussing blow-up at space infinity is the work of Lacey [**10**]. He considered the Dirichlet problem in a half line. He studied various nonlinear terms and proved that a solution blows up only at space infinity.

In particular, his result implies that the solution of

$$\begin{cases} u_t = u_{xx} + f(u), & x > 0, \quad t > 0, \\ u(0,t) = 1, & t > 0, \\ u(x,0) = u_0(x) \geq 1, & x > 0 \end{cases}$$

blows up only at space infinity, where u_0 satisfies $0 \leq u_0 \leq M$ with $M > 1$, and $f(s) = s^p$ and e^s.

His method is based on construction of suitable subsolutions and supersolutions. However, the construction heavily depends on the Dirichlet condition at $x = 0$ and does not apply to the Cauchy problem even for the case $n = 1$.

As previously described, the authors [**7**] proved the statement of Theorems 1 and 2 assuming that $u_0(x) \leq M$ for sufficiently large M for positive solutions of $u_t = \Delta u + u^p$. Later, Shimojyo [**12**] had the same results as in [**7**] by relaxing the assumptions of initial data $u_0 \geq 0$ which is similar to that in the present paper. His approach is a construction of a suitable supersolution which implies that $a \in \mathbb{R}^n$ is not a blow-up point. Although he restricted himself to $f(s) = s^p$, his idea works for our f under slightly stronger assumption on u_0. Here we give a different approach.

From Shimojyo's results [**12**], there arises a problem of "blow-up direction" defined in (8). We next study this "blow-up direction" for the value $+\infty$. Our third result is on this blow-up direction. It is convenient to introduce the function A_m defined by

$$A_m(s) = \frac{1}{|B_s(y_m)|} \int_{B_s(y_m)} u_0(z)\,\mathrm{d}z \tag{11}$$

for a given sequence $\{y_m\}_{m=1}^{\infty}$. This $A_m(s)$ represents the mean value of u_0 over the ball $B_s(y_m)$.

Theorem 3. *Assume the same hypotheses as in Theorem 1 and let $\{s_m\}_{m=1}^{\infty}$ be a sequence diverging to ∞ in $\mathbb{R}$. For a given direction $\psi \in S^{n-1}$, the following alternatives hold.*

(i) *If there exists a sequence $\{y_m\}_{m=1}^{\infty}$ satisfying $\lim_{m\to\infty} y_m/|y_m| = \psi$ it holds that*

$$\limsup_{m\to\infty} \inf_{s\in(1,s_m)} A_m(s) = M,$$

then ψ is a blow-up direction.

(ii) *If for any sequence $\{y_m\}_{m=1}^{\infty}$ satisfying $\lim_{m\to\infty} y_m/|y_m| = \psi$ there exists a constant $c \in (1/(M+N), \infty)$ such that*

$$\limsup_{m\to\infty} \inf_{s\in(1,c)} A_m(s) \leq M - \frac{1}{c},$$

then ψ is not a blow-up direction.

This characterizes blow up directions by profiles of initial data. This is a new result even if $f(u) = |u|^{p-1}u$ or $n = 1$.

Here are the main ideas of the proofs. To prove Theorem 1 we construct a suitable subsolution. To prove Theorem 2 we derive a non blow-up criterion. We do not appeal any energy arguments for rescaled function as is done in our previous paper [**7**]. Our argument consists of two parts. First we observe that

$$u(x,t) \leq \delta v(t)$$

near a point $a \in \mathbb{R}^n$ with some $\delta \in (0,1)$ when t is close to the blow-up time. By a bootstrap argument we derive that u is actually bounded near a when t is close to the blow-up time. To prove Theorem 3 we use a comparison argument as in Theorems 1 and 2 and a non blow-up criterion as in the proof of Theorem 2. Moreover, we give conditions on the direction $\psi \in S^{n-1}$ for being the blow-up direction or not cover all of S^{n-1} exclusively.

The detailed proofs will be discussed in paper [**8**].

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