Jiří Bouchala

Landesman–Lazer type conditions and quasilinear elliptic equations

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 45--51.

Persistent URL: http://dml.cz/dmlcz/700381

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 45–51

Landesman–Lazer Type Conditions and Quasilinear Elliptic Equations

Jiří Bouchala*

Department of Applied Mathematics, Faculty of Electrical Engineering and Computer Science, VŠB-Technical University,
Tř. 17. listopadu, 708 33 Ostrava, Czech Republic,
Email: jiri.bouchala@vsb.cz

Abstract. We study the existence of the weak solutions of nonlinear boundary value problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + g(u) - h(x) \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ is continuous function, $h \in L^{p'}(\Omega)$ ($p' = \frac{p}{p-1}$), Δ_p is the p-Laplacian, i.e. $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\lambda \in \mathbb{R}$.
Our sufficient conditions generalize all previously published results.

MSC 2000. 35J20, 35P30, 47H15

Keywords. The p-Laplacian, Ekeland variational principle, saddle point theorem, the strong unique continuation property

1 Introduction. The variational eigenvalues.

We study the existence of the weak solutions of nonlinear boundary value problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + g(u) - h(x) \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

* Research supported by the Grant Agency of Czech Republic 201/00/0376 and grant of the Ministry of Education, Youth and Sports J17/98: 272400019

This is the preliminary version of the paper.

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $p > 1$, $g: \mathbb{R} \to \mathbb{R}$ is continuous function, $h \in L^{p'}(\Omega)$ ($p' = \frac{p}{p-1}$), $\lambda \in \mathbb{R}$ and Δ_p is the p-Laplacian, i.e.

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

We recall that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1) if and only if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}x = \lambda \int_\Omega |u|^{p-2}uv \,\mathrm{d}x + \int_\Omega g(u)v \,\mathrm{d}x - \int_\Omega hv \,\mathrm{d}x$$

for all $v \in W_0^{1,p}(\Omega)$.

It is possible to achieve that the weak solutions of our BVP (1) corresponding with the critical points of the functional

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega G(u)\,\mathrm{d}x + \int_\Omega hu \,\mathrm{d}x \;:\; W_0^{1,p}(\Omega) \to \mathbb{R},$$

where

$$G(t) := \int_0^t g(s)\,\mathrm{d}s.$$

Now we are going to investigate how the choice of λ, g and h (and their relation) influence the *geometry* of our functional J. The great part in that has the information, if λ is the eigenvalue of the operator $-\Delta_p$ or not; i.e. if there exists a weak nontrivial solution of BVP

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Now we define the even functional

$$I(u) := \frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega |u|^p \,\mathrm{d}x} \;:\; W_0^{1,p}(\Omega) \setminus \{0\} \to \mathbb{R},$$

and for any $k \in \mathbb{N}$ we consider set

$$\mathcal{F}_k := \Big\{ \mathcal{A} \subset \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\} : \\ \text{there exists a continuous odd surjection } h: S^k \to \mathcal{A} \Big\},$$

where S^k represents the unit sphere in $\mathbb{R}^k$.

Pavel Drábek and Stephen B. Robinson proved in 1999 that for any $k \in \mathbb{N}$ the number

$$\lambda_k := \inf_{\mathcal{A} \in \mathcal{F}_k} \sup_{u \in \mathcal{A}} I(u)$$

is an eigenvalue of $-\Delta_p$. This situation is very interesting, because it is not known if this represents a complete list of eigenvalues [1] but it is known that:

[1] Nobody knows how to obtain all eigenvalues of $-\Delta_p$; we only know that we have complete list of eigenvalues if $N = 1$ or $p = 2$.

- λ_1 is the first eigenvalue,

  $\lambda_1 = \min\{\int_\Omega |\nabla u|^p \,\mathrm{d}x;\ u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p \,\mathrm{d}x = 1\}$,

  there exists a unique positive corresponding eigenfunction φ_1 whose norm in $W_0^{1,p}(\Omega)$ is 1,

- λ_2 is the second eigenvalue,
- $\forall k \in \mathbb{N} \setminus \{1,2\} :\ 0 < \lambda_1 < \lambda_2 \le \lambda_k \le \lambda_{k+1}$,
- $\lambda_k \to +\infty$.

Pavel Drábek and Stephen B. Robinson assumed in their paper that function g is bounded and they found some sufficient conditions for solvability of our BVP (1). Now we are going to generalize these results for some not bounded function g.

2 The case $\lambda < \lambda_1$.

Theorem 1. *If we suppose in addition that*

$$\boxed{\lim_{x\to\pm\infty} \tfrac{g(x)}{|x|^{p-1}} = 0,}$$

then the BVP (1) *has at least one weak solution.*

(It follows from Ekeland variational principle (see [6] and [1]) that the energy functional J has a global minimum in this case.)

3 The case $\lambda = \lambda_1$.

Theorem 2. *Let us define*

$$F(x) := \frac{p}{x}\int_0^x g(s)\,ds - g(x).$$

We suppose

$$\boxed{\lim_{x\to\pm\infty} \tfrac{g(x)}{|x|^{p-1}} = 0}$$

and

$$\boxed{\overline{F(-\infty)} \int_\Omega \varphi_1(x)\,dx < (p-1)\int_\Omega h(x)\varphi_1(x)\,dx < \underline{F(+\infty)} \int_\Omega \varphi_1(x)\,dx,} \tag{2}$$

or

$$\boxed{\overline{F(+\infty)}\int\limits_{\Omega}\varphi_1(x)\,dx < (p-1)\int\limits_{\Omega}h(x)\varphi_1(x)\,dx < \underline{F(-\infty)}\int\limits_{\Omega}\varphi_1(x)\,dx,} \tag{3}$$

where

$$\overline{F(-\infty)} = \limsup_{x\to-\infty} F(x),\ \underline{F(+\infty)} = \liminf_{x\to+\infty} F(x),$$
$$\overline{F(+\infty)} = \limsup_{x\to+\infty} F(x),\ \underline{F(-\infty)} = \liminf_{x\to-\infty} F(x).$$

Then the BVP (1) has at least one weak solution.

(If (2) is satisfied that the energy functional J has a saddle point geometry while if (3) holds this functional attains its global minimum … see [3].)

4 The case $\lambda_k < \lambda < \lambda_{k+1}$.

Theorem 3. *We suppose*

$$\boxed{\lim_{x\to\pm\infty}\frac{g(x)}{|x|^{p-1}} = 0}$$

and

$$\boxed{\begin{gathered}\forall v \in Ker(-\Delta_p - \lambda)\setminus\{0\}:\\ (p-1)\int\limits_{\Omega}h(x)v(x)\,dx < \underline{F(+\infty)}\int\limits_{\substack{\{x\in\Omega:\\ v(x)>0\}}}v(x)\,dx + \overline{F(-\infty)}\int\limits_{\substack{\{x\in\Omega:\\ v(x)<0\}}}v(x)\,dx,\end{gathered}} \tag{4}$$

or

$$\boxed{\begin{gathered}\forall v \in Ker(-\Delta_p - \lambda)\setminus\{0\}:\\ (p-1)\int\limits_{\Omega}h(x)v(x)\,dx > \overline{F(+\infty)}\int\limits_{\substack{\{x\in\Omega:\\ v(x)>0\}}}v(x)\,dx + \underline{F(-\infty)}\int\limits_{\substack{\{x\in\Omega:\\ v(x)<0\}}}v(x)\,dx,\end{gathered}} \tag{5}$$

and that

$$\boxed{\begin{gathered}\forall v \in Ker(-\Delta_p - \lambda)\setminus\{0\},\ \|v\| = 1:\\ (\forall\delta\in\mathbb{R}^+)\,(\exists\eta(\delta)\in\mathbb{R}^+):\ meas\{x\in\Omega:\ |v(x)|\le\eta(\delta)\} < \delta\\ \text{(“\textbf{the strong unique continuation property}”).}\end{gathered}} \tag{6}$$

Then the BVP (1) has at least one weak solution. [2]

[2] Notice that if $\lambda \in \mathbb{R}$ is not an eigenvalue of the $-\Delta_p$, i.e. if does not exist function $v \in \mathrm{Ker}(-\Delta_p - \lambda)\setminus\{0\}$, then the conditions (4), (5) and (6) are vacuously true.

(Proof of this theorem is based on application a saddle point theorem for linked sets ... see [1].)

5 The case $\lambda = \lambda_k$.

Theorem 4 ([1]). *We suppose*

$$\boxed{\lim_{x\to\pm\infty} \frac{g(x)}{|x|^{p-1}} = 0}$$

and

$$\boxed{\begin{gathered}\forall v \in Ker(-\Delta_p - \lambda) \setminus \{0\} : \\ (p-1)\int_\Omega h(x)v(x)\,dx < \underline{F(+\infty)} \int_{\substack{\{x\in\Omega:\\ v(x)>0\}}} v(x)\,dx + \overline{F(-\infty)} \int_{\substack{\{x\in\Omega:\\ v(x)<0\}}} v(x)\,dx,\end{gathered}} \tag{7}$$

or

$$\boxed{\begin{gathered}\forall v \in Ker(-\Delta_p - \lambda) \setminus \{0\} : \\ (p-1)\int_\Omega h(x)v(x)\,dx > \overline{F(+\infty)} \int_{\substack{\{x\in\Omega:\\ v(x)>0\}}} v(x)\,dx + \underline{F(-\infty)} \int_{\substack{\{x\in\Omega:\\ v(x)<0\}}} v(x)\,dx,\end{gathered}} \tag{8}$$

and that

$$\boxed{\begin{gathered}\forall v \in Ker(-\Delta_p - \lambda) \setminus \{0\},\ \|v\| = 1 : \\ (\forall \delta \in \mathbb{R}^+)\,(\exists \eta(\delta) \in \mathbb{R}^+) :\ meas\{x \in \Omega :\ |v(x)| \le \eta(\delta)\} < \delta.\end{gathered}}$$

Further, we assume that there exists sequence

$$\boxed{\mu_n \searrow \lambda_k \ \textit{(if we assume (7))}} \ \textit{or}\ \boxed{\mu_n \nearrow \lambda_k \ \textit{(if we assume (8))}}$$

such that

$$\boxed{\begin{gathered}\forall n \in \mathbb{N}\ \forall v \in Ker(-\Delta_p - \mu_n) \setminus \{0\},\ \|v\| = 1 : \\ (\forall \delta \in \mathbb{R}^+)\,(\exists \eta(\delta) \in \mathbb{R}^+) :\ meas\{x \in \Omega :\ |v(x)| \le \eta(\delta)\} < \delta.\end{gathered}}$$

Then the BVP (1) has at least one weak solution.

6 The one dimensional case.

At the finish we note: if we consider one dimensional problem and – for example – $\Omega = (0, \pi)$, i.e. if we consider BVP

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda|u|^{p-2}u + g(u) - h(x) \text{ in } (0,\pi), \\ u(0) = u(\pi) = 0, \end{cases} \tag{9}$$

then situation is easier: we know all eigenvalues of the p-Laplacian [3] and we know that any eigenfunction satisfies "the strong unique continuation property". Therefore we can rewrite our results in this form:

Theorem 5. *We suppose*

$$\boxed{\lim_{x\to\pm\infty} \frac{g(x)}{|x|^{p-1}} = 0}$$

and

$$\boxed{\begin{array}{c} \forall v \in Ker(-\Delta_p - \lambda) \setminus \{0\} : \\ (p-1)\int_0^\pi h(x)v(x)\,dx < \underline{F(+\infty)}\int_0^\pi v^+(x)\,dx + \overline{F(-\infty)}\int_0^\pi v^-(x)\,dx, \end{array}}$$

or

$$\boxed{\begin{array}{c} \forall v \in Ker(-\Delta_p - \lambda) \setminus \{0\} : \\ (p-1)\int_0^\pi h(x)v(x)\,dx > \overline{F(+\infty)}\int_0^\pi v^+(x)\,dx + \underline{F(-\infty)}\int_0^\pi v^-(x)\,dx, \end{array}}$$

where

$$v^+ := \max\{0, v\}, \; v^- := \min\{0, v\}.$$

Then the BVP (9) has at least one weak solution $u \in W_0^{1,p}(0,\pi)$.

---

[3] All eigenvalues are described by the equalities

$$\lambda_k := \left(\frac{k\pi_p}{\pi}\right)^p = k^p\lambda_1, \quad k \in \mathbb{N},$$

where

$$\pi_p := 2(p-1)^{\frac{1}{p}} \int_0^1 \frac{\mathrm{d}s}{(1-s^p)^{\frac{1}{p}}}.$$

References

1. Bouchala J.,*Strong resonance for quasilinear boundary value problems*, PhD Thesis, FAV ZČU Plzeň, 2000.
2. Bouchala J.,*Strong resonance problems for the p – Laplacian*, Proceedings of Seminar in Differential Equations, ZČU Plzeň, 2000, 115–122.
3. Bouchala J., Drábek P., *Strong resonance for some quasilinear elliptic equations*, Journal of Mathematical Analysis and Applications **245**, 2000, 7–19.
4. Drábek P., Robinson S.B., *Resonance problems for the p – Laplacian*, Journal of Functional Analysis **169**, 1999, 189–200.
5. Tang C.L., *Solvability for two–point boundary value problems*, Journal of Mathematical Analysis and Applications **216**, 1997, 368–374.
6. Willem M., *Minimax theorems*, Birkhäuser, Boston, 1996.