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Regularity of Minimizers in Optimal Control

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Abstract. We consider the Lagrange problem of optimal control with unrestricted controls – given a Lagrangian L, a dynamical equation $\dot{x}(t) = \varphi\left(t, x(t), u(t)\right)$, and boundary conditions $x(a) = x_a$, $x(b) = x_b \in \mathbb{R}^n$, find a control $u(\cdot) \in L_1\left([a, b]; \mathbb{R}^r\right)$ such that the corresponding trajectory $x(\cdot) \in W_{1,1}\left([a, b]; \mathbb{R}^n\right)$ of the dynamical equation satisfies the boundary conditions, and the pair $(x(\cdot), u(\cdot))$ minimizes the functional $J[x(\cdot), u(\cdot)] := \int_a^b L\left(t, x(t), u(t)\right) \mathrm{d}t$. We address the question: under what conditions we can assure optimal controls are bounded? This question is related to the one of Lipschitzian regularity of optimal trajectories, and the answer to it is crucial for closing the gap between the conditions arising in the existence theory and necessary optimality conditions. Rewriting the Lagrange problem in parametric form, we obtain a relation between the applicability conditions of the Pontryagin maximum principle to the later problem and the Lipschitzian regularity conditions for the original problem. Under the standard hypotheses of coercivity of the existence theory, the conditions imply that the optimal controls are essentially bounded, assuring the applicability of the classical necessary optimality conditions like the Pontryagin maximum principle. The result extends previous Lipschitzian regularity results to cover optimal control problems with general nonlinear dynamics.

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1 Introduction

We establish Lipschitzian regularity conditions for the minimizing trajectories of optimal control problems. Lipschitzian regularity has a number of important implications. For example in control engineering applications, where optimal strategies

This is the preliminary version of the paper.

are implemented by computer, the choice of discretization and numerical procedures depends on minimizer regularity [3,33]. Lipschitzian regularity of optimal trajectories also precludes occurrence of the undesirable Lavrentiev phenomenon [25,6,23,22] and provides the validity of known necessary optimality conditions under hypotheses of existence theory [11]. The techniques of the existence theory use compactness arguments which require to work with measurable control functions from L_p, $1 \leq p < \infty$ [5]. On the other hand, standard necessary conditions for optimality, such as the classical Pontryagin maximum principle [27], put certain restrictions on the optimal controls – namely, *a priori* assumption that they are essentially bounded. Examples are known, even for polynomial Lagrangians and linear dynamics [3], for which optimal controls predicted by the existence theory are unbounded and fail to satisfy the Pontryagin maximum principle [12]. If we are able to assure that a minimizer $(\tilde{x}(t),\, \tilde{u}(t))$, $a \leq t \leq b$, of our problem is such that $\tilde{u}(\cdot)$ is essentially bounded, then the solutions can be identified via the Pontryagin maximum principle. As far as $\varphi(t,\, \tilde{x}(t),\, \tilde{u}(t))$ is bounded, it also follows that the optimal trajectory $\tilde{x}(\cdot)$ is Lipschitzian. Similarly, the Hamiltonian adjoint multipliers $\tilde{\psi}(\cdot)$ of the Pontryagin maximum principle turn out to be Lipschitzian either. Thus, regularity theory justifies searching for minimizers among extremals and establishes a weaker form of the maximum principle in which the Hamiltonian adjoint multipliers are not required to be absolutely continuous but merely Lipschitzian.

The study of Lipschitzian regularity conditions has received few attention when compared with existence theory or necessary conditions, which have been well studied since the fifties and sixties. The question of Lipschitzian regularity, for the general Lagrange problem of optimal control, seems difficult, and attention have been on particular dynamics. Most part of results in this direction refers to problems of the calculus of variations. For a survey see [9, Ch. 2] or [34, Ch. 11]. Less is known for the Lagrange problem of optimal control. Problems whose dynamics is linear and time invariant – $\varphi(x,\, u) = Ax + Bu$ – were addressed in [14] and recently the case of control-affine dynamics – $\varphi(t,\, x,\, u) = f(t,\, x) + g(t,\, x)\, u$ – was studied [28]. For a survey see [29]. Results for general nonlinear dynamics, which is nonlinear both in state and control variables, are lacking. To deal with the problem we make use of an idea of time reparameterization that proved to be useful in many different contexts – see e.g. [19, Sec. 10], [16,17], [20, Lec. 13], [5, p. 46], [2], [1], [10], [24, Ch. 5], [21, p. 29], and [32]. Examples which possess minimizers according to the existence theory and to which our results are applicable while previously known Lipschitzian regularity conditions fail are provided.

2 Formulation of Problems (P), (P_τ) and $(\mathrm{P}_\tau[\mathrm{w}(\cdot)])$

We are interested in the study of Lipschitzian regularity conditions for the Lagrange problem of optimal control with arbitrary boundary conditions. For that is enough to consider the case when the boundary conditions are fixed: $x(a) = A$ and $x(b) = B$. Indeed, if $\tilde{x}(\cdot)$ is a minimizing trajectory for a Lagrange problem with any other kind of boundary conditions, then $\tilde{x}(\cdot)$ is also a minimizing trajectory for the corresponding fixed boundary problem with $A = \tilde{x}(a)$ and $B = \tilde{x}(b)$. The data for our problem is then

$$\left[\begin{array}{c} a,\, b \in \mathbb{R} \quad (a < b) \\ A,\, B \in \mathbb{R}^n \\ L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \longrightarrow \mathbb{R} \\ \varphi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \longrightarrow \mathbb{R}^n . \end{array}\right. \tag{1}$$

We assume $L(\cdot,\cdot,\cdot)$, $\varphi(\cdot,\cdot,\cdot) \in C$, and $\varphi(\cdot,\cdot,u)$, $\varphi(\cdot,\cdot,u) \in C^1$. (Smoothness hypotheses on L and φ can be weakened, as is discussed later in connection with the Pontryagin maximum principle.) The Lagrange problem of optimal control is defined as follows.

Problem (P).

$$I\left[x(\cdot),\, u(\cdot)\right] = \int_a^b L\left(t,\, x(t),\, u(t)\right)\, \mathrm{d}t \longrightarrow \min$$

$$\left[\begin{array}{c} x(\cdot) \in W_{1,\,1}\left([a,\, b];\, \mathbb{R}^n\right),\, u(\cdot) \in L_1\left([a,\, b];\, \mathbb{R}^r\right) \\ \dot{x}(t) = \varphi\left(t,\, x(t),\, u(t)\right), \quad a.e.\ t \in [a,\, b] \\ x(a) = A,\, x(b) = B. \end{array}\right. \tag{2}$$

The overdot denotes differentiation with respect to t, while the prime will be used in the sequel to denote differentiation in order to τ. To derive conditions assuring that the optimal controls $\tilde{u}(\cdot)$ of problem (P) are essentially bounded, $\tilde{u}(\cdot) \in L_\infty$, two auxiliary problems, defined with the same data (1), will be used.

Problem ($\mathbf{P}_\tau$).

$$J\left[t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot)\right] = \int_a^b L\left(t(\tau),\, z(\tau),\, w(\tau)\right)\, v(\tau)\, \mathrm{d}\tau \longrightarrow \min$$

$$\left[\begin{array}{c} t(\cdot) \in W_{1,\,\infty}\left([a,\, b];\, \mathbb{R}\right),\, z(\cdot) \in W_{1,\,1}\left([a,\, b];\, \mathbb{R}^n\right) \\ v(\cdot) \in L_\infty\left([a,\, b];\, [0.5,\, 1.5]\right),\, w(\cdot) \in L_1\left([a,\, b];\, \mathbb{R}^r\right) \\ \begin{cases} t'(\tau) = v(\tau) \\ z'(\tau) = \varphi\left(t(\tau),\, z(\tau),\, w(\tau)\right)\, v(\tau) \end{cases} \\ t(a) = a,\, t(b) = b; \\ z(a) = A,\, z(b) = B. \end{array}\right. \tag{3}$$

Remark 1. The fact that the control variable $v(\cdot)$ takes on its values in the set $[0.5,\, 1.5]$, guarantees that $t(\tau)$ has an inverse function $\tau(t)$.

The following problem is the same as problem (P_τ) except that $w(\cdot) \in L_1([a,\, b];\, \mathbb{R}^r)$ is fixed and the functional is to be minimized only over $t(\cdot)$, $z(\cdot)$ (the state variables) and $v(\cdot)$ (the control variable).

Problem $(\mathbf{P}_\tau[\mathbf{w}(\cdot)])$**.**

$$K\left[t(\cdot),\, z(\cdot),\, v(\cdot)\right] = \int_a^b F\left(\tau,\, t(\tau),\, z(\tau),\, v(\tau)\right)\, \mathrm{d}\tau \longrightarrow \min$$

$$\left[\begin{array}{c} t(\cdot) \in W_{1,\,\infty}\left([a,\, b];\, \mathbb{R}\right),\; z(\cdot) \in W_{1,\,1}\left([a,\, b];\, \mathbb{R}^n\right) \\ v(\cdot) \in L_\infty\left([a,\, b];\, [0.5,\, 1.5]\right) \\ \begin{cases} t'(\tau) = v(\tau) \\ z'(\tau) = f\left(\tau,\, t(\tau),\, z(\tau),\, v(\tau)\right) \end{cases} \\ t(a) = a,\; t(b) = b; \\ z(a) = A,\; z(b) = B. \end{array}\right.$$

where $F(\tau,\, t,\, z,\, v) = L\left(t,\, z,\, w(\tau)\right) v$, $f(\tau,\, t,\, z,\, v) = \varphi\left(t,\, z,\, w(\tau)\right) v$.

Remark 2. Problem (P_τ) is autonomous while (P) and $(P_\tau[w(\cdot)])$ are not.

The relation between problem (P) and problem $(P_\tau[w(\cdot)])$ is discussed in the following two sections.

3 Relation Between the Solutions of the Problems

Let us begin to determine the relation between admissible pairs for problem (P) and admissible quadruples for problem (P_τ).

Definition 3. The pair $(x(\cdot),\, u(\cdot))$ is said to be *admissible for* (P) if all conditions in (2) are satisfied. Similarly, $(t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot))$ is said to be *admissible for* (P_τ) if all conditions in (3) are satisfied.

Lemma 4. *Let* $(x(\cdot), u(\cdot))$ *be admissible for* (P)*. Then, for any function* $v(\cdot)$ *satisfying*

$$v(\cdot) \in L_\infty\left([a,\, b];\, [0.5,\, 1.5]\right), \tag{4}$$

$$\int_a^b v(s)\, \mathrm{d}s = b - a, \tag{5}$$

$t(\tau) = a + \int_a^\tau v(s)\, \mathrm{d}s$, $z(\tau) = x\left(t(\tau)\right)$ *and* $w(\tau) = u\left(t(\tau)\right)$*, are such that*

$$(t(\cdot), z(\cdot), v(\cdot), w(\cdot))$$

is admissible for (P_τ)*. Moreover,*

$$J\left[t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot)\right] = I\left[x(\cdot),\, u(\cdot)\right]. \tag{6}$$

Proof. All conditions in (3) become satisfied:

- Function $t(\cdot)$ is Lipschitzian: $\frac{\mathrm{d}t(\cdot)}{\mathrm{d}\tau} = v(\cdot) \in L_\infty([a,\, b];\, [0.5,\, 1.5])$;
- Function $z(\cdot)$ is absolutely continuous since it is a composition of the absolutely continuous function $x(\cdot)$ with the strictly monotonous Lipschitzian continuous function $t(\cdot)$:
$$\frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau} = v(\tau) > 0; \tag{7}$$
- Function $w(\cdot)$ is Lebesgue measurable, $w(\cdot) \in L_1$, because $u(\cdot)$ is measurable and $t(\cdot)$ is a strictly monotonous absolutely continuous function;
- Differentiating $z(\cdot)$ we obtain:
$$z'(\tau) = \frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}x\,(t(\tau))}{\mathrm{d}t}\,\frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}.$$
In view of (2) and (7), one concludes from this last equality that
$$\begin{aligned} z'(\tau) &= \varphi\,(t(\tau),\, x(t(\tau)),\, u(t(\tau)))\; v(\tau) \\ &= \varphi\,(t(\tau),\, z(\tau),\, w(\tau))\; v(\tau); \end{aligned}$$
- From (5) and from the definition of $t(\tau)$ we have $t(a) = a$ and $t(b) = b$. It follows that
$$\begin{aligned} z(a) &= x\,(t(a)) = x(a) = A; \\ z(b) &= x\,(t(b)) = x(b) = B. \end{aligned}$$

It remains to prove equality (6). Since

$$\begin{aligned} J\,[t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot)] &= \int_a^b L\,(t(\tau),\, z(\tau),\, w(\tau))\; v(\tau)\,\mathrm{d}\tau \\ &= \int_a^b L\,(t(\tau),\, x(t(\tau)),\, u(t(\tau)))\; v(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{8}$$

from the change of variable $t(\tau) = t$,

$$\left[\begin{array}{l} \mathrm{d}t = v(\tau)\,\mathrm{d}\tau \\ \tau = a \Leftrightarrow t = a \\ \tau = b \Leftrightarrow t = b, \end{array}\right. \tag{9}$$

it follows from (8) the pretended conclusion:

$$J\,[t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot)] = \int_a^b L\,(t,\, x(t),\, u(t))\;\mathrm{d}t = I\,[x(\cdot),\, u(\cdot)]\,.$$

Lemma 5. *Let* $(t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot))$ *be admissible for* (P_τ). *Then the pair*

$$(x(\cdot),\, u(\cdot)) = (z\,(\tau(\cdot))\,,\, w\,(\tau(\cdot)))\;,$$

where $\tau(\cdot)$ *is the inverse function of* $t(\cdot)$*, is admissible for* (P). *Moreover*

$$I\,[x(\cdot),\, u(\cdot)] = J\,[t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot)]\,. \tag{10}$$

Proof. All conditions in (2) become satisfied:

- Since defined by the composition of the absolutely continuous function $z(\cdot)$ with the strictly monotonous absolutely continuous function $\tau(\cdot)$,

$$\frac{\mathrm{d}\tau(t)}{\mathrm{d}t} = \frac{1}{v\left(\tau(t)\right)} > 0, \tag{11}$$

  function $x(\cdot)$ is absolutely continuous;
- We have that $u(\cdot) = w\left(\tau(\cdot)\right) \in L_1\left([a,\, b];\, \mathbb{R}^r\right)$, as it results from the composition of $w(\cdot) \in L_1\left([a,\, b];\, \mathbb{R}^r\right)$ with the continuous function $\tau : [a,\, b] \longrightarrow [a,\, b]$;
- Differentiating $x(\cdot)$ we obtain:

$$\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}z\left(\tau(t)\right)}{\mathrm{d}\tau}\,\frac{\mathrm{d}\tau(t)}{\mathrm{d}t}.$$

  From (3) and (11), one concludes from the last equality that

$$\dot{x}(t) = \frac{\varphi\left(t(\tau(t)),\, z(\tau(t)),\, w(\tau(t))\right)\, v\left(\tau(t)\right)}{v(\tau(t))} = \varphi\left(t,\, x(t),\, u(t)\right);$$

- As far as $\tau(a) = a$ and $\tau(b) = b$, it comes

$$\begin{aligned} x(a) &= z\left(\tau(a)\right) = z(a) = A; \\ x(b) &= z\left(\tau(b)\right) = z(b) = B. \end{aligned}$$

To finish we prove equality (10). By definition

$$\begin{aligned} I\left[x(\cdot),\, u(\cdot)\right] &= \int_a^b L\left(t,\, x(t),\, u(t)\right)\, \mathrm{d}t \\ &= \int_a^b L\left(t(\tau(t)),\, z(\tau(t)),\, w(\tau(t))\right)\, \mathrm{d}t. \end{aligned} \tag{12}$$

Doing the change of variable $\tau(t) = \tau$, it follows from (9) and (12) the pretended conclusion:

$$I\left[x(\cdot),\, u(\cdot)\right] = \int_a^b L\left(t(\tau),\, z(\tau),\, w(\tau)\right)\, v(\tau)\, \mathrm{d}\tau = J\left[t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot)\right].$$

From Lemmas 4 and 5, the following two corollaries are obvious.

Corollary 6. *If $(\tilde{x}(\cdot),\, \tilde{u}(\cdot))$ is a minimizer of problem (P), then, for any function $\tilde{v}(\cdot)$ satisfying (4) and (5) (e.g. $\tilde{v}(\tau) \equiv 1$), the 4-tuple*

$$\left(\tilde{t}(\cdot),\, \tilde{z}(\cdot),\, \tilde{v}(\cdot),\, \tilde{w}(\cdot)\right),$$

defined by $\tilde{t}(\tau) = a + \int_a^\tau \tilde{v}(s)\,\mathrm{d}s$, $\tilde{z}(\tau) = \tilde{x}\left(\tilde{t}(\tau)\right)$, $\tilde{w}(\tau) = \tilde{u}\left(\tilde{t}(\tau)\right)$, is a minimizer to problem (P_τ).

Corollary 7. *If $\left(\tilde{t}(\cdot),\, \tilde{z}(\cdot),\, \tilde{v}(\cdot),\, \tilde{w}(\cdot)\right)$ is a minimizer of problem (P_τ), then the pair $(\tilde{x}(\cdot),\, \tilde{u}(\cdot))$ defined from $\left(\tilde{t}(\cdot),\, \tilde{z}(\cdot),\, \tilde{v}(\cdot),\, \tilde{w}(\cdot)\right)$ as in Lemma 5 is a minimizer to problem (P).*

Thus, let $(\tilde{x}(\cdot),\, \tilde{u}(\cdot))$ be a minimizer of problem (P). From Corollary 6 we know how to construct a minimizer $\left(\tilde{t}(\cdot),\, \tilde{z}(\cdot),\, \tilde{v}(\cdot),\, \tilde{w}(\cdot)\right)$ to problem (P_τ). Obviously, as far as problem $(P_\tau[\tilde{w}(\cdot)])$ is the same as problem (P_τ) except that $\tilde{w}(\cdot)$ is fixed, $\left(\tilde{t}(\cdot),\, \tilde{z}(\cdot),\, \tilde{v}(\cdot)\right)$ furnishes a minimizer to problem $(P_\tau\,[\tilde{w}(\cdot)])$. Choosing $\tilde{v}(\tau) \equiv 1$ we obtain.

Proposition 8. *If $(\tilde{x}(\cdot),\, \tilde{u}(\cdot))$ is a minimizer of problem (P), then the triple $\left(\tilde{t}(\tau),\, \tilde{z}(\tau),\, \tilde{v}(\tau)\right) = (\tau,\, \tilde{x}(\tau),\, 1)$ furnishes a minimizer to problem $(P_\tau[\tilde{u}(\cdot)])$.*

It is also important to know how the extremals of the problems are related. This will be addressed in the next section.

4 Relation Between the Extremals of the Problems

At the core of optimal control theory is the celebrated Pontryagin maximum principle. The maximum principle is a first order necessary optimality condition for the optimal control problems. It first appear in the book [27]. Since then, several versions have been obtained by weakening the hypotheses. For example, in [27] it is assumed that functions $L(\cdot,\cdot,\cdot)$ and $\varphi(\cdot,\cdot,\cdot)$ are continuous, and have continuous derivatives with respect to the state variables x: $L(t,\cdot,u)$, $\varphi(t,\cdot,u) \in C^1$. Instead of the continuity assumption of $L(\cdot,\cdot,\cdot)$ and $\varphi(\cdot,\cdot,\cdot)$, a version only requiring that functions $L(\cdot,x,\cdot)$ and $\varphi(\cdot,x,\cdot)$ are Borel measurable can be found in book [4, Ch. 5]. There, in order to assure the applicability of the maximum principle, the following assumption is imposed: there exists an integrable function $\alpha(\cdot)$ defined on $[a,\, b]$ such that the bound

$$\left\| \frac{\partial L}{\partial x}\,(t,x,u(t)) \right\| \leq \alpha(t) \tag{13}$$

$$\left\| \frac{\partial \varphi_i}{\partial x}\,(t,x,u(t)) \right\| \leq \alpha(t) \tag{14}$$

$(i = 1,\, \ldots,\, n)$ holds for all $(t,\, x) \in [a,\, b] \times \mathbb{R}^n$. The existence and integrability of $\alpha(\cdot)$, and the bound (13)–(14), are guaranteed under the hypotheses that L and φ possess derivatives $\frac{\partial L}{\partial x}$ and $\frac{\partial \varphi}{\partial x}$ which are continuous in (t,x,u), and $u(\cdot)$ is essentially bounded (these are the hypotheses found in [27]). Alternative hypotheses are the following growth conditions (see [8, Sec. 4.4 and p. 212]):

$$\left\| \frac{\partial L}{\partial x} \right\| \leq c\,|L| + k\,, \quad \left\| \frac{\partial \varphi_i}{\partial x} \right\| \leq c\,|\varphi_i| + k\,, \tag{15}$$

with constants c and k, $c > 0$. Those who are familiar with the Lipschitzian regularity conditions for the basic problem of the calculus of variations, will recognize (15) as a generalization of the classical Tonelli–Morrey Lipschitzian regularity condition (cf. e.g. [29]). From this fact, one can guess a link between the applicability conditions of the maximum principle and the Lipschitzian regularity conditions. The link between the applicability conditions of the classical Pontryagin maximum principle [27] and the Lipschitzian regularity conditions for optimal control problems with control-affine dynamics, was established in [28]. Here, to deal with general nonlinear dynamics, we will need to apply the maximum principle under weaker hypotheses than those in [27]. This is due to the fact that when we fix $w(\cdot) \in L_1([a, b]; \mathbb{R}^r)$, functions $F(\tau, t, z, v)$ and $f(\tau, t, z, v)$ of problem $(P_\tau[w(\cdot)])$ are not continuous in τ but only measurable. Hypotheses (15) are suitable, as far as they can be directly verifiable for a given problem. Weaker hypotheses than (13) and (14) can also be considered. In this respect, important improvements are obtained from the use of nonsmooth analysis. For example, one can substitute (13) and (14) by the weaker conditions

$$\begin{gathered}
|L(t, x_1, u(t)) - L(t, x_2, u(t))| \leq \alpha(t) \|x_1 - x_2\| \\
|\varphi_i(t, x_1, u(t)) - \varphi_i(t, x_2, u(t))| \leq \alpha(t) \|x_1 - x_2\|
\end{gathered}$$

and formulate the maximum principle in a nonsmooth setting, in terms of generalized gradients (see [7,8]). Proving general versions of the maximum principle under weak hypotheses is still in progress and the interested reader is referred to the recent paper [30].

Definition 9. Let $(x(\cdot), u(\cdot))$ be admissible for (P). We say that the quadruple $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$, $\psi_0 \in \mathbb{R}_0^-$ and $\psi(\cdot) \in W_{1,1}([a, b]; \mathbb{R}^n)$, is an *extremal* of (P), if the following two conditions are satisfied for almost all $t \in [a, b]$:

the adjoint system

$$\dot{\psi}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \psi_0, \psi(t)); \tag{16}$$

the maximality condition

$$H(t, x(t), u(t), \psi_0, \psi(t)) = \sup_{u \in \mathbb{R}^r} H(t, x(t), u, \psi_0, \psi(t)); \tag{17}$$

where the Hamiltonian equals

$$H(t, x, u, \psi_0, \psi) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u).$$

Definition 10. Let $(t(\cdot), z(\cdot), v(\cdot), w(\cdot))$ be admissible for (P_τ). The 7-tuple

$$(t(\cdot), z(\cdot), v(\cdot), w(\cdot), p_0, p_t(\cdot), p_z(\cdot)),$$

$p_0 \in \mathbb{R}_0^-$, $p_t(\cdot) \in W_{1,\infty}([a, b]; \mathbb{R})$ and $p_z(\cdot) \in W_{1,1}([a, b]; \mathbb{R}^n)$, is said to be an *extremal* of (P_τ), if the following two conditions are satisfied for almost all $\tau \in [a, b]$:

the adjoint system

$$\begin{cases} p_t'(\tau) = -\dfrac{\partial \mathcal{H}}{\partial t}\left(t(\tau),\, z(\tau),\, v(\tau),\, w(\tau),\, p_0,\, p_t(\tau),\, p_z(\tau)\right)\,, \\ p_z'(\tau) = -\dfrac{\partial \mathcal{H}}{\partial z}\left(t(\tau),\, z(\tau),\, v(\tau),\, w(\tau),\, p_0,\, p_t(\tau),\, p_z(\tau)\right)\,; \end{cases} \tag{18}$$

the maximality condition

$$\begin{aligned} &\mathcal{H}\left(t(\tau),\, z(\tau),\, v(\tau),\, w(\tau),\, p_0,\, p_t(\tau),\, p_z(\tau)\right) \\ &\qquad = \sup_{\substack{v\in[0.5,\,1.5] \\ w\in\mathbb{R}^r}} \mathcal{H}\left(t(\tau),\, z(\tau),\, v,\, w,\, p_0,\, p_t(\tau),\, p_z(\tau)\right); \end{aligned} \tag{19}$$

where the Hamiltonian equals

$$\mathcal{H}\left(t,\, z,\, v,\, w,\, p_0,\, p_t,\, p_z\right) = \left(p_0\, L\left(t,\, z,\, w\right) + p_t + p_z \cdot \varphi\left(t,\, z,\, w\right)\right) v.$$

Remark 11. Functions H and $\mathcal{H}$, respectively the Hamiltonians in Definitions 9 and 10, are related by the following equality:

$$\mathcal{H}\left(t,\, z,\, v,\, w,\, p_0,\, p_t,\, p_z\right) = \left(H\left(t,\, z,\, w,\, p_0,\, p_z\right) + p_t\right) v\,. \tag{20}$$

From it one concludes that

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\partial H}{\partial t}\, v\,, \tag{21}$$

$$\frac{\partial \mathcal{H}}{\partial z} = \frac{\partial H}{\partial x}\, v\,. \tag{22}$$

Definition 12. An extremal is called *normal* if the cost multiplier (ψ_0 in the Definition 9 and p_0 in the Definition 10) is different from zero and *abnormal* if it vanishes.

Remark 13. As far as the Hamiltonian is homogeneous with respect to the Hamiltonian multipliers, for normal extremals one can always consider, by scaling, that the cost multiplier takes value -1.

Remark 14. The (Pontryagin) maximum principle give conditions, as those discussed in the introduction of this section, under which to each minimizer of the problem there corresponds an extremal with Hamiltonian multipliers not vanishing simultaneously ($(\psi_0,\, \psi) \neq 0$ in the Definition 9 and $(p_0,\, p) \neq 0$ in the Definition 10).

One can expect the set of extremals of problem (P_τ) to be richer than the set of extremals of problem (P). Nevertheless, there is a relationship between the extremals of the problems. Next lemma shows that to each extremal of problem (P) there corresponds extremals of problem (P_τ) lying on the zero level of the maximized Hamiltonian $\mathcal{H}$.

Lemma 15. *Let* $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot))$ *be an extremal of* (P)*. Then, for any function* $v(\cdot) \in L_\infty([a,\, b];\, [0.5,\, 1.5])$ *satisfying* $\int_a^b v(s)\,\mathrm{d}s = b - a$*, the 7-tuple*

$$(t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot),\, p_0,\, p_t(\cdot),\, p_z(\cdot))$$

defined by

$$\begin{gathered} t(\tau) = a + \int_a^\tau v(s)\,\mathrm{d}s\,, \\ z(\tau) = x(t(\tau))\,, \quad w(\tau) = u(t(\tau))\,, \\ p_0 = \psi_0\,, \quad p_z(\tau) = \psi(t(\tau))\,, \\ p_t(\tau) = -H\left(t(\tau),\, x(t(\tau)),\, u(t(\tau)),\, \psi_0,\, \psi(t(\tau))\right) \end{gathered}$$

is an extremal of (P_τ) *with* $\mathcal{H}\left(t(\tau),\, z(\tau),\, v(\tau),\, w(\tau),\, p_0,\, p_t(\tau),\, p_z(\tau)\right) \equiv 0$.

Proof. From Lemma 4 we know that such 7-tuple is admissible for (P_τ). The maximality condition (19) is trivially satisfied since we are in the singular case: from (20) the Hamiltonian $\mathcal{H}$ vanishes for $p_t = -H\left(t,\, z,\, w,\, p_0,\, p_z\right)$. It remains to prove the adjoint system (18). Since $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ along the extremals (see e.g. [27] or [4]) the derivative of $p_t(\tau)$ with respect to τ is given by

$$\frac{dp_t}{d\tau} = -\frac{dH}{d\tau} = -\frac{dH}{dt}\,\frac{dt}{d\tau} = -\frac{\partial H}{\partial t}\,\frac{dt}{d\tau} = -\frac{\partial H}{\partial t}\,v.$$

From relation (21) the first of the equalities (18) is proved: $p_t' = -\frac{\partial \mathcal{H}}{\partial t}$. Similarly, as far as $p_z(\tau) = \psi\left(t(\tau)\right)$ and from (16) $\frac{d}{dt}\psi(t) = -\frac{\partial H}{\partial x}$, it follows from (22) that $p_z' = \frac{d\psi(t)}{dt}\,\frac{dt}{d\tau} = -\frac{\partial H}{\partial x}\,v = -\frac{\partial \mathcal{H}}{\partial z}$.

It is also possible to construct an extremal of problem (P) given an extremal of (P_τ).

Lemma 16. *Let* $(t(\cdot),\, z(\cdot),\, v(\cdot),\, w(\cdot),\, p_0,\, p_t(\cdot),\, p_z(\cdot))$ *be an extremal of* (P_τ)*. Then* $(x(\cdot),\, u(\cdot),\, \psi_0,\, \psi(\cdot)) = \left(z\left(\tau(\cdot)\right),\, w\left(\tau(\cdot)\right),\, p_0,\, p_z\left(\tau(\cdot)\right)\right)$ *is an extremal of* (P) *with* $\tau(\cdot)$ *the inverse function of* $t(\cdot)$.

Proof. From Lemma 5 we know that the pair $(x(\cdot),\, u(\cdot))$ is admissible for (P_τ). Direct calculations show that

$$\dot{\psi} = \frac{\mathrm{d}}{\mathrm{d}t}p_z(\tau) = \frac{\mathrm{d}p_z(\tau)}{\mathrm{d}\tau}\,\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial z}\frac{1}{v}\,.$$

From (22) the required adjoint system is obtained: $\dot{\psi} = -\frac{\partial H}{\partial x}$. Maximality condition (19) implies that

$$\begin{aligned} &\mathcal{H}\left(t(\tau),\, z(\tau),\, v(\tau),\, w(\tau),\, p_0,\, p_t(\tau),\, p_z(\tau)\right) \\ &\qquad = \sup_{w \in \mathbb{R}^r} \mathcal{H}\left(t(\tau),\, z(\tau),\, v(\tau),\, w,\, p_0,\, p_t(\tau),\, p_z(\tau)\right) \end{aligned}$$

for almost all $\tau \in [a, b]$. Given the relation (20) one can write that

$$H\left(t(\tau),\, z(\tau),\, w(\tau),\, p_0,\, p_z(\tau)\right) = \sup_{w \in \mathbb{R}^r} H\left(t(\tau),\, z(\tau),\, w,\, p_0,\, p_z(\tau)\right) .$$

Putting $\tau = \tau(t)$ we obtain the maximality condition (17).

Lemmas 15 and 16 establish a correspondence between abnormal extremals of problems (P) and (P_τ).

Corollary 17. *If there are no abnormal extremals of problem (P) then there are no abnormal extremals of problem (P_τ). If there are no abnormal extremals of (P_τ) then there are also no abnormal extremals of (P).*

Definition 18. We call a control an *abnormal extremal control* if it corresponds to an abnormal extremal.

Proposition 19. *If $(\tilde{x}(\cdot),\, \tilde{u}(\cdot))$ is a minimizer of problem (P) and $\tilde{u}(\cdot)$ is not an abnormal extremal control, then the minimizing control $\tilde{v} \equiv 1$ of Proposition 8 is not an abnormal extremal control too.*

In the next section we show that the Lipschitzian regularity conditions we are looking for, assuring that all minimizing controls predicted by Tonelli's existence theorem are indeed bounded, appear from the applicability conditions of the maximum principle to problem $(P_\tau[\tilde{u}(\cdot)])$.

5 The General Regularity Result

Filippov [18] gave the first general existence theorem for optimal control (the original paper, in russian, appear in 1959). There exist now an extensive literature on the existence of solutions to problems of optimal control. We refer the interested reader to the book [5] for significant results, various formulations, and detailed discussions. Follows a set of conditions, of the type of Tonelli [31], that guarantee existence of minimizer for problem (P).

"Tonelli" existence theorem for (P). Problem (P) has a minimizer $(\tilde{x}(\cdot),\, \tilde{u}(\cdot))$ with $\tilde{u}(\cdot) \in L_1\left([a,\, b];\, \mathbb{R}^r\right)$, provided there exists at least one admissible pair, functions $L(\cdot,\cdot,\cdot)$ and $\varphi(\cdot,\cdot,\cdot)$ are continuous, and the following convexity and coercivity conditions hold:

(convexity) Functions $L(t, x, \cdot)$ and $\varphi(t, x, \cdot)$ are convex for all $(t,\, x)$;
(coercivity) There exists a function $\theta : \mathbb{R}_0^+ \to \mathbb{R}$, bounded below, such that

$$L(t,\, x,\, u) \geq \theta\left(\|\varphi(t,\, x,\, u)\|\right) \quad \text{for all } (t, x, u); \tag{23}$$

$$\lim_{r \to +\infty} \frac{\theta(r)}{r} = +\infty; \tag{24}$$

$$\lim_{\|u\| \to +\infty} \|\varphi(t,\, x,\, u)\| = +\infty \quad \text{for all } (t,\, x). \tag{25}$$

Remark 20. For the basic problem of the calculus of variations one has $\varphi = u$ and the theorem above coincides with the classical Tonelli existence theorem.

Analyzing the hypotheses of both necessary optimality conditions and existence theorem, one comes to the conclusion that the requirements of existence theory do not imply those of the maximum principle. Given a problem, it may happen that the necessary optimality conditions are valid while existence is not guarantee; or it may happen that the minimizers predicted by the existence theory fail to be extremals. Follows the main results of the paper.

Theorem 21. *Under the above hypothesis of coercivity, all control minimizers $\tilde{u}(\cdot)$ of (P) which are not abnormal extremal controls are essentially bounded on $[a, b]$ provided the applicability of the maximum principle to problem $(P_\tau\,[\tilde{u}(\cdot)])$ is assured.*

Remark 22. Convexity is not required in the regularity theorem. This is important since existence theorems without the convexity assumptions are a question of great interest (see e.g. [26] and the references therein).

Applying the hypotheses (15) of the maximum principle to functions F and f of problem $(P_\tau\,[\tilde{u}(\cdot)])$, the following result is trivially obtained.

Theorem 23. *Under the hypothesis of coercivity, the growth conditions: there exist constants $c > 0$ and k such that*

$$\left|\frac{\partial L}{\partial t}\right| \leq c\,|L| + k\,, \quad \left\|\frac{\partial L}{\partial x}\right\| \leq c\,|L| + k\,,$$
$$\left\|\frac{\partial \varphi}{\partial t}\right\| \leq c\,\|\varphi\| + k\,, \quad \left\|\frac{\partial \varphi_i}{\partial x}\right\| \leq c\,|\varphi_i| + k \quad (i = 1, \ldots, n)\,;$$

imply that all minimizers $\tilde{u}(\cdot)$ of (P), which are not abnormal extremal controls, are essentially bounded on $[a, b]$.

A minimizer $\tilde{u}(\cdot)$ which is not essentially bounded may fail to satisfy the Pontryagin Maximum Principle. As far as essentially bounded minimizers are concerned, the Pontryagin Maximum Principle is valid.

Corollary 24. *Under the hypotheses of Theorem 23, all minimizers of (P) are Pontryagin extremals.*

Proof. (Theorem 21) Let $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ be a minimizer of (P). From Propositions 8 and 19 and by the assumptions of the theorem, we know that there exist absolutely continuous functions $\tilde{p_t}(\cdot)$ and $\tilde{p_z}(\cdot)$ such that for almost all points $\tau \in [a, b]$

$$v \longmapsto \left[-L\left(\tau, \tilde{x}(\tau), \tilde{u}(\tau)\right) + \tilde{p_t}(\tau) + \tilde{p_z}(\tau) \cdot \varphi\left(\tau, \tilde{x}(\tau), \tilde{u}(\tau)\right)\right] v$$

is maximized at $v = 1$ on the interval $[0.5, 1.5]$. This implies that

$$L\left(\tau, \tilde{x}(\tau), \tilde{u}(\tau)\right) = \tilde{p_t}(\tau) + \tilde{p_z}(\tau) \cdot \varphi\left(\tau, \tilde{x}(\tau), \tilde{u}(\tau)\right)\,. \tag{26}$$

Let $|\tilde{p_t}(\tau)| \leq M$ and $\|\tilde{p_z}(\tau)\| \leq M$ on $[a,\, b]$. Dividing both sides of inequality (26) by $\|\varphi\,(\tau,\, \tilde{x}(\tau),\, \tilde{u}(\tau))\|$ and using the coercivity hypothesis (23), one obtains

$$\frac{\theta\,(\|\varphi\,(\tau,\, \tilde{x}(\tau),\, \tilde{u}(\tau))\|)}{\|\varphi\,(\tau,\, \tilde{x}(\tau),\, \tilde{u}(\tau))\|} \leq M\, \frac{1 + \|\varphi\,(\tau,\, \tilde{x}(\tau),\, \tilde{u}(\tau))\|}{\|\varphi\,(\tau,\, \tilde{x}(\tau),\, \tilde{u}(\tau))\|}\,.$$

The coercivity condition (24)–(25) yields the essential boundedness of $\tilde{u}(\cdot)$ on $[a,\, b]$.

6 An Example

As far as Theorem 23 is able to cover optimal control problems with dynamics which is nonlinear both in the state and in the control variables, plenty of examples possessing minimizers according to the existence theory can be easily constructed for which our result is applicable while previously known Lipschitzian regularity conditions, such as those in [14] and [28], fail. Follows one such example with $n = r = 2$.

Example 25.

$$\int_0^1 \left(u_1^2(t) + u_2^2(t)\right)\, \left(\mathrm{e}^{2\,(x_1(t)+x_2(t))} + 1\right)\, \mathrm{d}t \longrightarrow \min$$

$$\begin{cases} \dot{x_1}(t) = \sqrt{u_1^2(t) + u_2^2(t)} \\ \dot{x_2}(t) = u_2(t)\,\mathrm{e}^{x_1(t)+x_2(t)} \end{cases}$$

$$x_1(0) = 0,\; x_1(1) = 1,\; x_2(0) = 1,\; x_2(1) = 1.$$

Here we have:

$$L(x_1, x_2, u_1, u_2) = \left(u_1^2 + u_2^2\right)\, \left(\mathrm{e}^{2\,(x_1+x_2)} + 1\right);$$

$$\varphi(x_1, x_2, u_1, u_2) = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} \sqrt{u_1^2 + u_2^2} \\ u_2\,\mathrm{e}^{x_1+x_2} \end{bmatrix}.$$

The problem has a solution for $x_1(\cdot)\,, x_2(\cdot) \in W_{1,1}\,([0,\, 1];\, \mathbb{R})$ and $u_1(\cdot),\, u_2(\cdot) \in L_1\,([0,\, 1];\, \mathbb{R})$, as far as all conditions of Tonelli's existence theorem are satisfied:

- An admissible quadruple is $(x_1(t),\, x_2(t),\, u_1(t),\, u_2(t)) = (t,\, 1,\, 1,\, 0)$.
- Functions $L(\cdot,\cdot,\cdot,\cdot)$ and $\varphi(\cdot,\cdot,\cdot,\cdot)$ are continuous in $\mathbb{R}^4$.
- Function $L(x_1, x_2, \cdot, \cdot)$ is strictly convex as far as the matrix

$$\frac{\partial^2 L}{\partial u \partial u} = \begin{bmatrix} 2\left(\mathrm{e}^{2(x_1+x_2)} + 1\right) & 0 \\ 0 & 2\left(\mathrm{e}^{2(x_1+x_2)} + 1\right) \end{bmatrix}$$

is positive-definite for all $(x_1,\, x_2) \in \mathbb{R}^2$. The matrices

$$\frac{\partial^2 \varphi_1}{\partial u \partial u} = \begin{bmatrix} \frac{u_2^2}{\sqrt{\left(u_1^2+u_2^2\right)^3}} & -\frac{u_1 u_2}{\sqrt{\left(u_1^2+u_2^2\right)^3}} \\ -\frac{u_1 u_2}{\sqrt{\left(u_1^2+u_2^2\right)^3}} & \frac{u_1^2}{\sqrt{\left(u_1^2+u_2^2\right)^3}} \end{bmatrix}; \quad \frac{\partial^2 \varphi_2}{\partial u \partial u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are non-negative and it follows that $\varphi(x_1, x_2, \cdot, \cdot)$ is convex.

– From the inequality

$$L = \left(u_1^2 + u_2^2\right)\left(e^{2\,(x_1+x_2)} + 1\right) \geq u_1^2 + u_2^2 + u_2^2 e^{2\,(x_1+x_2)},$$

we have quadratic coercivity ($\theta(r) = r^2$);

Smooth assumptions on data (1) are satisfied, since $L(\cdot,\cdot,\cdot,\cdot)$ and $\varphi(\cdot,\cdot,u_1,u_2)$ are of class C^∞. Theorem 23 allow us to conclude that all minimizing controls, which are not abnormal extremal controls, are bounded:

– The conditions on $\frac{\partial L}{\partial t}$ and $\frac{\partial \varphi}{\partial t}$ are trivially satisfied as far as the problem is autonomous: L and φ do not depend explicitly on the time variable.
– The growth conditions on $\frac{\partial L}{\partial x}$ and $\frac{\partial \varphi}{\partial x}$ are also satisfied:

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = 2e^{2(x_1+x_2)}\left(u_1^2 + u_2^2\right) \leq 2L\,;$$
$$\frac{\partial \varphi_1}{\partial x_1} = \frac{\partial \varphi_1}{\partial x_2} = 0\,;$$
$$\frac{\partial \varphi_2}{\partial x_1} = \frac{\partial \varphi_2}{\partial x_2} = \varphi_2\,.$$

7 Final Remarks

In this paper we study properties of minimizing trajectories for general problems of optimal control in the cases where controls are unconstrained (like in the calculus of variations). We provide new conditions which guarantee Lipschitzian regularity of the minimizing trajectories for the Lagrange problem of optimal control in the general nonlinear case. These conditions solve the discrepancy between the optimality and existence results, assuring that minimizers predicted by the existence theory satisfy the optimality conditions. At the same time, undesirable phenomena, like the Lavrentiev one, are naturally precluded. We show that the conditions of Lipschitzian regularity are related with the applicability conditions of Pontryagin's maximum principle. To deal with dynamics which are control-affine, the classical Pontryagin maximum principle [27] is enough (see [28]). To treat the general case, a maximum principle under weak assumptions, like the one in [4], is necessary. Our approach is based on the relationship of the extremals of the Lagrange problem with the extremals of an auxiliary problem, and on the subsequent utilization of Pontryagin's maximum principle to the later problem. The maximality condition of Pontryagin's maximum principle together with the coercivity assumption of the existence theorem imply the Lipschitzian regularity of the corresponding minimizer of the original problem. This approach allows us to deal with more general class of problems of optimal control with nonlinear dynamics.

It remains to clarify the interconnection between Lipschitzian regularity and abnormal extremality. For the problems of the calculus of variations studied in [13] and [15] no abnormal extremals exist. For the optimal control problems considered in [14] and [28], abnormal extremals are, like here, put aside. The question of how to establish Lipschitzian regularity for the abnormal minimizing trajectories seems to be a completely open question.

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