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Convergence, via summability, of formal power series solutions to a certain class of completely integrable Pfaffian systems

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Abstract. A theory of $\mathbf{k}$-summability in a direction ($\mathbf{k} = (k_1, k_2, \ldots, k_n)$) has been put forward for formal power series of several complex variables. It involves the study of multidimensional Laplace and Borel transforms, and their effect on series (resp. functions) subject to Gevrey-like bounds (resp. admitting Gevrey strongly asymptotic expansion). As an example of the application of this tool to some questions in PDE's, a new proof is given of a result of R. Gérard and Y. Sibuya stating the convergence of the formal power series solutions to certain completely integrable Pfaffian systems.

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1 Introduction

The theory of k-summability in one variable was developed by Ramis [10,11], with the aim of building up analytic solutions to ODE's in sectors, departing from a formal power series solution which, in fact, will asymptotically represent the former. The introduction by Ecalle [4,5] of a more powerful tool, multisummability, led Braaksma [3] to prove that every formal power series solution to a nonlinear system of ODE's at an irregular singular point is multisummable, which allows to compute actual solutions from formal ones. A complete treatment of this subject can be found in the books by Balser [1,2]. The main tools for summability theory

This is an overview article.

are Gevrey asymptotics and (formal and analytic) Laplace-Borel transforms, while multisummability is a kind of "recurrent summability".

As a particular case, consider the system of nonlinear ODE's

$$z^{k+1}\frac{df}{dz} = \varphi_0(z) + A(z)f + \sum_{|\boldsymbol{\alpha}|\geq 2} f^{\boldsymbol{\alpha}}\varphi_{\boldsymbol{\alpha}}(z).$$

where f, φ_0 and $\varphi_{\boldsymbol{\alpha}}$ are p-vectors, and A is a $p\times p$ matrix. Suppose all data are holomorphic in $D=\{|z|<r\}$ and the series in the right member uniformly converge on compact subsets of $D\times\{\|f\|<\rho\}$. If k is a positive integer, $\varphi_0(0)=\mathbf{0}$ and $A(0)$ is invertible, then the system has a unique formal power series solution $\widehat{f}=\sum_{n\in\mathbb{N}} a_n z^n$ that may certainly diverge, though it is always k-summable.

Regarding the case of two variables, let us consider the Pfaffian system

$$\begin{aligned} z_1^{k_1+1}\frac{\partial f}{\partial z_1} &= \varphi_0(z_1,z_2) + A(z_1,z_2)f + \sum_{|\boldsymbol{\alpha}|\geq 2} f^{\boldsymbol{\alpha}}\varphi_{\boldsymbol{\alpha}}(z_1,z_2),\\ z_2^{k_2+1}\frac{\partial f}{\partial z_2} &= \psi_0(z_1,z_2) + B(z_1,z_2)f + \sum_{|\boldsymbol{\alpha}|\geq 2} f^{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}(z_1,z_2). \end{aligned} \tag{1}$$

where f, φ_0, $\varphi_{\boldsymbol{\alpha}}$, ψ_0 and $\psi_{\boldsymbol{\alpha}}$ are p-vectors, and A, B are $p\times p$ matrices. Suppose all data are holomorphic in $D=\{|z_1|+|z_2|<r\}$ and the series in the right members uniformly converge on compact subsets of $D\times\{\|f\|<\rho\}$. If k_1 and k_2 are positive integers, $\varphi_0(0,0)=\psi_0(0,0)=\mathbf{0}$, $A(0,0)$ and $B(0,0)$ are invertible and the system is completely integrable, then the system has a unique formal power series solution

$$\widehat{f} = \sum_{n,m\in\mathbb{N}} a_{nm} z_1^n z_2^m, \qquad a_{nm}\in\mathbb{C}^p.$$

Gérard and Sibuya [6] proved that $\widehat{f}$ is convergent.

This fact was not readily accepted, so Sibuya has given several proofs of this result. In one of them [14] the point of view of summability is adopted, by considering one of the variables as a parameter and summing the series in the other variable "uniformly" with respect to the parameter. However, it seems desirable to find a summation method that treats all variables equally and at the same time, as it has been explicitly pointed out by Sibuya [14] and Balser [2, Chapter 13]. Our definition of summability in a direction for series of several variables relies on (i) the concept of (Gevrey) strongly asymptotically developable functions (see Majima [8,9] and Haraoka [7]), that shares the usual stability properties –in particular, with respect to differentiation– of H. Poincaré's definition in one variable, and (ii) the definition and study of multidimensional Laplace and Borel transforms. As an application of this theory, we provide a new proof of the result by Gérard and Sibuya. For a detailed treatment of all these topics, see [13].

2 Notation

For $\boldsymbol{k} = (k_1, k_2) \in (0, \infty)^2$, $A = (A_1, A_2) \in (0, \infty)^2$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we put

$$\Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) = \Gamma(1 + \alpha_1/k_1)\Gamma(1 + \alpha_2/k_2), \quad A^{\boldsymbol{\alpha}} = A_1^{\alpha_1} A_2^{\alpha_2},$$

where the $\Gamma^{'s}$ on the right stand for the Gamma function. Consider, for $j = 1, 2$, an *open sector* (on the Riemann surface of the Logarithm) with vertex at the origin,

$$S_j = S(d_j, \theta_j, \rho_j) = \{ z = re^{i\varphi} : 0 < r < \rho_j, \ |\varphi - d_j| < \theta_j/2 \},$$

where $d_j \in \mathbb{R}$, $\theta_j > 0$ and $\rho_j \in (0, \infty]$ are the *bisecting direction*, the *width* and the *radius* of S_j, respectively. The *polysector* $\prod_{j=1}^2 S_j \subset \mathbb{R}^2$ will be denoted by $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$, where $\boldsymbol{d} = (d_1, d_2)$, $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and $\boldsymbol{\rho} = (\rho_1, \rho_2)$. In case $\rho_j = +\infty$ for $j = 1, 2$, we write $S = S(\boldsymbol{d}, \boldsymbol{\theta})$.

A polysector $T = \prod_{j=1}^2 T(d'_j, \theta'_j, \rho'_j)$ on $\mathbb{R}^2$ is a *bounded proper subpolysector* of $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$, and we write $T \ll S$, if for $j = 1, 2$ we have $\rho'_j < \rho_j$ (so that $\rho'_j < +\infty$) and

$$[d'_j - \rho'_j/2, d'_j + \rho'_j/2] \subset (d_j - \rho_j/2, d_j + \rho_j/2).$$

3 $\boldsymbol{k}$-summability of power series in two variables

Let us begin stating the basic definitions and results about Gevrey asymptotics in the sense of Majima.

Definition 1. A function $f \colon S = S_1 \times S_2 \subset \mathcal{R}^2 \to \mathbb{C}$ is *Gevrey strongly asymptotically developable of order* $\boldsymbol{k}$ ($f \in \mathcal{A}_{\boldsymbol{k}}(S)$) if there exists a family $\mathrm{TA}(f) = \{h_m, g_n, a_{nm} : n, m \in \mathbb{N}\}$, where h_m (resp. g_n) is a holomorphic function from S_1 (resp. S_2) to $\mathbb{C}$ and $a_{nm} \in \mathbb{C}$, $n, m \in \mathbb{N}$, such that the following holds: if we define the *approximate function of order* $\boldsymbol{\alpha} = (n, m) \in \mathbb{N}^2$ by

$$\mathrm{App}_{\boldsymbol{\alpha}}(f)(\boldsymbol{z}) = \sum_{j=0}^{n-1} g_j(z_2) z_1^j + \sum_{\ell=0}^{m-1} h_\ell(z_1) z_2^\ell - \sum_{j,\ell=0}^{j=n-1,\,\ell=m-1} a_{j\ell} z_1^j z_2^\ell,$$

then for every $T \ll S$, there exist $C_T > 0$, $A_T \in (0, \infty)^2$ such that for every $\boldsymbol{\alpha} \in \mathbb{N}^2$,

$$|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(f)(\boldsymbol{z})| \le C_T\, \Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) A_T^{\boldsymbol{\alpha}} |\boldsymbol{z}|^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} \in T.$$

$\mathrm{FA}(f) := \sum_{(n,m)\in\mathbb{N}^2} a_{nm} z_1^n z_2^m$ is the (formal) *series of strongly asymptotic expansion* of f.

Whenever $f \in \mathcal{A}_{\boldsymbol{k}}(S)$ one has $\mathrm{FA}(f)$ is *Gevrey of order* $\boldsymbol{k}$ ($\mathrm{FA}(f) \in \mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$), i.e., there exist $C > 0$, $A = (A_1, A_2) \in (0, \infty)^2$ such that for every $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, $|a_{\boldsymbol{\alpha}}| \le C\, \Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) A^{\boldsymbol{\alpha}}$.

$\mathcal{A}_{\boldsymbol{k}}(S)$ and $\mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$ are differential algebras, and the map $\mathrm{FA} \colon \mathcal{A}_{\boldsymbol{k}}(S) \to \mathbb{C}[\![\boldsymbol{z}]\!]_{\boldsymbol{k}}$ is a homomorphism. The next result will be crucial for our purposes.

Proposition 2 (Watson's lemma [12,13]). *Let* $S = S(d_1, \theta_1, \rho_1) \times S(d_2, \theta_2, \rho_2)$ *be a polysector such that* $\theta_j > \pi/k_j$ *for* $j = 1, 2$*. Then, the mapping* FA *is injective.*

We are now ready to introduce a concept of summability for formal power series in two variables.

Definition 3. A formal power series $\widehat{f} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^2} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ is said to be $\boldsymbol{k}$*-summable in direction* $\boldsymbol{d} \in \mathbb{R}^2$ if there exist a polysector $S = S(\boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\rho})$, with $\boldsymbol{\theta} > \dfrac{\boldsymbol{\pi}}{\boldsymbol{k}}$, and a function $f \in \mathcal{A}_{\boldsymbol{k}}(S)$ such that $\mathrm{FA}(f) = \widehat{f}$. In this case, we write $f = \mathcal{S}_{\boldsymbol{k},\boldsymbol{d}}(\widehat{f}\,)$, and call f the $\boldsymbol{k}$*-sum of* $\widehat{f}$ *in direction* $\boldsymbol{d}$.

It seems natural to ask whether the analogue to Fubini's theorem holds in this context. The fact is that basic results remain unaltered when one considers:

1. (Strong) asymptotic expansions for functions with values in a complex Banach space $(E, \|\cdot\|)$, and
2. k-summability of formal power series in one variable with coefficients in E.

Suppose $S = S_1 \times S_2$ and $B \in (0, \infty)^2$. Although $\mathcal{A}_{\boldsymbol{k}}(S, E)$ (with the expected meaning) is not a Banach space, we can consider its subspace

$$\mathcal{W}_{\boldsymbol{k},B}(S,E) := \{f \colon S \to E : \|f\|_{\boldsymbol{k},B} := \sup_{\boldsymbol{z} \in S,\, \boldsymbol{\alpha} \in \mathbb{N}^2} \frac{\|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})\|}{\boldsymbol{\alpha}!\, \Gamma(1 + \boldsymbol{\alpha}/\boldsymbol{k}) B^{\boldsymbol{\alpha}}} < \infty\}.$$

Lemma 4. *$(\mathcal{W}_{\boldsymbol{k},B}(S,E), \|\cdot\|_{\boldsymbol{k},B})$ is a Banach space, and the restriction to any $T \ll S$ of the functions in $\mathcal{A}_{\boldsymbol{k}}(S,E)$ provides elements of $\mathcal{W}_{\boldsymbol{k},B}(T,E)$ for some suitable $B = B(T)$. Also, the map*

$$f \in \mathcal{W}_{\boldsymbol{k},B}(S,E) \to f^* \in \mathcal{W}_{k_1,B_1}(S_1, \mathcal{W}_{k_2,B_2}(S_2,E))$$

defined for every $z_1 \in S_1$ by $f^*(z_1) = f(z_1, \cdot)$ is an isomorphism.

These facts together lead to the following definition and result.

Definition 5. A formal power series $\widehat{f} = \sum_{n,m=0}^{\infty} a_{nm} z_1^n z_2^m$ is *iteratively* $\boldsymbol{k}$*-summable in direction* $\boldsymbol{d}$ (in a certain order, but this will turn out to be irrelevant) if, when we write

$$\widehat{f} = \sum_{n=0}^{\infty} \widehat{g}_n z_1^n, \qquad \text{where} \quad \widehat{g}_n = \sum_{m=0}^{\infty} a_{nm} z_2^m,$$

the following hold:

(i) Every $\widehat{g}_n$ is k_2-summable in direction d_2, and the sum g_n belongs to $\mathcal{A}_{k_2}(S_2)$, where the sector $S_2 = S_2(d_2, \theta_2, \rho_2)$ does not depend on n.

(ii) There exist $T_2 = T_2(d_2, \varphi_2, r_2) \ll S_2$, with $\varphi_2 > \pi/k_2$, and $B_2(T_2) > 0$ such that $g_n \in \mathcal{W}_{k_2,B_2}(T_2)$ for every $n \in \mathbb{N}$, and the series $\widehat{g} = \sum_{n=0}^{\infty} g_n z_1^n$ (with coefficients in that Banach space) is k_1-summable in direction d_1.

Proposition 6. *A formal power series $\widehat{f}$ is $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$ if and only if it is iteratively $\boldsymbol{k}$-summable in direction $\boldsymbol{d}$ (in any order).*

It will also be interesting to consider the case that convergence appears in some of the summation steps. Let $S_1 = S(d_1, \theta_1, \rho_1)$ be a sector, D_2 a disk around 0 and $k_1 > 0$.

Definition 7. The space $\mathcal{A}_{(k_1,\infty)}(S_1 \times D_2)$ consists of the holomorphic functions $f\colon S_1 \times D_2 \to \mathbb{C}$ for which there exists a family $\mathrm{TA}(f) = \{h_m, g_n, a_{nm} : n, m \in \mathbb{N}\}$, where h_m (resp. g_n) is a holomorphic function from S_1 (resp. D_2) to $\mathbb{C}$ and $a_{nm} \in \mathbb{C}$, $n, m \in \mathbb{N}$,such that, if we define $\mathrm{App}_{\boldsymbol{\alpha}}(f)$ as before, then for every $T_1 \ll S_1$ and every compact $K_2 \subset D_2$ there exist $C > 0$ and $A \in (0,\infty)^2$ (both depending on T_1 and K_2) such that for every $\boldsymbol{\alpha} = (n, m) \in \mathbb{N}^2$ and $\boldsymbol{z} \in T_1 \times K_2$,

$$|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(f)(\boldsymbol{z})| \le C\,\Gamma(1 + n/k_1)A^{\boldsymbol{\alpha}}|\boldsymbol{z}|^{\boldsymbol{\alpha}}.$$

Definition 8. A formal power series $\widehat{f}$ is (k_1,∞)-*summable in direction* $d_1 \in \mathbb{R}$ if there exist $S_1 = S(d_1, \theta_1, \rho_1)$, with $\theta_1 > \pi/k_1$, a disk D_2 and a function $f \in \mathcal{A}_{(k_1,\infty)}(S_1 \times D_2)$ such that $\mathrm{FA}(f) = \widehat{f}$.

For a complex Banach space E, a disk D and $B > 0$, we consider the space

$$\mathcal{W}_{\infty,B}(D, E) := \{f\colon D \to E\colon \|f\|_{\infty,B} := \sup_{z\in D,\, p\in\mathbb{N}} \frac{\|f^{(p)}(z)\|}{p!\,B^p} < \infty\}.$$

$(\mathcal{W}_{\infty,B}(D, E), \|\cdot\|_{\infty,B})$ is a Banach space, and it plays a role in the characterization, similar to that in Proposition 6, of (k_1,∞)-summability in terms of iterative summability.

4 New proof of Gérard-Sibuya's result

The information in the paper of Sibuya [14] can be seen to imply that the formal solution to (1), $\widehat{f} = \sum_{n,m=0}^{\infty} a_{nm} z_1^n z_2^m$, is:

(a) (iteratively) (k_1,∞)-summable in every direction d_1 except for those in a finite set E_1, and
(b) (iteratively) (∞, k_2)-summable in every direction d_2 except for those in a finite set E_2.

By (b), $\widehat{h}_m := \sum_{n=0}^{\infty} a_{nm} z_1^n$ converges in a disk D_1 for every m, with sum h_m. By (a), we can choose directions $d_1^1, \dots, d_1^\ell \notin E_1$ in such a way that the sectors T_j where the sums $\mathcal{S}_{k_1,d_1^j}\widehat{h}_m$ are defined cover a whole disk $\widehat{D}_1 \subset D_1$ around 0 (of course, these sums glue together and agree with h_m); also, there exists $A_j > 0$ such that $h_m \in \mathcal{W}_{k_1,A_j}(T_j)$ for every m, and the series $\sum_{m=0}^{\infty} h_m z_2^m$, with coefficients in $\mathcal{W}_{k_1,A_d}(T_d)$, converges. Since

$$\sup_{z_1\in\widehat{D}_1} |h_m(z_1)| \le \sup_{d\in F_1} \|\mathcal{S}_{k_1,d}\widehat{h}_m\|_{k_1,A_d},$$

we see that $\sum_{m=0}^{\infty} h_m z_2^m$ converges in a disk D_2 when its coefficients are considered in the space $\mathcal{H}(\widehat{D}_1))$ of holomorphic functions in $\widehat{D}_1$ (with the compact open topology). In order to conclude, it suffices to take into account that $\mathcal{H}(D_2, \mathcal{H}(\widehat{D}_1))$ and $\mathcal{H}(\widehat{D}_1 \times D_2)$ are isomorphic.

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