EQUADIFF 10

Nikolaos S. Papageorgiou; George Smyrlis

A multiplicity theorem for the Neumann problem for nonlinear hemivariational inequalities

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 327--337.

Persistent URL: http://dml.cz/dmlcz/700364

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 327–337

# A multiplicity theorem for the Neumann problem for Nonlinear Hemivariational Inequalities

Nikolaos S. Papageorgiou[1] and George Smyrlis[2]

[1] Department of Mathematics, National Technical University,
Zographou Campus, Athens 15780, Greece
Email: npapg@math.ntua.gr

[2] Department of Mathematics, National Technical University,
Zographou Campus, Athens 15780, Greece
Email: grsmir@math.ntua.gr

**Abstract.** In this paper we study a nonlinear hemivariational inequality driven by the $p$-Laplacian with Neumann boundary conditions. We prove a multiplicity theorem which produces at least three distinct solutions. We employ a Landesman-Lazer type condition and our approach is based on the nonsmooth critical point theory for locally Lipschitz functions.

**MSC 2000.** 35J20, 35J85

**Keywords.** Hemivariational inequality, Neumann problem, locally Lipschitz function, Clarke subdifferential, nonsmooth Cerami condition, nonsmooth critical point theory, multiplicity theorem, $p$-Laplacian.

## 1 Introduction

Hemivariational inequalities are a new type of variational expressions, which arise in physical and engineering problems, when we deal with nonsmooth, nonconvex energy functionals. Generally speaking, mechanical problems involving nonmonotone, possibly multivalued stress-strain laws or boundary conditions derived by nonconvex superpotentials, lead to hemivariational inequalities. For concrete applications we refer to the books of Naniewicz -Panagiotopoulos [22] and Panagiotopoulos [23]. Hemivariational inequalitites have intrinsinc mathematical interest as a new form of variational expression. They include as a particular case problems with discontinuities. In the last five years hemivariational inequalities

*This is the final form of the paper.*

have been studied from a mathematical viewpoint primarily for semilinear Dirichlet problems. We refer to the works of Goeleven-Motreanu-Panagiotopoulos [12], Motreanu-Panagiotopoulos [21] and the references therein.Quasilinear Dirichlet problems were studied recently by Gasinski-Papageorgiou [8],[9], [10], [11]. The study of the Neumann problem is lagging behind. In the past Neumann problems with a $C^1$-energy functional (i.e. continuous forcing term) were studied by Mawhin-Ward-Willem [20], Drabek-Tersian [7] (semilinear problems) and Huang [16], Arcoya-Orsina [2] and Hu-Papageorgiou [13] (quasilinear problems). The only work on the Neumann problem with a discontinuous forcing term is that of Costa-Goncalves [6], where the right hand side of the semilinear equation is independent of $z \in Z$, it is bounded and it has mean value zero.

The aim of this paper is to prove a multiplicity result for a quasilinear hemivariational inequality with Neumann boundary condition, using conditions of Landesman-Lazer type. Similar conditions were employed by Goeleven -Motreanu-Panagiotopoulos [12] (semilinear Dirichlet problems) and by Arcoya-Orsina [2] (quasilinear Neumann problems with a $C^1$-potential function). In [12], the approach is degree theoretic and the authors make a rather restrictive hypothesis, namely they assume that there exists a continuous map $W : L^2(Z) \longrightarrow L^2(Z)$ such that $W(x)(z) \in \partial j(z, x(z))$, a.e. on $Z$ (here $\partial j(z, \cdot)$ denotes the subdifferential in the sense of Clarke). Given that the Clarke's subdifferential is only strong -to-weak upper semicontinuous, we realize that this is a quite restrictive hypothesis. In Arcoya-Orsina [2] the approach is variational. However, we think that there is a gap in the proof of the existence theorem (theorem 3). Namely, the claim (p. 1631) that the proof of lemma 2.1 extends to the Neumann problem is not precise, since we no longer can appeal to Poincaré's inequality. A more careful analysis is needed.

Our approach is variational and it is based on the nonsmooth critical point theory for locally Lipschitz functionals as this was formulated by Chang [4] and extended recently by Kourogenis-Papageorgiou [18].

## 2 Mathematical background

Let $X$ be a Banach space and $X^*$ its dual. A function $\varphi : X \longrightarrow \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighborhood $\mathcal{U}$ of $x$ and a constant $k_{\mathcal{U}}$ such that $|\varphi(z) - \varphi(y)| \leq k_{\mathcal{U}} \|z - y\|$, for all $z, y \in \mathcal{U}$. Recall that if $\psi : X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous, it is locally Lipschitz in the interior of its effective domain $dom\psi = \{x \in X : \psi(x) < +\infty\}$. Given $x, h \in X$ we can define the generalized directional derivative of $\varphi$ at $x$ in the direction $h$ by

$$\varphi^0(x; h) = \limsup_{x' \to x,\ t \downarrow 0} \frac{\varphi(x' + th) - \varphi(x')}{t}.$$

It is easy to see that $h \mapsto \varphi^0(x;h)$ is sublinear continuous, so by the Hahn-Banach theorem it is the support function of a nonempty, convex and $w^*$-compact set

$$\partial\varphi(x) = \left\{x^* \in X^* : (x^*, h) \leq \varphi^0(x;h), \text{ for all } h \in X\right\}.$$

The set $\partial\varphi(x)$ is called the generalized (Clarke) subdifferential of $\varphi$ at $x \in X$.If $\varphi$ is also convex, Clarke's subdifferential coincides with the subdifferential in the sense of convex analysis. If $\varphi, \psi$ are both locally Lipschitz functions, then for all $x \in X$ and all $\lambda \in \mathbb{R}$, we have $\partial(\varphi+\psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x)$ and $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. This fact makes the nonsmooth critical point theory an extension of the smooth theory. A point $x \in X$ is said to be a critical point of $\varphi$, if $0 \in \partial\varphi(x)$, i.e. $\varphi^0(x;h) \geq 0$, for all $h \in X$. Evidently, if $x \in X$ is a local extremum of $\varphi$, then $0 \in \partial\varphi(x)$. It is well known that the smooth critical point theory uses a compactness-type condition known as the Palais-Smale condition or the more general Cerami condition. In the present nonsmooth setting this condition takes the form:" We say that $\varphi$ satisfies the nonsmooth $C_c$-condition, if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and $(1+\|x_n\|)m(x_n) \to 0$, as $n \to +\infty$, has a strongly convergent subsequence". Here $m(x) = \inf\{\|x^*\| : x^* \in \partial\varphi(x)\}$, for all $x \in X$.

Let $Z \subseteq \mathbb{R}^N$ be a bounded open domain with a $C^1$-boundary. We consider the following direct sum decomposition : $W^{1,p}(Z) = \mathbb{R} \oplus V$, $V = \{v \in W^{1,p}(Z) : \int_Z v(z)dz = 0\}$, $2 \leq p < \infty$. Let $\lambda_1 = \inf\left\{\frac{\|Dv\|_p^p}{\|v\|_p^p} : v \in V, v \neq 0\right\}$. We can show that $\lambda_1 > 0$ and that it is the first nonzero eigenfunction of the $p$-Laplacian $-\Delta_p x = -div\,(\|Dx\|^{p-2}\,Dx)$ with Neumann boundary condition (note that $\lambda_0 = 0$ is also an eigenvalue).

## 3 Multiplicity Theorem

We study the following quasilinear hemivariational inequality:

$$\left\{\begin{array}{l} -div\,(||Dx(z)||^{p-2}\,Dx(z)) \in \partial j(z,x(z)) \quad \text{a.e. on } Z \\[2ex] \dfrac{\partial x}{\partial n_p}(z) = 0, \quad \text{on } \Gamma. \end{array}\right\} \tag{1}$$

Here $\dfrac{\partial x}{\partial n_p}(z) = ||Dx(z)||^{p-2}\,(Dx(z), n(z))_{\mathbb{R}^N}$, with $n(z)$ the outward normal to $\Gamma$ (= the boundary of $Z$) and $2 \leq p < \infty$. Our hypotheses on the nonsmooth potential $j(z,x)$ are the following:

$\underline{H(j)}$ : $j : Z \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that $j(\cdot, 0) \in L^1(Z)$ and

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable;

(ii) for almost $z \in Z$, $x \mapsto j(z,x)$ is locally Lipschitz;

(iii) for almost $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \leq a(z) + c|x|^{r-1}, \text{ with } a \in L^q(Z) \ \left(\frac{1}{p} + \frac{1}{q} = 1\right), \ c > 0, \ 1 \leq r < p;$$

(iv) for almost $z \in Z$ and all $x \in \mathbb{R}$, $pj(z,x) \leq \lambda_1 \, |x|^p$

(v) there exist two functions $j_\pm \in L^1(Z)$ such that $\lim_{x\to\pm\infty} j(z,x) = j_\pm(z)$, uniformly for almost all $z \in Z$

(vi) there exist $c_- < 0 < c_+$ such that

$$\int_Z j(z,c_+)dz, \int_Z j(z,c_-)dz > \int_Z j_\pm(z)dz > 0.$$

We consider the energy functional $\varphi : W^{1,p}(Z) \longrightarrow \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j(z,x(z))dz.$$

We know that $\varphi$ is locally Lipschitz (see Hu-Papageorgiou [15], p.313).

**Proposition 1.** *If hypotheses $H(j)$ hold, then $\varphi$ satisfies the nonsmooth $C_c$ -condition for $c \neq -\int_Z j_\pm(z)dz$ .*

*Proof.* Let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ be a sequence such that

$$\varphi(x_n) \to c, \text{ with } c \neq -\int_Z j_\pm(z)dz \quad \text{and } (1+\|x_n\|)m(x_n) \to 0, \quad \text{as } n \to \infty.$$

We will show that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded. Suppose not. Then by passing to a subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(Z), \;\; y_n \to y \text{ in } L^p(Z), \;\; \text{a.e. on } Z$$

and

$$|y_n(z)| \leq k(z), \;\; \text{a.e. on } Z, \text{ for all } n \geq 1, \text{ with } k \in L^p(Z).$$

From the choice of the sequence $\{x_n\}_{n\geq 1}$ we have that

$$\left|\frac{1}{p}\|Dx_n\|_p^p - \int_Z j(z,x_n(z))dz\right| \leq M_1, \;\; \text{for all } n \geq 1, \text{ with } M_1 > 0.$$

Dividing by $\|x_n\|^p$ we obtain

$$\left|\frac{1}{p}\|Dy_n\|_p^p - \int_Z \frac{j(z,x_n(z))}{\|x_n\|^p}dz\right| \leq \frac{M_1}{\|x_n\|^p}. \tag{2}$$

Using Lebourg's mean value theorem (see Clarke [5], p.41), we see that for almost all $z \in Z$ we can find $u_\lambda^* \in \partial j(z,\lambda x)$ with $0 < \lambda < 1$, such that

$$\begin{aligned} &|j(z,x) - j(z,0)| = |u_\lambda^* x| \\ \Longrightarrow\; &|j(z,x)| \leq |j(z,0)| + (a(z) + c|x|^{r-1})|x| \leq a_1(z) + c_1|x|^r, \;\; a_1 \in L^1(Z), \; c_1 > 0 \end{aligned}$$

(see hypothesis $H(j)$(iii) and recall that $j(\cdot,0) \in L^1(Z)$). So we obtain

$$\int_Z \frac{j(z,x_n(z))}{\|x_n\|^p}dz \le \int_Z \frac{a_1(z)}{\|x_n\|^p}dz + c_1 \int_Z \frac{|y_n(z)|^r}{\|x_n\|^{p-r}}dz \to 0, \ \text{ as } n \to \infty (\text{ since } r < p).$$

Thus by passing to the limit in (2) and using the weak lower semicontinuity of the norm functional, we obtain $\|Dy\|_p^p = 0 \Longrightarrow y = \xi \in \mathbb{R}$. Note that $y_n \to \xi$ in $W^{1,p}(Z)$ and because $\|y_n\| = 1$, for $n \ge 1$, we infer that $\xi \ne 0$. We may assume without loss of generality that $\xi > 0$. Then $x_n(z) \to +\infty$, a.e. on $Z$. Also let $x_n^* \in \partial\varphi(x_n)$, $n \ge 1$, such that $m(x_n) = \|x_n^*\|$. The existence of such an element follows from the fact that $\partial\varphi(x_n)$ is $w$-compact and $x^* \mapsto \|x^*\|$ is weakly lower semicontinuous on $W^{1,p}(Z)^*$. From the choice of the sequence $\{x_n\}_{n\ge1} \subseteq W^{1,p}(Z)$ we have $|\langle x_n^*, \ x_n\rangle| \le \varepsilon_n$, with $\varepsilon_n \downarrow 0$ and $\langle\cdot,\cdot\rangle$ being the duality brackets for the pair $\big(W^{1,p}(Z), \ W^{1,p}(Z)^*\big)$. Let $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2} \left(Dx(z), Dy(z)\right)_{\mathbb{R}^N} dz, \text{ for all } x, y \in W^{1,p}(Z).$$

It is easy to check that $A$ is demicontinuous, monotone, thus it is maximal (see Hu-Papageorgiou [14], p. 309). For every $n \ge 1$, we have $x_n^* = A(x_n) - u_n^*$, with $u_n^* \in \partial J(x_n)$, where $J : W^{1,p}(Z) \longrightarrow \mathbb{R}$ is the integral functional defined by $J(x) = \displaystyle\int_Z j(z,x(z))dz$. If $J_1 : L^p(Z) \longrightarrow \mathbb{R}$ is defined by $J_1(x) = \displaystyle\int_Z j(z,x(z))dz$, then $J = J_1 \mid_{W^{1,p}(Z)}$ and both are locally Lipschitz. Moreover, from Chang [4] (theorem 2.2) we have that $\partial J(x) \subseteq \partial J_1(x) = S^q_{\partial j(\cdot,x(\cdot))} = \{u^* \in L^q(Z) : u^*(z) \in \partial j(z,x(z)) \ \text{ a.e. on } Z\} \subseteq L^q(Z)$ (see Clarke [5], p. 83). So we have

$$\langle x_n^*, x_n\rangle = \|Dx_n\|_p^p - \int_Z u_n^*(z)x_n(z)dz \le \varepsilon_n.$$

We consider the direct -sum decomposition $W^{1,p}(Z) = \mathbb{R} \oplus V$, with $V = \{v \in W^{1,p}(Z) : \displaystyle\int_Z v(z)dz = 0\}$. So we can write $x_n = \overline{x}_n + \hat{x}_n$ with $\overline{x}_n \in \mathbb{R}$ and $\hat{x}_n \in V$, $n \ge 1$ and we we have

$$\|D\hat{x}_n\|_p^p - \int_Z u_n^*(z)x_n(z)dz \le \varepsilon_n. \tag{3}$$

From the definition of the Clarke subdifferential (see section 2) we have

$$u_n^*(z)x_n(z) \le j^0(z,x_n(z)) = \limsup_{\substack{v_n \to x_n(z)\\ \varepsilon\downarrow 0}} \frac{j(z, v_n + \varepsilon\, x_n(z)) - j(z,v_n)}{\varepsilon}.$$

Recall that for almost all $z \in Z$, $x_n(z) \to +\infty$ as $n \to \infty$. So $v_n \to +\infty$ as $n \to \infty$. Hence by virtue of hypothesis $H(j)$ (v), given $\varepsilon > 0$ we can find $n_0(\varepsilon) \ge 1$ such that for all $n \geq n_0$ and all $z \in Z \setminus N_1$, $|N_1| = 0$ ($|\cdot|$ being the Lebesgue measure on $\mathbb{R}^N$), we have $j_+(z) - \varepsilon^2 \leq j(z, v_n + \varepsilon x_n(z)) \leq j_+(z) + \varepsilon^2$ and $j_+(z) - \varepsilon^2 \leq j(z, v_n) \leq j_+(z) + \varepsilon^2$. So for all $n \geq n_0$ and all $z \in Z \setminus N_1$, we have

$$u_n^*(z)x_n(z) \leq \frac{j_+(z) + \varepsilon^2 - j_+(z) + \varepsilon^2}{\varepsilon} = \frac{2\varepsilon^2}{\varepsilon} = 2\varepsilon$$

and

$$u_n^*(z)x_n(z) \geq \frac{j_+(z) - \varepsilon^2 - j_+(z) - \varepsilon^2}{\varepsilon} = \frac{-2\varepsilon^2}{\varepsilon} = -2\varepsilon.$$

Therefore for all $n \geq n_0$ and all $z \in Z \setminus N_1$, we have $|u_n^*(z)x_n(z)| \leq \varepsilon$, hence $u_n^*(z)x_n(z) \to 0$ as $n \to \infty$ uniformly for almost all $z \in Z$ and so it is fulfilled that $\int_Z u_n^*(z)x_n(z)dz \to 0$ as $n \to \infty$. Thus if we pass to the limit as $n \to \infty$ in (3), we obtain $\|D\hat{x}_n\|_p^p \to 0$ as $n \to \infty$. This by virtue of the Poincaré-Wirtinger inequality (see Hu-Papageorgiou [15], p. 866) implies that $\hat{x}_n \to 0$ in $W^{1,p}(Z)$ as $n \to \infty$.

Let $\Gamma_n(z) = \{(v^*, \lambda) \in \mathbb{R} \times (0,1) : v^* \in \partial j(z, \overline{x}_n + \lambda\hat{x}_n(z)),\ j(z, \overline{x}_n + \hat{x}_n(z)) - j(z, \overline{x}_n) = v^* x_n(z)$. From the Lebourg mean value theorem, we have that for almost all $z \in Z$, $\Gamma_n(z) \neq \emptyset$. By redefining $\Gamma_n$ on the exceptional Lebesgue-null set (setting $\Gamma_n$ to be equal to $\{0\}$, for example), we may assume without any loss of generality that $\Gamma_n(z) \neq \emptyset$ for all $z \in Z$. We claim that for every $h \in \mathbb{R}$, the function $(z, \lambda) \mapsto j^0(z, \overline{x}_n + \lambda\hat{x}_n(z); h)$ is measurable. Indeed, from the definition of the generalized directional derivative, $j^0(z, \overline{x}_n + \lambda\hat{x}_n(z); h)$ equals to

$$\inf_{m\geq 1} \sup_{r,s\, \in\, \mathbb{Q}\, \cap\, (-1/m,\, 1/m)} \frac{j(z, \overline{x}_n + \lambda\hat{x}_n(z) + r + sh) - j(z, \overline{x}_n + \lambda\hat{x}_n(z) + r)}{s}.$$

But $j(z,x)$ is jointly measurable (see Hu-Papageorgiou [14], p.142). So it follows that $(z, \lambda) \mapsto j^0(z, \overline{x}_n + \lambda\hat{x}_n(z); h)$ is measurable. Let $S_n(z,\lambda) = \partial j(z, \overline{x}_n + \lambda\hat{x}_n(z))$ and $\{h_m\}_{m\geq 1} \subseteq \mathbb{R}$ be a countable dense set. Because $j^0(z, \overline{x}_n + \lambda\hat{x}_n(z); \cdot)$ is continuous, we have

$$\begin{aligned} GrS_n &= \{(z,\lambda,u) \in Z \times (0,1) \times \mathbb{R} : u \in S_n(z,\lambda)\} \\ &= \bigcap_{m\geq 1} \{(z,\lambda,u) \in Z \times (0,1) \times \mathbb{R} : uh_m \leq j^0(z, \overline{x}_n + \lambda\hat{x}_n(z); h_m)\}, \end{aligned}$$

so $Gr\Gamma_n = GrS_n \bigcap \{(z, v^*, \lambda) \in Z \times \mathbb{R} \times (0,1) : j(z, \overline{x}_n + \hat{x}_n(z)) - j(z, \overline{x}_n) = v^*\hat{x}_n(z)\} \in \mathcal{L} \times B(\mathbb{R}) \times B(0,1)$, with $\mathcal{L}$ being the Lebesgue $\sigma$-field of $Z$. Invoking the Yankov-von-Neumann-Aumann selection theorem (see Hu-Papageorgiou [14], p. 158), we obtain measurable functions $v_n^* : Z \longrightarrow \mathbb{R}$ and $\lambda_n : Z \longrightarrow (0,1)$ such that $(v_n^*(z), \lambda_n(z)) \in \Gamma_n(z)$ a.e. on $Z$. Therefore we have

$$j(z, \overline{x}_n + \hat{x}_n(z)) - j(z, \overline{x}_n) = v_n^*(z)\hat{x}_n(z),\ \ v_n^*(z) \in \partial j(z, \overline{x}_n + \lambda\hat{x}_n(z)),\ \text{a.e. on } Z,$$

for all $n \geq 1$. Thus we can write that

$$\varphi(x_n) = \frac{1}{p}\|Dx_n\|_p^p - \int_Z v_n^*(z)\hat{x}_n(z)dz - \int_Z j(z, \overline{x}_n)dz. \tag{4}$$

Arguing as before, we can show that $\int_Z v_n^*(z)\hat{x}_n(z)dz \to 0$ as $n \to \infty$ (note that $\overline{x}_n \to +\infty$ since $x_n(z) \to +\infty$ a.e. on $Z$ and $\{\hat{x}_n(z)\}_{n\geq 1}$ is bounded for almost all $z \in Z$ (recall that $\hat{x}_n \to 0$ in $W^{1,p}(Z)$)). Also we know that $\|Dx_n\|_p = \|D\hat{x}_n\|_p \to 0$ as $n \to \infty$. So by passing to the limit as $n \to \infty$ in (4), we obtain that $c = -\int_Z j_+(z)dz$, a contradiction. Similarly, if we assume that $\xi < 0$, we reach the contradiction that $c = -\int_Z j_-(z)dz$.

Therefore $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$. We have $|\langle x_n^*,\ x_n - x\rangle| \leq \varepsilon_n$, with $\varepsilon_n \downarrow 0$, which implies that $\langle A(x_n),\ x_n - x\rangle - \int_Z u_n^*(x_n - x)dz \leq \varepsilon_n,\ \ n \geq 1$.

By virtue of hypothesis $H(j)$ (iii), $\{u_n^*\}_{n\geq 1} \subseteq L^{r'}(Z) \subseteq L^q(Z)$ is bounded ($1/r + 1/r' = 1$ and $r < p$). So $\int_Z u_n^*(x_n - x)dz \to 0$. It follows that $\limsup\langle A(x_n),\ x_n - x\rangle \leq 0$ and because $A$ is maximal monotone, it is generalized pseudomonotone (see Hu-Papageorgiou [14], p. 365) and so $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$, hence $\|Dx_n\|_p \to \|Dx\|_p$. Because $Dx_n \xrightarrow{w} Dx$ in $L^p(Z,\mathbb{R}^N)$ and the latter is a uniformly convex Banach space, it follows that $Dx_n \to Dx$ in $L^p(Z,\mathbb{R}^N)$ (see Hu-Papageorgiou[14], p. 28) and so $x_n \to x$ in $W^{1,p}(Z)$.

**Proposition 2.** *If hypotheses $H(j)$ hold, then $\varphi$ is bounded below and $\varphi\,|_V \geq 0$.*

*Proof.* By virtue of hypothesis $H(j)$ (v), we can find $N_2 \subseteq Z$ Lebesgue-null set and $M_2 > 0$ such that

$$|j(z,x) - j_+(z)| \leq 1,\ \ \text{for all } z \in Z \setminus N_2 \ \text{ and all } x \geq M_2$$

and

$$|j(z,x) - j_-(z)| \leq 1,\ \ \text{for all } z \in Z \setminus N_2 \ \text{ and all } x \leq -M_2\,.$$

Also because $|j(z,x)| \leq a_1(z) + c_1|x|^r$ a.e. on $Z$ with $a_1 \in L^1(Z),\ c_1 > 0$ (see the proof of proposition 1) we have that for almost all $z \in Z \setminus N_2$ and all $|x| \leq M_2$, $|j(z,x)| \leq a_2(z)$ with $a_2 \in L^1(Z)$. So for all $x \in W^{1,p}(Z)$ we have

$$\begin{aligned}\varphi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z j(z,x(z))dz \\ &\geq -\int_{\{x(z)\geq M_2\}} j(z,x(z))dz - \int_{\{x(z)\leq -M_2\}} j(z,x(z))dz - \\ &\quad - \int_{\{|x(z)|\leq M_2\}} j(z,x(z))dz \geq \\ &\geq -\|j_+\|_1 - \|j_-\|_1 - 2|Z| - \|a_2\|_1,\end{aligned}$$

which implies that $\varphi$ is bounded from below.

For $v \in V$, recall that $\|Dv\|_p^p \geq \lambda_1 \|v\|_p^p$ (see section 2). So using hypothesis $H(j)$ (iv) for all $v \in V$ we have

$$\varphi(v) = \frac{1}{p}\|Dv\|_p^p - \int_Z j(z, v(z))dz \geq \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_1}{p}\|v\|_p^p \geq 0.$$

Using these two auxiliary results we can prove the following multiplicity theorem for problem (1).

**Theorem 3.** *If hypotheses $H(j)$ hold, then problem (1) has at least three distinct solutions.*

*Proof.* We introduce the following open subsets of $W^{1,p}(Z)$

$$\mathcal{U}^{\pm} = \{x = \pm\eta + v : \eta > 0, v \in V\}.$$

Let $m_{\pm} = \inf\left[\varphi(x) : x \in \mathcal{U}^{\pm}\right] > -\infty$ (see proposition 2). Also let $\overline{\varphi}_{\pm} : W^{1,p}(Z) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\overline{\varphi}_{\pm}(x) = \begin{cases} \varphi(x), & \text{if } x \in \overline{\mathcal{U}}^{\pm} \\ +\infty, & \text{otherwise} \end{cases}$$

Both functions $\overline{\varphi}_{\pm}$ are lower semicontinuous and bounded below. In what follows we shall work with $\overline{\varphi}_{+}$ but a similar analysis can be conducted using $\overline{\varphi}_{-}$. Invoking theorem 1.1 of Zhong [25] with $\varepsilon = \dfrac{1}{n}$, $n \geq 1$, we generate a sequence $\{x_n\}_{n\geq 1} \subseteq \mathcal{U}^{+}$ such that $\overline{\varphi}(x_n) \downarrow m_{+}$ and

$$\overline{\varphi}_{+}(x_n) \leq \overline{\varphi}_{+}(y) + \frac{\frac{1}{n}\,\|x_n - y_n\|}{1 + \|x_n\|}, \quad \text{for all } y \in W^{1,p}(Z),$$

$$\Longrightarrow \frac{-\frac{1}{n}\,\|x_n - y_n\|}{1 + \|x_n\|} \leq \overline{\varphi}_{+}(y) - \overline{\varphi}_{+}(x_n), \quad \text{for all } y \in W^{1,p}(Z).$$

Let $u \in W^{1,p}(Z)$ and set $y = x_n + tu$, $t > 0$. Since $\mathcal{U}^{+}$ is open for $t \in (0, \delta)$ we have $y \in \mathcal{U}^{+}$. So

$$\frac{-\frac{1}{n}\,\|u\|}{1 + \|x_n\|} \leq \frac{\varphi(x_n + tu) - \varphi(x_n)}{t}, \quad t \in (0, \delta), \quad (\text{recall } \overline{\varphi}_{+}\,|_{\overline{\mathcal{U}}^{+}} = \varphi_{+}),$$

$$\Longrightarrow \frac{-\frac{1}{n}\,\|u\|}{1 + \|x_n\|} \leq \varphi^0(x_n; u), \quad (\text{see section 2}), \text{ for all } u \in W^{1,p}(Z).$$

Introduce $\vartheta_n(u) = \dfrac{1+\|x_n\|}{\frac{1}{n}}\varphi^0(x_n;u),\ n \geq 1$. Evidently, $\vartheta_n$ is continuous, sublinear ( hence $\vartheta_n(0)=0$ ) and for all $u \in W^{1,p}(Z)$ we have $-\|u\| \leq \vartheta_n(u)$. So we can apply lemma 1.3 of Szulkin [24] to obtain $y_n^* \in W^{1,p}(Z)^*$ such that $\|y_n^*\| \leq 1$ and $\langle y_n^*, u\rangle \leq \vartheta_n(u)$, for all $u \in W^{1,p}(Z)$ and all $n \geq 1$. If $x_n^* = \dfrac{\frac{1}{n}}{1+\|x_n\|}y_n^*$ we have $\langle x_n^*, u\rangle \leq \varphi^0(x_n;u)$ for all $u \in W^{1,p}(Z),\ n \geq 1$. So $x_n \in \partial\varphi(x_n)$ for all $n \geq 1$. We have

$$(1+\|x_n\|)m(x_n) \leq (1+\|x_n\|)\|x_n^*\| \leq \frac{1}{n} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Recall that $\varphi(x_n) \downarrow m_+$ and $m_+ \leq \int_Z j(z,c_+)dz < \int_Z j_+ dz < 0$ (see hypothesis $H(j)$ (vi)). From proposition 1 we know that $\varphi$ satisfies the nonsmooth $C_{m_+}$ -condition. So by passing to a subsequence if necessary, we may assume that $x_n \to y_1$ in $W^{1,p}(Z)$. We have $\varphi(x_n) \to \varphi(y_1) = m_+ < 0$. If $y_1 \in bd\,\mathcal{U}^+ = V$, then $\varphi(y_1) = m_+ \geq 0$ (see proposition 2), a contradiction. Therefore $y_1 \in \mathcal{U}^+$ and it follows that $y_1$ is a local minimum of $\varphi$, hence $0 \in \partial\varphi(y_1)$. With a similar argument we obtain $y_2 \in \mathcal{U}^-$ such that $0 \in \partial\varphi(y_2)$ and of course $y_1 \neq y_2 \neq 0$.

Finally because of hypothesis $H(j)$ (vi) and the fact that $\varphi\mid_V \geq 0$, we can apply the nonsmooth Saddle Point Theorem (see Kourogenis-Papageorgiou [18]) and produce $y_3 \in W^{1,p}(Z)$ such that $\varphi(y_3) = c \geq 0 - \int_Z j_\pm(z)dz > m_\pm$ and $0 \in \partial\varphi(y_3)$.

Now let $y = y_k,\ \ k=1,2,3$. We have $0 \in \partial\varphi(y)$ and so

$$A(y) = u^*, \quad \text{for some } u^* \in \partial J(x) \subseteq L^q(Z) \tag{5}$$

(see the proof of proposition 1).

Let $\psi \in C_0^\infty(Z)$. Since $-div(\|Dy\|^{p-2}\ Dy) \in W^{-1,q}(Z) = W_0^{1,p}(Z)^*$ (see for example Adams [1], p. 50), by integration by parts we obtain

$$\langle A(y),\psi\rangle = \langle -div(\|Dy\|^{p-2}\ Dy),\psi\rangle = \int_Z u^*\psi\ dz = \langle u^*,\psi\rangle.$$

But $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, so we obtain

$$-div(\|Dy(z)\|^{p-2}\ Dy(z)) = u^*(z) \in \partial j(z,y(z)), \quad \text{a.e. on } Z. \tag{6}$$

Also from the "quasilinear " Green's identity (see Kenmochi [17], Casas-Fernandez [3] or Hu-Papageorgiou [15], p. 867), for every $v \in W^{1,p}(Z)$ we have

$$\int_Z (-div(\|Dy\|^{p-2}\ Dy))\ v\ dz + \int_Z \|Dy\|^{p-2}(Dy,Dv)_{\mathbb{R}^N}\ dz = \left\langle \frac{\partial x}{\partial n_p},\ \gamma(v)\right\rangle_{\Gamma}$$

with $\langle\cdot,\ \cdot\rangle_\Gamma$ being the duality brackets for the pair $\left(W^{1/q,\,p}(\Gamma),\ W^{-1/q,\,q}(\Gamma)\right)$ and $\gamma : W^{1,p}(Z) \longrightarrow L^p(\Gamma)$ is the trace operator. From (5) and (6) we obtain

$$0 = \int_Z -u^*\, v\, dz + \langle A(y), v\rangle = \left\langle \frac{\partial x}{n_p},\ \gamma(v)\right\rangle_\Gamma .$$

But $\gamma\left(W^{1,p}(Z)\right) = W^{1/q,\,p}(\Gamma)$ (see Kufner-John- Fučik [19], p. 338). So it follows that $\dfrac{\partial x}{\partial n_p} = 0$. Therefore $y_1,\ y_2,\ y_3$ are distinct solutions of (1).

As a simple example of a function which satisfies hypotheses $H(j)$, consider the following locally Lipschitz nonsmooth potential $j(x)$ (for simplicity we drop the $z$- dependence):

$$j(x) = \begin{cases} \dfrac{\lambda_1}{p}|x|^p, & \text{if } |x| \le 1 \\ \\ \dfrac{a}{x^2} + \dfrac{\lambda_1}{p} - a, & \text{if } \ |x| \ge 1 \end{cases}, \qquad \text{with} \quad p\, a < \lambda_1,\ a > 0,$$

$$\Longrightarrow\ \partial j(x) = \begin{cases} \lambda_1 |x|^{p-2}x, & \text{if } |x| < 1 \\ \\ [-2a, \lambda_1], & \text{if } |x| = 1 \\ \\ -\dfrac{2a}{x^3}, & \text{if } |x| > 1. \end{cases}$$

Clearly, $j_\pm = \dfrac{\lambda_1}{p} - a > 0$ and so we can have $c_\pm = \pm 1$. Then $j(c_\pm) = \dfrac{\lambda_1}{p} > \dfrac{\lambda_1}{p} - a = j_\pm$. Also clearly $pj(x) \le \lambda_1 |x|^p$ for all $x \in \mathbb{R}$ and finally for all $x \in \mathbb{R}$ and all $u^* \in \partial j(x)$, we have $|u^*| \le M_0,\ M_0 > 0$. So hypotheses $H(j)$ are satisfied.

## References


1. R.Adams, *Sobolev Spaces* Academic Press, New York, (1975).
2. D. Arcoya-L.Orsina, *Landesman-Lazer conditions and quasilinear elliptic equations*, Nonlin. Anal. **28**(1997), 1623-1632.
3. E.Casas-L. Fernandez, *A Green's formula for quasilinear elliptic operators*, J.Math.Anal.Appl. **142**(1989), 62-73.
4. K.-C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J.Math.Anal.Appl. **80**(1981), 102-129.
5. F.H.Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York (1983).
6. D.Costa-J.V.A.Goncalves, *Critical point theory for nondifferntiable functionals and applications*, J.Math. Anal.Appl. **153**(1990), 470-485.
7. P. Drabek-S. Tersian, *Characterization of the range of Neumann problem for semilinear elliptic equations*, Nonlin. Anal. **11**(1987), 733-739.

8. L.Gasinski-N.S.Papageorgiou, *Nonlinear hemivariational inequalities at resonance*, Bull. Austr.Math. Soc. **60**(1999), 353-364.
9. L.Gasinski-N.S.Papageorgiou, *An existence theorem for nonlinear hemivariational inequalities at resonance*, Bull. Austr.Math. Soc. **63**(2001), 1-14.
10. L.Gasinski-N.S.Papageorgiou, *Multiple solutions for nonlinear hemivariational inequalities near resonance*, Funkcialaj Ekvaciaj **43**(2000), 271-284.
11. L.Gasinski-N.S.Papageorgiou, *Solutions and multiple solutions for quasilinear hemivariational inequalities at resonance*, Proc.Royal Soc. Edinburg(Math) **131A**(2001), 1-21.
12. D.Goeleven-D.Motreanu-P.Panagiotopoulos, *Eigenvalue problems for variational-hemivariational inequalities at resonance*, Nonlin. Anal. **33**(1998), 161-180.
13. S.Hu-N.S.Papageorgiou, *Nonlinear elliptic problems of Neumann type*, Rendiconti Circolo Matem. di Palermo **L**(2001), 47-66.
14. S.Hu-N.S.Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory* Kluwer, Dordrecht, The Netherlands(1997).
15. S.Hu-N.S.Papageorgiou, *Handbook of Multivalued Analysis, Volume II: Applications* Kluwer, Dordrecht, The Netherlands(2000).
16. Y.X.Huang, *On eigenvalue problems of the p-Laplacian with Neumann boundary conditions*, Proc.AMS **109**(1990), 177-184.
17. N.Kenmochi, *Pseudomonotone operators and nonlinear elliptic boundary value problems*, J.Math.Soc. Japan **27** (1975), 121-149.
18. N.Kourogenis-N.S.Papageorgiou, *Nonsmooth critical point theory and nonlinear elliptic equations at resonance*, J.Austr.Math.Soc.(SerA) **69**(2000), 245-271.
19. A.Kufner-O.John-S.Fučik, *Function Spaces*, Noordhoff Publ.Co., Leyden, The Netherlands(1997).
20. J.Mawhin-D. Ward-M.Willem, *Variational methods and semilinear elliptic equations*, Arch. Rat. Mech.Anal. **95**(1986),269-277.
21. D.Motreanu-P.Panagiotopoulos, *A minimax approach to the eigenvalue problem of hemivariational inequalities and applications*, Appl.Anal. **58**(1995), 53-76.
22. Z.Naniewicz-P.Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, New York(1995).
23. P.Panagiotopoulos, *Hemivariational Inequalities. Applications to Mechanics and Engineering*, Springer-Verlag, New York(1993).
24. A.Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann.Inst.H.Poincaré, Analyse Non Lineaire **3**(1986), 77-109.
25. C.K.Zhong, *On Ekeland's variational principle and a minimax theorem*, J.Math.Anal.Appl. **205**(1997), 239-250.