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A Partial Generalization of Diliberto's Theorem for Certain DEs of Higher Dimension.

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Abstract. For analysis of bifurcation of planar systems is sometimes used a result first obtained probably by Diliberto. This result is here partially extended to certain class of autonomous ordinary differential equations in $\mathbb{R}^3$.

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1 Main results

Andronov and his coworkers were able, in their famous book [1], to derive an effective theorem about continuation of periodic solutions of a two dimensional system. Chicone [2] achieved similar result but with substantially greater elegance using a result first published by Diliberto [3]:

Theorem 1. *Let $F = (p, q)$ be a C^1 2-vector function defined on an open subset of $\mathbb{R}^2$. Let $(\varphi(t, x_0, y_0), \psi(t, x_0, y_0))$ be the solution of plane initial problem*

$$\begin{aligned} \dot{x} &= p(x, y) \qquad x(0) = x_0, \\ \dot{y} &= q(x, y) \qquad y(0) = y_0. \end{aligned} \tag{1}$$

If $|p(x_0, y_0)| + |q(x_0, y_0)| > 0$, then the principal fundamental matrix $Y(t)$ of (1) at $t = 0$ of the homogeneous variational equation

$$\dot{u} = \frac{\partial p(\varphi(t, x_0, y_0), \psi(t, x_0, y_0))}{\partial x} u + \frac{\partial p(\varphi(t, x_0, y_0), \psi(t, x_0, y_0))}{\partial y} v$$

This is the shortened preliminary version of the paper.

$$\dot{v} = \frac{\partial q(\varphi(t,x_0,y_0),\psi(t,x_0,y_0))}{\partial x}\,u + \frac{\partial q(\varphi(t,x_0,y_0),\psi(t,x_0,y_0))}{\partial y}\,v$$

is such that

$$Y(t)F^\top(x,y) = a(t,x,y)F((\varphi(t,x,y),\psi(t,x,y)) + b(t,x,y)F^\top(\varphi(t,x,y),\psi(t,x,y))$$

and $Y(t)F(x,y) = F((\varphi(t,x,y),\psi(t,x,y))$, *where*

$$\begin{aligned} b(t,x,y) = {} & \frac{p^2(x,y)+q^2(x,y)}{p^2(\varphi(t,x,y),\psi(t,x,y))+q^2(\varphi(t,x,y),\psi(t,x,y))}\times \\ & \times \exp\left\{\int_0^t \frac{\partial p(\varphi(s,x,y),\psi(s,x,y))}{\partial x} + \frac{\partial q(\varphi(s,x,y),\psi(s,x,y))}{\partial y}\,ds\right\}. \end{aligned} \tag{2}$$

In his book Chicone [2] was able to obtain an interesting geometrical identification for the function $b(t,x,y)$.

Nowadays we are able to extend Diliberto's result on many differential systems in $\mathbb{R}^n$ and the results will be published elsewhere. In this short announcement we shall limit ourselves to a three-dimensional system

$$\dot{x} = f(x), \tag{3}$$

fulfilling the following hypotheses:

H1 the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function with open domain,

H2 all solutions of (3) are defined on $[0,\infty)$,

H3 the system (3) has a nondegenerate first integral ([4, p.114]) $g : \mathbb{R}^n \to \mathbb{R}^n$ and $g \in C^2$.

If $\varphi(t,x)$ is the solution of the initial problem (3), $\varphi(t,0) = x$, then the first integral g yields a two-dimensional submanifold M generated as the level set of g containing the point x. Clearly the unit surface normal $n(x) := \|\mathrm{grad} g(x)\|^{-1}\mathrm{grad} g(x)$ is well defined and is C^1 on M. Moreover we may suppose that

H4 there are two C^1 functions $a_1(x)$, $a_2(x)$ on M such that $\|a_1(x)\| = \|a_2(x)\| = 1$ and $T_xM := \mathrm{span}\{a_1(x),a_2(x)\}$.

Finally denoting the usual inner product as $\langle .|.\rangle$ and the usual cross-product as $[.|.]$, we may state Diliberto's theorem for three-dimensional systems (3) with a first integral.

Theorem 2. *Let hypotheses H1, H2, H3, H4 be fulfilled and $\varphi(t,x)$ denote the solution of the differential equation (3), $\varphi(0,x) = x$. If $f(x) \neq 0$, then the principal fundamental matrix $Y(t)$ at $t = 0$ of the variational equation $\dot{y} = Df(\varphi(t,x))y$ is such that $Y(t)f(x) = f(\varphi(t,x))$,*

$$Y(t)[n(x)|f(x)] = a(t,x)f(\varphi(t,x)) + b(t,x)[n(\varphi(t,x))|f(\varphi(t,x))],$$

and

$$b(t,x) = \frac{\|f(x)\|^2}{\|f(\varphi(t,x))\|^2} \exp \int_0^t \Big\{ \langle a_1 | (Df) a_1 \rangle + \langle a_2 | (Df) a_2 \rangle - \langle a_1 | (Da_1) f \rangle + \langle a_2 | (Da_2) f \rangle \Big\} (\varphi(s,x))\, ds. \quad (4)$$

As an application let us present the following theorem concerning the Poincaré mapping of the system (3):

Theorem 3. *Let the hypotheses H1, H2, H3 and H4 be fulfilled. Let* $x_1 \in M$*,* $f(x_1) \neq 0$ *and* $x_1 = \varphi(p, x_1)$*, where* $0 < p < \infty$ *is the first time with this property. Let* Σ *be a plane containing* x_1 *and orthogonal to* $f(x_1)$*. Let* $\Psi \colon U \subset \Sigma \to \Sigma$ *be the Poincaré mapping. If* $v \in T_{x_1}\Sigma \cap T_{x_1}M$*, then*

$$D\Psi(x_1)v = \frac{\langle v_1 | [n(x_1) | f(x_1)] \rangle}{\|f(x_1)\|^2} b(p, x_1) [n(x_1) | f(x_1)].$$

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