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About One Property of the Generalized Liénard Differential Equations

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Abstract. A special property for establishing boundedness and periodicity of the oscillatory properties of the solutions of equation $x'' = F(x, x')$ is defined and investigated.

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In this note I will consider a certain property, called property (B), introduced in the paper [1] for the equation

$$x'' = F(x, x'), \tag{1}$$

where $F\colon R \to R^2$ is a continuous function which guarantees the existence and unicity of the solutions defined by the Cauchy conditions. This property, which we will later explicitely describe, gives good conditions for establishing the oscillatory properties of the solutions, mainly the boundedness and periodicity of the solutions.

We will substitute the equation (1) by the system

$$\begin{aligned} x' &= y, \\ y' &= F(x, y). \end{aligned} \tag{2}$$

We put $g(x) = -F(x, 0)$ and we will assume that

$$xg(x) > 0 \quad \text{for} \quad x \neq 0. \tag{3}$$

This guarantees that only the origin will be a singular point of the system (2) and that the trajectories of our system go clockwise. Moreover, we will assume that the function $F(x,y)/y$ is bounded for $|x| < \alpha$, $|y| > \beta$, $\alpha > 0$, $\beta > 0$. This assumption guarantees the nonexistence of the vertical asymptotes.

If $P = (x_0, y_0) \in R^2$, then $\gamma^+(P)$ will denote positive and $\gamma^-(P)$ the negative parts of the trajectory of the system (2) going through the point P.

Definition We will say that the system (2) has the property (B) if there exists a point $P = (x_0, y_0)$, where $y_0 \neq 0$ such that $\gamma^+(P)$ crosses the x-axis but $\gamma^-(P)$ does not.

In the paper [1] the authors prove the following theorem.

Theorem 1. The system (2) has the property (B) in the positive halfplane if and only if there exists a differentiable function $\Phi(x)$ and $\bar{x} > 0$ such that $\Phi(x) > 0$ for $x < \bar{x}$, $\Phi(\bar{x}) = 0$ and

$$F(x, \Phi(x)) \leq \Phi'(x)\Phi(x) \quad \text{for each} \quad x < \bar{x}. \tag{4}$$

We will consider the Liénard differential equation

$$x'' = -f(x)x' - g(x). \tag{5}$$

The system (2) will have the form

$$\begin{aligned} x' &= y, \\ y' &= -f(x)'y - g(x). \end{aligned} \tag{6}$$

Theorem 2. Let $f(x)$ and $g(x)$ be continuous functions on the interval $(-\infty, \infty)$ and let be

$$xf(x) > 0, \; xg(x) > 0 \quad \text{for} \quad x \neq 0. \tag{7}$$

Moreover, let be guaranteed the existence and uniqueness of the Cauchy problem of the system (6). Let be $F(x) = \int_0^x f(s)ds$ and let be

$$F(-\infty) < +\infty, \; F(+\infty) > e^{F(-\infty)},$$

$$\frac{g(x)}{f(x)} \leq e^{F(-\infty)}[e^{F(-\infty)} - 1] \quad \text{for} \quad x \leq 0.$$

Then the system (6) has the property (B).

Proof: Respecting Theorem 1 it is sufficient to prove that there exists a differentiable function $\Phi(x)$ and $\bar{x} > 0$ such that

$$\Phi(x) > 0 \quad \text{for} \quad x < \bar{x}, \; \Phi(\bar{x}) = 0$$

and

$$F(x,\Phi(x)) = -f(x)\Phi(x) - g(x) \leq \Phi'(x)\Phi(x) \quad \text{for} \quad x < \bar{x}. \tag{8}$$

Let be $\bar{x} > 0$ such that

$$e^{-F(-\infty)}F(\bar{x}) > 0. \tag{9}$$

Put $\Phi_1(x) = F(\bar{x}) - F(x)$ for $0 \leq x \leq \bar{x}$. Then $\Phi_1'(x) = -f(x)$ and from (9) we get

$$-f(x)[F(\bar{x})-F(x)]-g(x)-[F(\bar{x})-F(x)](-f(x)) = -g(x) \leq 0 \quad \text{for} \quad 0 \leq x \leq \bar{x}.$$

For $x \leq 0$ put $\Phi_2(x) = e^{-F(x)}F(\bar{x})$. Then

$$\Phi_2'(x) = e^{-F(x)}F(\bar{x})(-f(x)).$$

It is easy to see that $\Phi_1(0) = \Phi_2(0) = F(\bar{x}) > 0$ and $\Phi_1'(0) = 0 = \Phi_2'(0)$. Put $\Phi(x) = \Phi_1(x)$ for $0 \leq x \leq \bar{x}$ and $\Phi(x) = \Phi_2(x)$ for $x < 0$. Setting $\Phi(x) = \Phi_2(x)$ we get

$$\begin{aligned} &- f(x)e^{-F(x)}F(\bar{x}) - g(x) - e^{-F(x)}F(\bar{x})e^{-F(x)}F(\bar{x})(-f(x)) = \\ &\qquad = -f(x)e^{-F(x)}F(\bar{x})[1 - e^{-F(x)}F(\bar{x})] - g(x) \leq \\ &\qquad \leq -f(x)e^{-F(x)}F(\bar{x})[1 - e^{-F(x)}F(\bar{x})] - e^{F(-\infty)}[e^{F(-\infty)} - 1]f(x) = \\ &\qquad = f(x)\{-e^{-F(x)}F(\bar{x})[1 - e^{-F(x)}F(\bar{x})] - e^{F(-\infty)}[e^{F(-\infty)} - 1]\}. \end{aligned} \tag{10}$$

Using (9) we get

$$e^{-F(x)}F(\bar{x})[e^{-F(x)}F(\bar{x}) - 1] \geq F(x)[F(x) - 1] \geq e^{F(-\infty)}[e^{F(-\infty)} - 1].$$

Thus the expression in the composed brackets in (10) is nonnegative. From this it follows that $\Phi(x) = \Phi_2(x) = e^{-F(x)}F(\bar{x})$ fulfills (8) for $x \leq 0$. This finishes the proof.

References

1. Bucci F., Villari G. *Phase portrait of the system* $x' = y,\ y = F(x,y)$, Bolletino U.M.I., 1990.