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# Non existence of standing waves for hyperbolic Davey-Stewartson Systems

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**Abstract.** We consider the problem of existence of standing wave solutions of the Davey-Stewartson (DS) system in the hyperbolic-hyperbolic case. We extend the result of non existence of standing wave solutions for the elliptic-hyperbolic case of the (DS) system ([8]). We show that there are no solutions of the form $e^{iwt}v(x,y)$ with $v \in H^1(\mathbb{R}^2)$ and homogeneous boundary conditions on $\varphi$ if $b \neq 0$. We finish with a result about non-existence of standing wave solutions which are smooth but with non-homogeneous boundary conditions on the velocity potential for both elliptic-hyperbolic and hyperbolic-hyperbolic cases.

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**Keywords.** Davey-Stewartson System, Standing Waves

## 1 Introduction

The Davey–Stewartson (DS) system models the evolution of water waves in a three dimensional flow that travels predominantly in one direction. The system can be written in the form:

$$iu_t + \delta u_{xx} + u_{yy} = \lambda |u|^2 u + bu\varphi_x, \quad (x,y) \in \mathbb{R},\ t \in \mathbb{R}, \tag{1}$$

$$\varphi_{xx} + m\varphi_{yy} = (|u|^2)_x, \tag{2}$$

for the (complex) wave amplitude $u(x,y,t)$ and the (real) mean velocity potential $\varphi$. The coefficients $(\delta, \lambda, m, b)$ depend on the fluid depth, surface tension and gravity and can take both signs [1,4,5]. The parameters $\lambda$ and $\delta$ are normalized such that, $|\lambda| = |\delta| = 1$.

*This is the final form of the paper.*

The character of the solution depends strongly on the signs of the above coefficients. It is useful to classify the system as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of $(\delta, m)$: $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$ ([6]). It has been known since the work of Ghidaglia and Saut ([6]) that the initial value problem of (DS) systems in the elliptic-elliptic and hyperbolic-elliptic cases has a unique solution in the spaces $L^2(\mathbb{R}^2)$, $H^1(\mathbb{R}^2)$ and $H^2(\mathbb{R}^2)$.

The Cauchy problem for the DS system in the elliptic-hyperbolic and the hyperbolic-hyperbolic cases has been studied by Hayashi and Saut [9]. The boundary conditions that have been imposed are, for the wave amplitude $u$:

$$u(x,y,t),\ D^\alpha u \to 0 \quad \text{as} \quad x^2+y^2 \to \infty, \tag{3}$$

and for the mean velocity $\varphi$ are of radiation type:

$$\lim_{\xi\to-\infty} \varphi(\xi,\eta,t) = 0, \qquad \lim_{\eta\to-\infty} \varphi(\xi,\eta,t) = 0 \tag{4}$$

where $(\xi,\eta)$ are the characteristic coordinates:

$$\xi = \frac{1}{2}(x+\sqrt{-m}y), \quad \eta(x,y) = \frac{1}{2}(x-\sqrt{-m}y). \tag{5}$$

More general boundary conditions for $\varphi$ may be the following:

$$\lim_{\xi\to-\infty} \varphi(\xi,\eta,t) = f(\eta), \qquad \lim_{\eta\to-\infty} \varphi(\xi,\eta,t) = g(\xi) \tag{6}$$

with

$$\lim_{\xi\to-\infty} f(\xi) = \lim_{\xi\to-\infty} g(\xi) = 0, \tag{7}$$

and $f, g \in L^\infty(\mathbb{R})$. Standing wave solutions for the DS system have been studied in the elliptic-elliptic and hyperbolic-elliptic cases. By extending the analysis developed for standing wave solutions of the Nonlinear Schrödinger equation

$$iu_t + u_{xx} + u_{yy} = \lambda |u|^2 u. \tag{8}$$

Cipolatti [2] proved existence, regularity and behavior at infinity of standing wave solutions in the elliptic-elliptic case, $(\delta = 1, m > 0)$. Moreover, he showed the existence and uniqueness of ground states (positive solutions). In [3], Cipolatti proved that the ground states are unstable. Ghidaglia and Saut ([7]) gave necessarily conditions for existence of standing waves in the hyperbolic–elliptic case $(\delta = -1, m > 0)$. They showed that solutions of the form $e^{i\omega t}v(x,y)$ exists only if $\lambda = -1$ and $b > 1$.

Recently, Guzmán-Gómez ([8]) showed that for elliptic-hyperbolic Davey–Stewartson system $(\delta = 1, m < 0)$, and boundary conditions as in (4) there are not standing wave solutions. This study was rather different from Cipollati [2] due to the lack of regularizing effect for the velocity potential $\varphi$ which satisfies a hyperbolic equation if $m < 0$. In [8], the author proved that if

$$\begin{aligned} u(x,y,t) &= e^{i\omega t}v(x,y) && (9)\\ \varphi(x,y,t) &= \phi(x,y), && (10)\\ \omega \in \mathbb{R}, v \in H^1 \ \text{ and } & \ \varphi(x,y) \in L^\infty(\mathbb{R}^2) && (11) \end{aligned}$$

is a solution of the system (1)-(2), $m < 0$, $v$ satisfies weakly the elliptic equation

$$(1-\frac{1}{m})(v_{xx}+v_{yy})+2(1+\frac{1}{m})v_{xy}-\omega v = F. \tag{12}$$

Due to the ellipticity of (12), if $F \in L^2(\mathbb{R}^2)$, then $v \in H^2(\mathbb{R}^2)$. Once $v$ is regular enough and decays at infinity it can be concluded that $v$ must be zero.

The aim of this paper is to show first that the hyperbolic-hyperbolic case of the (DS) system has no solutions of the form (9)-(11). Also, we approach the problem for non-homogeneous boundary conditions on $\varphi$ (6) and obtain conditions on $f$ and $g$ for which standing wave classical solutions does not exist. This latter result is valid for both: elliptic-hyperbolic and hyperbolic-hyperbolic cases.

In this work, we notice that if there is a solution of the system (1)-(2), in the form $(9)-(10)$, with $v \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ then $v$ is necessarily zero. We then extend the result to $H^1(\mathbb{R}^2)$ by density. This technique is more general that the one used in [8]; we do not need the regularity effect of the correspondent equation (12) and the density argument is valid for both: $\delta = 1$ and $\delta = -1$.

The non existence of standing wave classical solutions follow from the proof of Theorem 6 where homogeneous boundary conditions on $\varphi$ are considered; we approach the problem of existence of standing wave solutions in the classical sense but with non-homogeneous boundary conditions and provide conditions on $f$ and $g$ that no standing wave solutions may exist. Here $H^k(\mathbb{R}^2)$ denotes the Sobolev space of square integrable functions with square integrable derivatives up to order $k$ and

$$\|u\|_2 = \|u\|_{L^2(\mathbb{R}^2)}, \quad \|u\|_{H^k}^2 = \|u\|_2^2 + \sum_{|\alpha|\le k} \|D^\alpha u\|_2^2, \quad \text{and} \quad \langle f,g\rangle = \int_{\mathbb{R}^2} f\overline{g}$$

The paper is organized as follows: In section 2 we solve the wave equation (2) for the velocity potential $\varphi$ in terms of $u$ and substitute it in equation (1) to obtain a single equation of Schrödinger type with a nonlocal term (eq. 16). We also provide the main estimates for the nonlocal term that will be used in section 3. In section 3 we show that if $u(x,y,t) = e^{i\omega t}v(x,y),\ v \in H^1(\mathbb{R}^2)$ is a weak solution of the (DS) system then $v$ satisfies weakly (22); we obtain some estimates for the linear and nonlinear part of equation (22) to conclude that if

$$\{v_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^2), \quad \{v_n\} \to v \text{ in } H^1(\mathbb{R}^2)$$

then

$$\lim_{n\to+\infty} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} (v_n^2)_x(x,y)dy \right)^2 dx = 0. \tag{13}$$

We then prove the main Theorem (6), that is, we show that $v(x,y)=0$ a.e. In section 4 we prove Theorem 7. We show that under certain conditions on the boundary conditions for $\varphi$ if there is a classical solution of the form $e^{i\omega t}v(x,y)$ then $v(x,y)=0\forall(x,y)\in\mathbb{R}^2$.

## 2 Velocity Potential

We begin transforming the coupled system (1)-(2) into a single equation with a nonlocal term by solving equation (2) and substituting it in equation (1). In terms of the characteristic variables $(\xi,\eta)$ , (5), we can rewrite the equation for the mean velocity as

$$\varphi_{\xi\eta}=\frac{1}{4}\left((|u|^2)_\xi+(|u|^2)_\eta\right). \tag{14}$$

We will consider boundary conditions of radiation type (4) for $\varphi$. A similar problem can be stated with the boundary conditions defined at $+\infty$ instead of at $-\infty$, leading to the same results.

Integrating equation (14), we obtain

$$\begin{aligned}\varphi(\xi,\eta)&=\frac{1}{4}\int_{-\infty}^{\xi}\int_{-\infty}^{\eta}\left((|u|^2)_{\xi'}+(|u|^2)_{\eta'}\right)(\xi',\eta')d\xi'd\eta'\\&=\frac{1}{4}\left(\int_{-\infty}^{\eta}|u(\xi,\eta')|^2d\eta'+\int_{-\infty}^{\xi}|u(\xi',\eta)|^2d\xi'\right).\end{aligned}$$

Rewriting equation (1) in terms of the $\xi$-$\eta$ variables and using the above expression for $\varphi$ we obtain

$$\begin{aligned}&iu_t+\left(\delta-\frac{1}{m}\right)(u_{\xi\xi}+u_{\eta\eta})+2\left(\delta+\frac{1}{m}\right)u_{\xi\eta}\\&=\left(\lambda+\frac{b}{2\sqrt{-m}}\right)|u|^2u+\frac{b}{4\sqrt{-m}}u\left(\int_{-\infty}^{\eta}|u|_\xi^2d\eta'+\int_{-\infty}^{\xi}|u|_\eta^2d\xi'\right).\end{aligned} \tag{15}$$

By renaming the variables $\xi$, $\eta$ by $x$, $y$, and defining the new parameters $\alpha=\left(\delta-\frac{1}{m}\right)$, $\beta=2\left(\delta+\frac{1}{m}\right)$, $\gamma=\lambda+\frac{b}{2\sqrt{-m}}$, $\epsilon=\frac{b}{4\sqrt{-m}}$ we rewrite equation (15) as

$$\begin{aligned}iu_t+\alpha(u_{xx}+u_{yy})+\beta u_{xy}=\gamma|u|^2u+\ \epsilon u\biggl(&\int_{-\infty}^{y}(|u|^2)_x(x,y')dy'\\&+\int_{-\infty}^{x}(|u|^2)_y(x',y)dx'\biggr).\end{aligned} \tag{16}$$

In the next lemma we state the main estimate we will use for the second term of the right hand side of (16).

**Lemma 1.** *Let $f, g \in H^1(\mathbb{R}^2)$ and $h \in L^2(\mathbb{R}^2)$. Then*

$$a) \left\| f \int_{-\infty}^{y} gh(x, y')dy' \right\|_2 \le \|f\|_{H^1}\|g\|_{H^1}\|h\|_2. \tag{17}$$

$$b) \left\| f \int_{-\infty}^{x} gh(x', y)dx' \right\|_2 \le \|f\|_{H^1}\|g\|_{H^1}\|h\|_2. \tag{18}$$

*Proof.* We only prove *a)*. We notice that

$$\left\| f \int_{-\infty}^{y} (gh)(x, y')dy' \right\|_2^2 \le \|f\|^2_{L^2_y L^\infty_x} \left\| \int_{-\infty}^{y} (gh)(x, y')dy' \right\|^2_{L^2_x L^\infty_y} \tag{19}$$

where

$$\|f\|^2_{L^2_y L^\infty_x} = \int_{-\infty}^{\infty} (essup_x |f(x,y)|)^2 \, dy.$$

Thanks to the Sobolev inequality $\|u\|_{L^\infty(\mathbb{R})} \le \|u\|_{H^1(\mathbb{R})}$,

$$\begin{aligned} \|f\|^2_{L^2_y L^\infty_x} &= \int_{-\infty}^{\infty} \|f(x,y)\|^2_{L^\infty_x} dy \\ &\le \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |f(x,y)|^2 dx + \int_{-\infty}^{\infty} \left| \frac{\partial f}{\partial x}(x,y) \right|^2 dx \right) dy \\ &\le \|f\|^2_{H^1(\mathbb{R}^2)}. \end{aligned} \tag{20}$$

Also,

$$\begin{aligned} \left\| \int_{-\infty}^{y} (gh)(x, y')dy' \right\|^2_{L^2_x L^\infty_y} &= \int_{-\infty}^{+\infty} \left( essup_y \left| \int_{-\infty}^{y} (gh)(x, y')dy' \right| \right)^2 dx \\ &\le \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} |gh(x,y)| dy \right)^2 dx \\ &\le \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |g(x,y)|^2 dy \int_{-\infty}^{\infty} |h(x,y)|^2 dy \right) dx \\ &\le \|g\|^2_{L^2_y L^\infty_x} \|h\|_2^2. \end{aligned} \tag{21}$$

Using (19), (20), and (21), (17) is obtained. □

## 3 Standing Wave Solutions

We look for time-periodic solutions of equation (16) in the form $u(x, y, t) = e^{i\omega t} v(x, y)$ where $v$ is real valued and belongs to $H^1(\mathbb{R}^2)$. Therefore, the function $v$ must solve the following equality

$$\alpha(v_{xx} + v_{yy}) + \beta v_{xy} = \omega v + \gamma v^3 + \epsilon v \left( \int_{-\infty}^{y} (v^2)_x dy' + \int_{-\infty}^{x} v_y^2 dx' \right), \tag{22}$$

where $\alpha = -\left(1+\frac{1}{m}\right)$, $\beta = 2(-1+\frac{1}{m})$, $m<0$. In this paper we only consider weak solutions of equation (22), that is, $v \in H^1(\mathbb{R}^2)$ that satisfies equation

$$\begin{aligned} -\alpha\int_{\mathbb{R}^2} v_x f_x - \alpha\int_{\mathbb{R}^2} v_y f_x - \beta\int v_y f_x = \omega\int_{\mathbb{R}^2} vf + \gamma\int_{\mathbb{R}^2} v^3 f \\ +\,\epsilon\int_{\mathbb{R}^2} v\left(\int_{-\infty}^{y}(v^2)_x(x,y')dy' + \int_{-\infty}^{x}(v^2)_y(x',y)dx'\right) f, \quad \forall f \in H^1(\mathbb{R}^2). \end{aligned} \tag{23}$$

In [8], thanks to the regularity effect of the elliptic equation (12),($m<0$), the authors proved that any weak solution of (16), belongs to $H^2(\mathbb{R}^2)$ and they can conclude that $v=0$. In the hyperbolic-hyperbolic (DS) system we cannot use that $v \in H^2(\mathbb{R}^2)$. Instead, we use that $\mathcal{C}_0^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, $\{v_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$, $\{v_n\} \to v$ in $H^1(\mathbb{R}^2)$ and with the help of standard Sobolev estimates and lemma 1 we obtain that

$$\lim_{n\to+\infty}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}(v_n)_x^2\right)^2 = 0.$$

We then prove the main theorem.

We define by $\mathcal{L}$ and $\mathcal{N}$ to be the corresponding linear and nonlinear part of equation (23), that is,

$$\begin{aligned} \mathcal{L}(u) &= \alpha(u_{xx}+u_{yy}) + \beta u_{xy} - \omega u, \quad m<0, \\ \mathcal{N}(u) &= \lambda u^3 + \epsilon\left(\int_{-\infty}^{y}(u^2)_x dy' + \int_{-\infty}^{x}(u^2)_y dx'\right). \end{aligned}$$

We may conclude that $v \in H^1(\mathbb{R}^2)$ is a weak solution of (22) if and only if

$$\langle \mathcal{L}(v), f\rangle = \langle \mathcal{N}(v), f\rangle \quad \forall f \in L^2(\mathbb{R}^2). \tag{24}$$

We will use that whenever $\{v_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$, $v_n \to v$ in $H^1(\mathbb{R}^2)$

$$\lim_{n\to+\infty}\langle(\mathcal{L}+\mathcal{N})(v_n), v_{nx}\rangle = \langle(\mathcal{L}+\mathcal{N})(v), v_x\rangle \tag{25}$$

If $v$ is a weak solution of equation (22), the right hand side of equality (25) is zero. Also, $\langle\mathcal{L}(v_n),(v_n)_x\rangle = 0$ $\forall n>0$; on the other hand, after several integration by parts, we can prove that $\lim_{n\to+\infty}\langle\mathcal{N}(v_n), v_{nx}\rangle = 0$ only if $v_n \to 0$, that is $v=0$ a.e.. Equality (25) is a consequence of the following limits:

$$\begin{aligned} \lim_{n\to+\infty}\|\mathcal{L}(v_n)-\mathcal{L}(v)\|_2 = 0, \\ \lim_{n\to+\infty}\|\mathcal{N}(v_n)-\mathcal{N}(v)\|_2 = 0. \end{aligned}$$

To prove the two limits above is the purpose of the following two propositions.

**Proposition 2.** *Let $v \in H^1(\mathbb{R}^2)$ be a weak solution of equation (22),and $\{v_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\|v_n - v\|_{H^1} \to 0$, then*

$$\begin{aligned} &a)\ \mathcal{L}(v) \in L^2(\mathbb{R}^2), \\ &b)\ \lim_{n\to+\infty}\|\mathcal{L}(v_n)-\mathcal{L}(v)\|_2 = 0. \end{aligned}$$

*Proof.* Let $v \in H^1(\mathbb{R}^2)$ be a weak solution of equation 22. Thank's to the Sobolev embedding $H^1(\mathbb{R}^2) \subset L^6(\mathbb{R}^2)$, $v^3 \in L^2(\mathbb{R}^2)$ and from Lemma 1

$$\epsilon\left(v\int_{-\infty}^{y} v_x^2(x,y)dy + v\int_{-\infty}^{x} v_y^2(x,y)dx\right) \in L^2(\mathbb{R}^2),$$

therefore $\mathcal{N}(v) \in L^2(\mathbb{R}^2)$. From (24)

$$|\langle \mathcal{L}(v), f\rangle| \leq \|\mathcal{N}(v)\|_2\|f\|_2, \quad \forall f \in L^2(\mathbb{R}^2),$$

hence

$$\mathcal{L}(v) \in L^2(\mathbb{R}^2) \quad \text{and} \quad \|\mathcal{L}(v)\|_2 \leq \|\mathcal{N}(v)\|_2,$$

a) follows.

Now we prove b): Let $\{v_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\lim_{n\to+\infty}\|v_n - v\|_{H^1} = 0$. For any $f \in H^1(\mathbb{R}^2)$

$$\langle \mathcal{L}(v_n), f\rangle = -\alpha\int_{\mathbb{R}^2} v_{nx}f_x - \alpha\int_{\mathbb{R}^2} v_{ny}f_y - \beta\int_{\mathbb{R}^2} v_{ny}f_x - \omega\int_{\mathbb{R}^2} v_n f$$

therefore,

$$\begin{aligned}\lim_{n\to+\infty}\langle \mathcal{L}(v_n), f\rangle &= -\alpha\int_{\mathbb{R}^2} v_x f_x - \alpha\int_{\mathbb{R}^2} v_y f_y - \beta\int_{\mathbb{R}^2} v_y f_x - \omega\int_{\mathbb{R}^2} v f\\ &= \langle \mathcal{L}(v), f\rangle.\end{aligned}$$

Because $H^1(\mathbb{R}^2)$ is dense en $L^2(\mathbb{R}^2)$,

$$\lim_{n\to+\infty}\langle \mathcal{L}(v_n), f\rangle = \langle \mathcal{L}(v), f\rangle \quad \text{for any} \quad f \in L^2(\mathbb{R}^2). \tag{26}$$

Equation (26) together with *a)* implies that

$$\lim_{n\to+\infty}\|\mathcal{L}(v_n) - \mathcal{L}(v)\|_2 = 0.\square$$

**Proposition 3.** *Let $v \in H^1(\mathbb{R}^2)$ be a weak solution of equation (22), and $\{v_n\} \subset H^1(\mathbb{R}^2)$ such that $\lim_{n\to+\infty}\|v_n - v\|_{H^1} = 0$ then*

$$\lim_{n\to+\infty}\|\mathcal{N}(v_n) - \mathcal{N}(v)\|_2 = 0. \tag{27}$$

*Proof.* To prove (27) is enough to show the following three limits:

$$\begin{aligned}&a)\ \lim_{n\to+\infty}\|(v_n)^3 - v^3\|_2 = 0,\\ &b)\ \lim_{n\to+\infty}\left\|v_n\int_{-\infty}^{x}(v_n^2)_y dx' - v\int_{-\infty}^{x}(v^2)_y dx'\right\|_2 = 0,\\ &c)\ \lim_{n\to+\infty}\left\|v_n\int_{-\infty}^{y}(v_n^2)_y dy' - v\int_{-\infty}^{y}(v^2)_x dy'\right\|_2 = 0.\end{aligned}$$

The first limit follows from Cauchy-Schwartz inequality and the Sobolev embedding $H^1(\mathbb{R}^2) \subset L^p(\mathbb{R}^2),\ \forall p > 2$.

$$\begin{aligned}\|(v_n)^3 - (v)^3\|_2 &= \|(v_n^2 - v_n v + v^2)(v_n - v)\|_2\\ &\le \|(v_n)^2 - v_n v + v^2\|_4 \|v_n - v\|_4\\ &\le 2\left(\|v_n\|_8^2 + \|v\|_8^2\right)\|v_n - v\|_{H^1}.\end{aligned}$$

To prove limit b) we use Lemma 1:

$$\begin{aligned}\left\|v_n \int_{-\infty}^{y} (v_n^2)_x - v \int_{-\infty}^{y} (v^2)_x\right\|_2 &\le \left\|v_n \int_{-\infty}^{y} (v_n^2 - v^2)_x\right\|_2 + \left\|(v_n - v)\int_{-\infty}^{y} (v^2)_x\right\|_2\\ &\le \|v_n\|_H^1 \|v_n + v\|_H^1 \|v_n - v\|_{H^1} + \|v_n - v\|_H^1 \|v\|_{H^1}^2\\ &\le C\left(\|v_n\|_{H^1}^2 + \|v\|_{H^1}^2\right)\|v_n - v\|_{H^1}.\end{aligned}$$

Limit c) follows similarly. □

**Lemma 4.** *There exists a positive constant $C$ such that*

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}(f^2)_x(x,y)dy\right)^2 dx \le C\|f\|_{H^1}^4 \tag{28}$$

*for any $f \in H^1(\mathbb{R}^2)$.*

*Proof.* We observe that by Cauchy-Schwartz inequality

$$\begin{aligned}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}(f^2)_x\right)^2 dxdy &\le 4\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f^2 dy \int_{-\infty}^{+\infty}(f_x)^2 dy\right)dx\\ &\le 4\sup_{x\in\mathbb{R}}\int_{-\infty}^{+\infty} f^2(x,y)dy \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(f_x)^2(x,y)dxdy. \end{aligned}\tag{29}$$

We use the inequality

$$\|g\|_{L^\infty(\mathbb{R})} \le C\|g\|_{H^1(\mathbb{R})},$$

(see (20)) to estimate the right hand side of (29) and obtain (28). □

**Lemma 5.** *Let $f$ be in $H^1(\mathbb{R}^2)$ and $\{f_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that*

$$\lim_{n\to+\infty}\|f_n - f\|_{H^1} = 0,$$

*therefore*

$$\lim_{n\to+\infty}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}(f_n)_x^2(x,y)dy\right)^2 dx = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}(f)_x^2(x,y)dy\right)^2 dx. \tag{30}$$

*Proof.* Let

$$F_n(x)=\int_{-\infty}^{+\infty}(f_n^2)_x(x,y)dy \quad \text{and} \quad F(x)=\int_{-\infty}^{+\infty}(f)^2(x,y)dy$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\left(F_n^2(x)-F^2(x)\right)dx &= \int_{-\infty}^{+\infty}(F_n(x)+F(x))(F_n(x)-F(x))dx\\ &\le \left(\|F_n\|_{L^2(\mathbb{R})}+\|F\|_{L^2(\mathbb{R})}\right)\|F_n-F\|_{L^2(\mathbb{R})}.\end{aligned}$$

From Lemma 4, $\|F_n\|^2_{L^2(\mathbb{R})}\le C\|f_n\|^4_{H^1}$ and $\|F\|^2_{L^2(\mathbb{R})}\le C\|f\|^4_{H^1}$, therefore

$$\int_{-\infty}^{+\infty}\left(F_n^2(x)-F^2(x)\right)dx \le C\left(\|f_n\|^2_{H^1}+\|f\|^2_{H^1}\right)\|F_n-F\|_{L^2(\mathbb{R})}. \tag{31}$$

Now we estimate $\|F_n-F\|_{L^2(\mathbb{R})}$:

$$\begin{aligned}\|F_n-F\|^2_{L^2(\mathbb{R})} &\le \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}(f_n^2)_x(x,y)dy-\int_{-\infty}^{+\infty}(f^2)_x(x,y)dy\right)dx\\ &= \int_{-\infty}^{+\infty}\left|2\int_{-\infty}^{+\infty}(f_nf_{nx}-ff_x)(x,y)dy\right|^2 dx\\ &\le 4\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f_n(f_{nx}-f_x)(x,y)dy\right.\\ &\qquad \left.+\int_{-\infty}^{+\infty}f_x(f_n-f)(x,y)dy\right]^2 dx\\ &\le 8\left(\|f_n\|_2^2+\|f_x\|_2^2\right)\|f_n-f\|^2_{H^1}.\end{aligned} \tag{32}$$

Combining (31) and (32) and using that $\|f_n-f\|_{H^1}\to 0$, limit (30) follows. □

Now we prove the main theorem.

**Theorem 6.** *Let $v\in H^1(\mathbb{R}^2)$ be a weak solution of equation (22) with $b\neq 0$, then $v(x,y)=0$ almost every where.*

*Proof.* Let $\{v_n\}\subset \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\{v_n\}\to v$ in $H^1(\mathbb{R}^2)$, then $v$ satisfies Eq.(24) and

$$\langle\mathcal{L}(v)-\mathcal{L}(v_n),f\rangle+\langle\mathcal{L}(v_n),f\rangle=\langle\mathcal{N}(v)-\mathcal{N}(v_n),f\rangle+\langle\mathcal{N}(v_n),f\rangle$$

$\forall f\in L^2(\mathbb{R}^2)$.

Therefore,

$$\langle\mathcal{L}(v)-\mathcal{L}(v_n),v_{nx}\rangle+\langle\mathcal{L}(v_n),v_{nx}\rangle=\langle\mathcal{N}(v)-\mathcal{N}(v_n),v_{nx}\rangle+\langle\mathcal{N}(v_n),v_{nx}\rangle.$$

Thanks to $\{v_n\}\subset \mathcal{C}_0^\infty(\mathbb{R}^2)$, $\langle\mathcal{L}(v_n),v_{nx}\rangle=0$ and

$$|\langle\mathcal{N}(v_n),v_{nx}\rangle|\le\|\mathcal{L}(v)-\mathcal{L}(v_n)\|_2\|v_{nx}\|_2+\|\mathcal{N}(v)-\mathcal{N}(v_n)\|_2\|v_{nx}\|_2. \tag{33}$$

Because $v \in H^1(\mathbb{R}^2)$ and $\{v_n\} \to v$ in $H^1(\mathbb{R}^2)$, there exists a positive constant $M$, independent of $n$, such that

$$\|v_n\|_{H^1} \leq M. \tag{34}$$

Combining (33), (34) with Propositions 2 and 3, we obtain

$$\lim_{n\to+\infty} \langle \mathcal{N}(v_n), v_{nx}\rangle = 0. \tag{35}$$

On the other hand,

$$\langle \mathcal{N}(v_n), v_{nx}\rangle = \gamma \int_{\mathbb{R}^2} v_{nx}(v_n)^3 + \epsilon \int_{\mathbb{R}^2} v_{nx} v_n \int_{-\infty}^{x} (v_n^2)_y + \epsilon \int_{\mathbb{R}^2} v_{nx} v_n \int_{-\infty}^{y} (v_n^2)_x. \tag{36}$$

We observe that $\{v_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$ implies that the first integral in the right hand side of (36) is zero. Integrating by parts the second term in the right hand side of (36) we obtain

$$\begin{aligned}\epsilon \int_{\mathbb{R}^2} v_{nx} v_n(x, y \int_{-\infty}^{x} (v_n^2)_y(x', y)dx' &= \frac{\epsilon}{2} \int_{\mathbb{R}^2} \left(v_n^2(x,y)\right)_x \int_{-\infty}^{x} (v_n^2)_y(x', y)dx'dy \\ &= \frac{\epsilon}{2} \int_{-\infty}^{\infty} \lim_{x\to+\infty} \left( v_n^2(x,y) \int_{-\infty}^{x} (v_n^2)_y(x',y)dx' \right) dy \\ &\quad - \frac{\epsilon}{2} \int_{\mathbb{R}^2} (v_n)^2 (v_n^2)_y. \end{aligned} \tag{37}$$

Because $v_n \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and for any $y \in \mathbb{R}$,

$$\left| \int_{-\infty}^{x} (v_n^2)_y(x', y)dx' \right| \leq 2 \left( \int_{-\infty}^{\infty} |v_n(x,y)|^2 dx \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} |v_{ny}(x,y)|^2 dx \right)^{\frac{1}{2}},$$

the right hand side of equation (37) is zero and equality (36) becomes

$$\begin{aligned}\langle \mathcal{N}(v_n), v_{nx}\rangle &= \epsilon \int_{\mathbb{R}^2} v_{nx} v_n(x,y) \int_{-\infty}^{y} (v_n^2)_x(x, y')dy'. \\ &= \frac{\epsilon}{2} \int_{\mathbb{R}^2} \int_{-\infty}^{y} (v_n^2)_{xy}(x, y')dy' \int_{-\infty}^{y} (v_n^2)_x(x,y')dy' \\ &= \frac{\epsilon}{4} \int_{\mathbb{R}^2} \frac{\partial}{\partial y} \left( \int_{-\infty}^{y} (v_n^2)_x(x,y')dy' \right)^2 dy \\ &= \frac{\epsilon}{4} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} (v_n^2)_x(x,y)dy \right)^2 dx. \end{aligned} \tag{38}$$

Equation (38) together with equation (35) implies that

$$\lim_{n\to+\infty} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} (v_n^2)_x dy \right)^2 dx = 0. \tag{39}$$

and together with Lemma 5,

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}(v^2)_x dy\right)^2 dx = 0. \tag{40}$$

Hence,

$$\int_{-\infty}^{+\infty}(v^2)_x(x,y)dy = 0 \quad \text{almost everywhere} \quad \text{and} \quad \int_{-\infty}^{+\infty} v^2(x,y)dy = \text{constant}.$$

Because $v \in L^2(\mathbb{R}^2)$ the constant is necessarily zero and Theorem 6 follows. □

## 4 Standing Wave Solutions. Non-homogeneous boundary conditions.

In this section we prove the non-existence of standing wave solutions of elliptic-hyperbolic and hyperbolic-hyperbolic cases of the Davey-Stewartson system for classical solutions with some non-homogeneous boundary conditions of the mean velocity potential.

The Davey-Stewartson system with non-homogeneous boundary conditions (6) can be written in the form:

$$\begin{aligned} iu_t + \alpha(u_{xx}+u_{yy}) + \beta u_{xy} = \gamma|u|^2u + \ \epsilon u\Bigg(\int_{-\infty}^{y}(|u|^2)_x(x,y')dy' \\ + \int_{-\infty}^{x}(|u|^2)_y(x',y)dx'\Bigg) + buf'(y) + bug'(x) \end{aligned} \tag{41}$$

A standing wave solution for the equation (41) is a function $v \in \mathcal{C}^2(\mathbb{R}^2)$ that satisfies

$$\begin{aligned} \alpha(v_{xx}+v_{yy}) + \beta v_{xy} = \omega v + \gamma v^3 + \epsilon v\left(\int_{-\infty}^{y}(v^2)_x dy' + \int_{-\infty}^{x} v_y^2 dx'\right) \\ + bv(f'(y)+g'(x)), \end{aligned} \tag{42}$$

where $\alpha = \left(\delta - \frac{1}{m}\right)$, $\beta = 2(\delta + \frac{1}{m})$, $m < 0$.

**Theorem 7.** *Let $f$ and $g$ be bounded functions in $\mathcal{C}^2(\mathbb{R})$ such that $f'(x)$ and $g'(x)$ are also bounded . If*

$$f''(x) \le 0 \ \forall x \in \mathbb{R} \quad \text{or} \quad g''(x) \le 0 \forall x \in \mathbb{R}. \tag{43}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} g(x) = 0 \tag{44}$$

*If $v \in H^2(\mathbb{R}^2)$ is a classical solution of equation (42) with $b \neq 0$, then $v(x,y) = 0$ $\forall (x,y) \in \mathbb{R}^2$.*

*Proof.* We consider that $g''(x) \leq 0 \, \forall x \in \mathbb{R}$. The proof follows the ideas of the proof of Theorem 6. We use that $v$ is a solution in the classical sense with $v \in H^2(\mathbb{R}^2)$. We take the $L^2$ inner product of $v_x$ with each term of equation (42), integrate by parts and, similarly as in the proof of Theorem 6 we obtain that

$$0 = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} (v^2)_x dy \right)^2 dx - \frac{b}{2} \int v^2 (g''(x)) \tag{45}$$

Therefore, using the assumption on $f$, we conclude that

$$\int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} (v^2)_x dy \right)^2 dx = 0 \tag{46}$$

therefore we conclude similarly as in the proof of Thereom 6 that $v(x, y) = 0$ a.e., because $v$ is continuous, $v(x, y) = 0 \; \forall (x, y) \in \mathbb{R}^2$. □

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