Maria do Rosário Grossinho; Feliz Manuel Minhós

Solvability of some higher order two-point boundary value problems

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 183--189.

Persistent URL: http://dml.cz/dmlcz/700352

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 183–189

Solvability of Some Higher Order Two-Point Boundary Value Problems

Maria do Rosário Grossinho[1⋆] and Feliz Manuel Minhós[2]

[1] Dep. Matemática, ISEG, Universidade Técnica de Lisboa,
Rua do Quelhas, 6, 1200-781 LISBOA, Portugal,
CMAF, Universidade de Lisboa, Av. Prof. Gama Pinto, 2
1649-003 LISBOA Codex, Portugal
Email: mrg@lmc.fc.ul.pt
[2] Dep. Matemática. Universidade de Évora
Rua Romão Ramalho, 7000 ÉVORA, Portugal, and
CIMA-UE, Universidade de Évora,
Rua Romão Ramalho, 7000 ÉVORA, Portugal,
Email: fminhos@dmat.uevora.pt

Abstract. This paper is concerned with the study of some nonlinear *n-th* order differential equation

$$u^{(n)}(t) = f(t, u(t), u'(t), ..., u^{(n-1)}(t)),$$

with two-point boundary conditions, via upper and lower solutions.

MSC 2000. 34B15

Keywords. Higher order two-point BVP, upper and lower solutions, Leray-Schauder degree, Nagumo-type conditions

1 Motivation

Consider the n-th order nonlinear differential equation

$$u^{(n)}(t) = \arctan u^{(n-2)}(t) - [u(t)]^{2k+1} \left[u^{(n-1)}(t)\right]^2, \tag{1}$$

⋆ Supported by FCT.

This is the preliminary version of the paper.

$t \in [0,1]$, $k \in \mathbb{N}$, and the two-point boundary conditions

$$\begin{array}{c} u^{(i)}(0) = 0, \quad i = 0, ..., n-3, \\ a\ u^{(n-2)}(0) - b\ u^{(n-1)}(0) = A, \\ c\ u^{(n-2)}(1) + d\ u^{(n-1)}(1) = B. \end{array} \tag{2}$$

We observe that the results contained in the work [7] for higher order nonlinear differential problems cannot be applied to study the above problem. In fact, there, the equations involve nonlinearities that do not depend on the $(n-1)$-th order derivative of the solution. More precisely, [7] concerns equations of the following type

$$u^{(n)}(t) + f(t, u(t), u'(t), ..., u^{(n-2)}(t)) = 0.$$

Motivated by the above facts, we study the equation

$$u^{(n)}(t) = f(t, u(t), u'(t), ..., u^{(n-1)}(t)), \tag{3}$$

with the boundary conditions (2), where $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, a, b, c, d, A, $B \in \mathbb{R}$ and a, b, c and d satisfy $b, d \geq 0$, $a^2 + b > 0$ and $c^2 + d > 0$. Then we apply it to solve problem (1)–(2). The arguments used follow some ideas contained in [1] and [4], for second order problems, and [2] for third order.

In Section 2, we establish an existence result for problem (3)–(2) relying on the existence of upper and lower solutions. The function f is supposed to satisfy some Nagumo-type conditions. We sketch briefly the proof and refer [3] for details. In Section 3, we consider the problem (1)–(2), with a, b, c and d non-negative constants. We exhibit an upper and a lower solution for this problem and show that $f(t, x_0, ..., x_{n-1}) = \arctan(x_{n-2}) - (x_0)^{2k+1} (x_{n-1})^2$ satisfies Nagumo-type conditions. Then an existence result is derived by applying the theorem of Section 2. We end Section 3 with more one applied problem.

2 Existence Result

We begin by defining lower and upper solutions for problem (3)–(2) and Nagumo-type conditions.

Definition 1. **(i)** A function $\alpha(t) \in C^n(]0,1[) \cap C^{n-1}([0,1])$ is a lower solution of problem (3)–(2) if

$$\alpha^{(n)}(t) \geq f(t, \alpha(t), \alpha'(t), ..., \alpha^{(n-1)}(t)) \tag{4}$$

and

$$\begin{array}{c} \alpha^{(i)}(0) = 0, \ i = 0, ..., n-3, \\ a\ \alpha^{(n-2)}(0) - b\ \alpha^{(n-1)}(0) \leq A, \\ c\ \alpha^{(n-2)}(1) + d\ \alpha^{(n-1)}(1) \leq B. \end{array} \tag{5}$$

(**ii**)A function $\beta(t) \in C^n(]0,1[) \cap C^{n-1}([0,1])$ is an upper solution of problem (3)–(2) if

$$\beta^{(n)}(t) \leq f(t, \beta(t), \beta'(t), ..., \beta^{(n-1)}(t)) \tag{6}$$

and

$$\begin{array}{c} \beta^{(i)}(0) = 0, \ i = 0, ..., n-3, \\ a\ \beta^{(n-2)}(0) - b\ \beta^{(n-1)}(0) \geq A, \\ c\ \beta^{(n-2)}(1) + d\ \beta^{(n-1)}(1) \geq B. \end{array} \tag{7}$$

Definition 2. Let $E \subset [0,1] \times \mathbb{R}^n$. A continuous function $g : E \to \mathbb{R}$ satisfies the Nagumo-type conditions in E if there exists a real continuous function $h_E : \mathbb{R}_0^+ \to]0, +\infty[$, such that

$$|g(t, x_0, ..., x_{n-1})| \leq h_E(|x_{n-1}|), \ \forall (t, x_0, ..., x_{n-1}) \in E, \tag{8}$$

with

$$\int_0^{+\infty} \frac{s}{h_E(s)}\, ds = +\infty \ . \tag{9}$$

The following lemma will play a crucial role in establishing a priori estimates for the solutions of (3)–(2).

Lemma 3. *Let $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function verifying Nagumo-type conditions (8) and (9) in*

$$E = \{(t, x_0, ..., x_{n-1}) \in [0,1] \times \mathbb{R}^n : \gamma_i(t) \leq x_i \leq \Gamma_i(t), \ i = 0, ..., n-2\},$$

where $\gamma_i(t)$ and $\Gamma_i(t)$ are continuous functions such that, for each i and every $t \in [0,1]$,

$$\gamma_i(t) \leq \Gamma_i(t).$$

Then there is $r > 0$ (depending only on h_E, γ_{n-2} and Γ_{n-2}) such that every solution $u(t)$ of (3)–(2) and verifying

$$\gamma_i(t) \leq u^{(i)}(t) \leq \Gamma_i(t),$$

for $i = 0, ..., n-2$ and every $t \in [0,1]$, satisfies

$$\left\| u^{(n-1)} \right\|_\infty < r.$$

The following theorem contains an existence result. Some information about the location of the solution and its i-derivatives, with $i = 1, ..., n-2$, is also given.

Theorem 4. *Let $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Suppose that there are lower and upper solutions of (3)–(2), $\alpha(t)$ and $\beta(t)$, respectively, such that, for $t \in [0,1]$,*

$$\alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t) \tag{10}$$

and that f satisfies Nagumo-type conditions (8) *and* (9) *in*

$$E_* = \left\{(t, x_0, ..., x_{n-1}) \in [0,1] \times \mathbb{R}^n : \alpha^{(i)}(t) \le x_i \le \beta^{(i)}(t), i = 0, ..., n-2\right\},$$

where by $\alpha^{(0)}$ and $\beta^{(0)}$ we mean α and β. If f verifies

$$\begin{aligned} f(t, \alpha(t), ..., \alpha^{(n-3)}(t), x_{n-2}, x_{n-1}) \ge f(t, x_0, ..., x_{n-1}) \ge \\ \ge \ f(t, \beta(t), ..., \beta^{(n-3)}(t), x_{n-2}, x_{n-1}), \end{aligned} \tag{11}$$

for every $(t, x_0, ..., x_{n-1}) \in [0,1] \times \mathbb{R}^n$ such that $\alpha^{(i)}(t) \le x_i \le \beta^{(i)}(t)$ with $i = 0, ..., n-3$, then the problem (3)–(2) *has at least a solution $u(t) \in C^n([0,1])$ satisfying*

$$\alpha^{(i)}(t) \le u^{(i)}(t) \le \beta^{(i)}(t),$$

for $i = 0, ..., n-2$ and $t \in [0,1]$.

Remark: If the function $f(t, x_0, ..., x_{n-1})$ is decreasing on $(x_0, ..., x_{n-3})$ then (11) is satisfied.

Proof. We sketch briefly the proof. For $i = 0, ..., n-2$ define the auxiliary continuous functions

$$\delta_i(t, x_i) = \begin{cases} \beta^{(i)}(t) & \text{if } x_i > \beta^{(i)}(t) \\ x_i & \text{if } \alpha^{(i)}(t) \le x_i \le \beta^{(i)}(t) \\ \alpha^{(i)}(t) & \text{if } x_i < \alpha^{(i)}(t). \end{cases}$$

For $\lambda \in [0,1]$, consider the homotopic equation

$$\begin{aligned} u^{(n)}(t) = \lambda \ f(t, \delta_0(t, u(t)), ..., \delta_{n-2}(t, u^{(n-2)}(t)), u^{(n-1)}(t)) + \\ + u^{(n-2)}(t) - \lambda \ \delta_{n-2}(t, u^{(n-2)}(t)), \end{aligned} \tag{12}$$

with the boundary conditions

$$\begin{aligned} & u^{(i)}(0) = 0, \ i = 0, ..., n-3, \\ & u^{(n-2)}(0) = \lambda \left[A - a \ \delta_{n-2}(0, u^{(n-2)}(0)) + b \ u^{(n-1)}(0) + \right. \\ & \qquad \left. + \delta_{n-2}(0, u^{(n-2)}(0))\right], \\ & u^{(n-2)}(1) = \lambda \left[B - c \ \delta_{n-2}(1, u^{(n-2)}(1)) - d \ u^{(n-1)}(1) + \right. \\ & \qquad \left. + \delta_{n-2}(1, u^{(n-2)}(1))\right]. \end{aligned} \tag{13}$$

Take $r_1 > 0$ such that for every $t \in [0,1]$,

$$-r_1 < \alpha^{(n-2)}(t) \le \beta^{(n-2)}(t) < r_1 \ ,$$

$$f(t, \alpha(t), ..., \alpha^{(n-2)}(t), 0) - r_1 - \alpha^{(n-2)}(t) < 0,$$

$$f(t, \beta(t), ..., \beta^{(n-2)}(t), 0) + r_1 - \beta^{(n-2)}(t) > 0$$

and

$$\left|A - a\ \beta^{(n-2)}(0) + \beta^{(n-2)}(0)\right| < r_1,$$

$$\left|A - a\ \alpha^{(n-2)}(0) + \alpha^{(n-2)}(0)\right| < r_1,$$

$$\left|B - c\ \beta^{(n-2)}(1) + \beta^{(n-2)}(1)\right| < r_1,$$

$$\left|B - c\ \alpha^{(n-2)}(1) + \alpha^{(n-2)}(1)\right| < r_1.$$

The proof is based on the following steps (see [3] for details)

Step 1.*Every solution $u(t)$ of problem (12)–(13) satisfies*

$$\left|u^{(i)}(t)\right| < r_1,\ \forall t \in [0,1],$$

for $i = 0, ..., n-2$ and independently of $\lambda \in [0,1]$.

This statement follows easily by using the definitions of upper and lower solutions combined with the condition (11).

Step 2. *There is $r_2 > 0$ such that, for every solution $u(t)$ of problem (12)–(13),*

$$\left|u^{(n-1)}(t)\right| < r_2,\ \forall t \in [0,1],$$

independently of $\lambda \in [0,1]$.

This assertion can be derived by using Step 1 and the auxiliar Lemma 3.

Step 3. *For $\lambda = 1$, problem (12)–(13) has at least a solution $u_1(t)$.*

This statement follows by applying Leray-Schauder degree theory.

Step 4. *The function $u_1(t)$ is a solution of (3)–(2).*

By using the definitions of upper and lower solutions and condition (11), it can be shown that every solution of the problem (12)–(13) lies between α and β, and therefore is a solution of (3)–(2).

By integration one can easily deduce the location result that concerns the derivatives of $u_1(t)$.

3 Applications

Application 1. Consider the differential equation (1) and the boundary conditions

$$\begin{array}{c} u^{(i)}(0) = 0, \quad i = 0, ..., n-3, \\ a\ u^{(n-2)}(0) - b\ u^{(n-1)}(0) = A, \\ c\ u^{(n-2)}(1) + d\ u^{(n-1)}(1) = B, \end{array} \tag{14}$$

for $A, B \in \mathbb{R}$, $a, b, c, d \geq 0$ such that $a + b > 0$ and $c + d > 0$.

The function

$$f(t, x_0, ..., x_{n-1}) = \arctan(x_{n-2}) - (x_0)^{2k+1}(x_{n-1})^2$$

is continuous and decreasing on x_0. If A and B are such that $|A| \leq a$ and $|B| \leq c$ then functions $\alpha, \beta : [0,1] \to \mathbb{R}$ defined by

$$\alpha(t) = -\frac{t^{n-2}}{(n-2)!} \text{ and } \beta(t) = \frac{t^{n-2}}{(n-2)!}$$

are, respectively, lower and upper solutions of the problem (1)–(14).

Moreover, the function f satisfies the Nagumo-type conditions (8) and (9) in

$$E = \left\{ (t, x_0, ..., x_{n-1}) \in [0,1] \times \mathbb{R}^n : |x_0| \leq \frac{t^{n-2}}{(n-2)!} \right\},$$

for $h_E : \mathbb{R}_0^+ \to \mathbb{R}^+$ given by $h_E(x) = \frac{\pi}{2} + x^2$. As conditions (10) and (11) are satisfied then, by Theorem 4, there is at least a solution $u(t)$ for (1)–(14) such that

$$-\frac{t^{n-2-i}}{(n-2-i)!} \leq u^{(i)}(t) \leq \frac{t^{n-2-i}}{(n-2-i)!},$$

for $i = 0, ..., n-2$.

Observe that in this case the estimation for $u^{(n-2)}$ does not depend on n since by the above inequality $-1 \leq u^{(n-2)}(t) \leq 1$.

Next application shows a non-uniform estimation for $u^{(n-2)}$.

Application 2. For $n \geq 2$, consider the equation

$$u^{(n)}(t) = \arctan\left(\frac{u^{(n-2)}(t)}{(n-2)!}\right) \sqrt[k]{\left(u^{(n-1)}(t)\right)^2 + 1} - \arctan(u(t)), \qquad (15)$$

with $k \in \mathbb{N}$, and the boundary conditions (14).

If $A, B \in \mathbb{R}$ are such that $|A| \leq a(n-2)!$ and $|B| \leq c(n-2)!$, then functions $\alpha, \beta : [0,1] \to \mathbb{R}$ given by

$$\alpha(t) = -t^{n-2} \text{ and } \beta(t) = t^{n-2}$$

are, respectively, lower and upper solutions for (15)-(14), verifying (10).

The function

$$f(t, x_0, ..., x_{n-1}) = \arctan\left(\frac{x_{n-2}}{(n-2)!}\right) \sqrt[k]{(x_{n-1})^2 + 1} - \arctan(x_0)$$

is continuous. Moreover, it satisfies (11) and the Nagumo-type conditions (8) and (9) with

$$h(x) = \frac{\pi}{2} + \frac{\pi}{2} \sqrt[k]{(x)^2 + 1},$$

in every subset $E \subset [0,1] \times \mathbb{R}^n$.

So, by Theorem 4, there is at least a solution $u(t)$ for (15)–(14) such that, for every $t \in [0,1]$,

$$-(n-2)...(n-i-1)\ t^{n-2-i} \leq u^{(i)}(t) \leq (n-2)...(n-i-1)\ t^{n-2-i},$$

with $i = 0, ..., n-3$, and

$$-(n-2)! \leq u^{(n-2)}(t) \leq (n-2)!.$$

References

1. C. De Coster, *La méthode des sur et sous solutions dans l'étude de problèmes aux limites.* Université Catholique de Louvain. Faculté des Sciences. Départment de Mathématique. Février 1994.
2. M.R. Grossinho and F. Minhós, *Existence result for some third order separated boundary value problems.* Journal of Nonlinear Analysis: Series A. Theory and Methods, **47**, (2001), 2407-2418.
3. M.R. Grossinho and F. Minhós, *Upper and lower solutions for higher order boundary value problems,* preprint CMAF *(2001).*
4. C. De Coster and P. Habets. *Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results.* Institut de Mathématique Pure et Appliquée. Université Catholique de Louvain. Recherches de Mathématique n^{o}52. April 1996.
5. J. Mawhin, *Topological degree methods in nonlinear boundary value problems,* Regional Conference Series in Mathematics, n^{o}40, American Mathematical Society, Providence, Rhode Island. (1979).
6. M. Nagumo, *Über die differentialgleichung* $y'' = f(t, y, y')$, Proc. Phys-Math. Soc. Japan, **19**, (1937), 861-866.
7. F. H. Wong, *An application of Schauder's fixed point theorem with respect to higher order BVPs,* Proceedings of the American Mathematical Society, **126**, (1998), 2389-2397.