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# A doubly degenerate elliptic system

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**Abstract.** We consider the steady state of the thermistor problem, a coupled set of nonlinear elliptic equations governing the temperature and the electric potential. We study the existence of weak solutions under the assumption that the two diffusion coefficients are not bounded below far from zero, arising to a degenerate system.

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## 1 Introduction

The heat produced by an electrical current passing through a conductor device is governed by the so-called thermistor problem. This problem consists of a system of nonlinear parabolic-elliptic system describing the temperature, $u$, and the electric potential $\varphi$ ([1,7]). Here, we consider the steady state case, resulting in a coupled nonlinear elliptic system. Let $\mathcal{J}$ be the current density, $\mathcal{Q}$ the heat flux and $\mathcal{E} = -\nabla\varphi$ the electric field; then by Ohm's and Fourier's law we have

$$\mathcal{J} = \sigma(u)\mathcal{E}, \quad \mathcal{Q} = -a(u)\nabla u,$$

*This is the preliminary version of the paper.*

where $a(u)$ and $\sigma(u)$ are, respectively, the thermal and electric conductivities. Also, from the usual conservation laws $\nabla \cdot \mathcal{J} = 0$, $\nabla \cdot \mathcal{Q} = \mathcal{E} \cdot \mathcal{J}$ we obtain

$$\begin{cases} -\nabla \cdot (a(u)\nabla u) = \nabla \cdot (\sigma(u)\varphi\nabla\varphi) & \text{en } \Omega, \\ \nabla \cdot (\sigma(u)\nabla\varphi) = 0 & \text{en } \Omega, \\ u = 0 & \text{sobre } \partial\Omega, \\ \varphi = \varphi_0 & \text{sobre } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega$ is an open, bounded and smooth enough set in $\mathbb{R}^N$, $N \geq 1$. Usually, the right hand side of the equation for the temperature is written as $\sigma(u)|\nabla\varphi|^2$, which is equal to $\nabla \cdot (\sigma(u)\varphi\nabla\varphi)$ thanks to the equation verified by $\varphi$; this is true, for instance, if $\varphi \in H^1(\Omega)$.

The steady state themistor problem has been studied by several authors along the last two decades. Among them, we refer to Cimatti ([3,4,5,6]) and Cimatti–Prodi [8]. In these papers, the authors have obtained some existence results of weak solutions in both, two and three dimensions, using the so-called Diesselhorst transformation, and under the conditions $u = u_0$ on $\partial\Omega$, and $u_0$ being a constant value, or $u = u_0 \geq u_m > 0$ on $\partial\Omega$, together with the hypothesis $0 < a_m \leq a(u)$, or $a(u) = a_0$ constant, or even under the Wiedemann–Franz law (that is, $a(s) = Ls\sigma(s)$, $L > 0$ a constant value) with metallic conduction, and certain assumptions on $\sigma(u)$. We notice that in all these papers is assumed that $a(s) \geq a_0 > 0$, for all $s$.

In the present work we show an existence result of a weak solution to the steady state thermistor problem in divergence form (1) under the general assumption that both $a(s)$ and $\sigma(s)$ are not bounded below far from zero. In this way, system (1) becomes doubly degenerate; in particular, we cannot expect the regularity $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, or that $u$ belongs to some Sobolev space. We point out that the technique we use here is not based en the derivation of $L^\infty$-estimates for the temperature.

## 2 Setting of the problem

We consider the steady state thermistor problem in divergence form (1) under the following hypotheses on data:

(H.1) $\sigma \in C(\mathbb{R})$ and $0 < \sigma(s) \leq \bar{\sigma}$, for all $s \in \mathbb{R}$.

(H.2) $a \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\int_0^{+\infty} a(s)\,\mathrm{d}s = +\infty$, and $A(r) = \int_0^r a(s)\,\mathrm{d}s$ is a strictly increasing function.

(H.3) $\varphi_0 \in H^1(\Omega)$.

(H.4) There exist an integer $M > 1$ and a function $\alpha : [M, +\infty) \to \mathbb{R}$ such that $\alpha(s) > 0$, for all $s \geq M$, $\alpha$ is non-increasing and $\sigma(s) \geq \alpha(s) > 0$.

(H.5) Let $p \in \left(\frac{2N}{N+2}, 2\right)$ if $N \geq 2$, $p \in (1,2)$ if $N = 1$ and $p' = 2 - p$, then

$$\int_M^{+\infty} \frac{\mathrm{d}s}{\alpha(s)^{p/p'} A(s-1)^{\bar{q}/2}} < +\infty, \text{ with } \begin{cases} \bar{q} = 2^* & \text{if } N \geq 3, \\ \bar{q} \in [2, +\infty) & \text{if } N = 2, \\ \bar{q} \in [1, +\infty) & \text{if } N = 1. \end{cases} \tag{2}$$

The main result of this work now follows

**Theorem 1.** *Under assumptions* (H.1)–(H.5)*, problem*

$$\left.\begin{array}{rl} -\Delta A(u) = \nabla \cdot (\sigma(u)\varphi\nabla\varphi) & \textit{in } \mathcal{D}'(\Omega), \\ \nabla \cdot (\sigma(u)\nabla\varphi) = 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \\ \varphi = \varphi_0 & \textit{on } \partial\Omega, \end{array}\right\} \tag{3}$$

*has a weak solution* $(u, \varphi)$ *in the following sense*

$$\forall\, q < \frac{N}{N-1} \;\textit{ if } N \geq 2,\; q = 2 \textit{ if } N = 1, \;\; A(u) \in W_0^{1,q}(\Omega), \tag{4}$$

$$\varphi - \varphi_0 \in W_0^{1,p}(\Omega), \quad \sigma(u)^{1/2}\nabla\varphi \in L^2(\Omega), \tag{5}$$

$$\int_\Omega \nabla A(u)\nabla\xi = -\int_\Omega \sigma(u)\varphi\nabla\varphi\nabla\xi, \text{ for all } \xi \in \mathcal{D}(\Omega), \tag{6}$$

$$\int_\Omega \sigma(u)\nabla\varphi\nabla\phi = 0, \text{ for all } \phi \in H_0^1(\Omega). \tag{7}$$

*Furthermore, the term* $\nabla \cdot (\sigma(u)\varphi\nabla\varphi)$ *is a Radon measure and* $u \geq 0$ *almost everywhere in* $\Omega$*.*

### 2.1 Approximate problems

Let $n \in \mathbb{N}$ and introduce the functions $a_n(s) = a(s) + \dfrac{1}{n}$, $\sigma_n(s) = \sigma(s) + \dfrac{1}{n}$, then we set the approximate problem given as follows

$$\left\{\begin{array}{rl} -\nabla \cdot (a_n(u_n)\nabla u_n) = \sigma_n(u_n)|\nabla\varphi_n|^2 & \text{in } \Omega, \\ \nabla \cdot (\sigma_n(u_n)\nabla\varphi_n) = 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \\ \varphi_n = T_n(\varphi_0) & \text{on } \partial\Omega. \end{array}\right. \tag{8}$$

where $T_n(s) = \min(|s|, n)\,\mathrm{sign}\, s$. By virtue of the classical existence results ([1]), problem (8) has a solution such that $u_n \in H_0^1(\Omega)$, $\varphi_n - \varphi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

### 2.2 Estimates and passing to the limit

Since

$$\int_\Omega \sigma_n(u_n)\nabla\varphi_n\nabla\phi = 0, \text{ for all } \phi \in H_0^1(\Omega), \tag{9}$$

taking $\phi = \varphi_n - \varphi_0$ yield

$$\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2 = \int_\Omega \sigma_n(u_n)\nabla\varphi_n\nabla\varphi_0$$
$$\leq \left(\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2\right)^{1/2}\left(\int_\Omega \sigma_n(u_n)|\nabla\varphi_0|^2\right)^{1/2},$$

hence

$$\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2 \leq \tilde{\sigma}\int_\Omega |\nabla\varphi_0|^2 \leq \tilde{\sigma}\|\varphi_0\|_{H^1(\Omega)} = C(\tilde{\sigma},\varphi_0) = C_1, \tag{10}$$

therefore, $(f_n) = \big(\sigma_n(u_n)|\nabla\varphi_n|^2\big)$ is bounded in $L^1(\Omega)$.

Let $v_n = A_n(u_n)$, $A_n(r) = \int_0^r a_n(s)\,\mathrm{d}s$ and consider the elliptic problem

$$\left.\begin{aligned} -\Delta v_n &= f_n \text{ in } \Omega,\\ v_n &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

From Boccardo–Gallouët estimates ([2,9]), we deduce that

$$(v_n) \text{ is bounded in } W_0^{1,q}(\Omega), \text{ for all } q < \frac{N}{N-1} \text{ if } N \geq 2,\ q = 2 \text{ if } N = 1. \tag{11}$$

In this way, there exist a subsequence $(v_m) \subset (v_n)$ and $v \in W_0^{1,q}(\Omega)$ such that

$$v_m \rightharpoonup v \text{ in } W_0^{1,q}(\Omega)\text{-weakly.} \tag{12}$$

Since the embeddings $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$, for all $r < \frac{N}{N-2}$ if $N \geq 2$, or $W_0^{1,q}(\Omega) = H_0^1(\Omega) \hookrightarrow C(\bar{\Omega})$ if $N = 1$, are compacts, we may also assume that

$$v_m \to v \text{ in } L^r(\Omega)\text{-strongly, if } N \geq 2, \tag{13}$$
$$v_m \to v \text{ in } C(\bar{\Omega})\text{-strongly, if } N = 1, \tag{14}$$
$$v_m \to v \text{ a.e. in } \Omega. \tag{15}$$

Moreover, since $f_n \geq 0$ in $\Omega$, then $v_n \geq 0$ in $\Omega$. Since $A_n$ is strictly increasing, we also have $u_n \geq 0$ in $\Omega$. Now, we show that $(A(u_n)) \subset H_0^1(\Omega)$ is bounded in $W_0^{1,q}(\Omega)$. Indeed,

$$|\nabla A(u_n)| = |a(u_n)\nabla u_n| \leq |a_n(u_n)\nabla u_n| = |\nabla A_n(u_n)|$$

and by virtue of (11), $(A(u_n))$ is also bounded in $W_0^{1,q}(\Omega)$; then there exist a subsequence $(A(u_m)) \subset (A(u_n))$ and $z \in W_0^{1,q}(\Omega)$ such that

$$A(u_m) \rightharpoonup z \text{ in } W_0^{1,q}(\Omega)\text{-weakly,} \tag{16}$$
$$A(u_m) \to z \text{ in } L^r(\Omega)\text{-strongly, for all } r < \frac{N}{N-2} \text{ if } N \geq 2, \tag{17}$$
$$A(u_m) \to z \text{ in } C(\bar{\Omega})\text{-strongly if } N = 1, \tag{18}$$
$$A(u_m) \to z \text{ a.e. in } \Omega. \tag{19}$$

But, since $A$ is bijective, from (19) we deduce

$$u_m \to A^{-1}(z) = u \text{ a.e. in } \Omega, \tag{20}$$

with $u \geq 0$ a.e. in $\Omega$.

Thanks to the definition of $\sigma_n$ together with (20) we obtain

$$\sigma_m(u_m) \to \sigma(u) \text{ a.e. in } \Omega. \tag{21}$$

Also, by virtue of (H.1), $(\sigma_n(u_n))$ is bounded in $L^\infty(\Omega)$, and taking into account (21), we have

$$\sigma_m(u_m) \to \sigma(u) \text{ in } L^\infty(\Omega)\text{-weakly-}*. \tag{22}$$

Now, we seek for estimates to the sequence $(\varphi_n)$ in some Sobolev space $W^{1,p}(\Omega)$, with $1 < p < 2$. By virtue of (H.5), $\frac{2}{p'}$ is the conjugate exponent of $\frac{2}{p}$. Applying Young's inequality and taking into account (10), we obtain

$$\begin{aligned}\int_\Omega |\nabla\varphi_n|^p &\leq \left(\int_\Omega \sigma_n(u_n)^{-p/p'}\right)^{p'/2} \left(\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2\right)^{p/2} \\ &\leq C_1^{p/2} \left(\int_\Omega \sigma_n(u_n)^{-p/p'}\right)^{p'/2}.\end{aligned}$$

Let's show the following estimate

$$\int_\Omega \sigma_n(u_n)^{-p/p'} \leq C_2. \tag{23}$$

From $0 < \sigma(s) \leq \sigma_n(s) \leq \tilde{\sigma}$, for all $s \in \mathbb{R}$, it yields

$$\tilde{\sigma}^{-p/p'} \leq \sigma_n(s)^{-p/p'} \leq \sigma(s)^{-p/p'}, \text{ for all } s \in \mathbb{R},$$

hence

$$\int_\Omega \sigma_n(u_n)^{-p/p'} \leq \int_\Omega \sigma(u_n)^{-p/p'} \leq \int_{\{|u_n| \leq M\}} \sigma(u_n)^{-p/p'} + \int_{\{u_n > M\}} \sigma(u_n)^{-p/p'}.$$

Thanks to (H.1), $\sigma^{-1}$ is bounded on compact sets of $\mathbb{R}$, in particular, there exists a constant value $C_M > 0$ such that $\min_{|s| \leq M} \sigma(s) = C_M$, and this implies that $\sigma(u_n)^{-p/p'}\chi_{\{|u_n| \leq M\}} \leq C_M^{-p/p'}$, and

$$\int_{\{|u_n| \leq M\}} \sigma(u_n)^{-p/p'} \leq C_M^{-p/p'}|\Omega| = C(M, p, p', \Omega) = C_3.$$

On the other hand, by virtue of (H.4), we deduce

$$\begin{aligned}\int_{\{u_n > M\}} \sigma(u_n)^{-p/p'} &\leq \int_{\{u_n > M\}} \alpha(u_n)^{-p/p'} \leq \sum_{i \geq M} \int_{\{i \leq u_n < i+1\}} \alpha(u_n)^{-p/p'} \\ &\leq \sum_{i \geq M} \int_{\{i \leq u_n < i+1\}} \alpha(i+1)^{-p/p'} \leq \sum_{i \geq M} \alpha(i+1)^{-p/p'} \,|\{u_n \geq i\}|\end{aligned} \tag{24}$$

In order to derive some estimate to $|\{u_n \geq i\}|$, we first study $|\{v_n = A_n(u_n) \geq i\}|$. To do so, we take $T_i(v_n)$ as a test function in the equation of $u_n$; then

$$\int_\Omega \nabla v_n \nabla T_i(v_n) = \int_\Omega \sigma_n(u_n)|\nabla \varphi_n|^2 T_i(v_n) \leq C_1 i,$$

the left hand side can be written as $\int_\Omega \nabla v_n \nabla T_i(v_n) = \int_\Omega |\nabla T_i(v_n)|^2 = I_{i,n}$. By Sobolev's inequality we have

$$\begin{aligned} I_{i,n} &\geq C\left(\int_\Omega |T_i(v_n)|^{\bar{q}}\right)^{2/\bar{q}} \geq C\left(\int_{\{v_n \geq i\}} |T_i(v_n)|^{\bar{q}}\right)^{2/\bar{q}} \\ &= C\left(\int_{\{v_n \geq i\}} i^{\bar{q}}\right)^{2/\bar{q}} = Ci^2 \left|\{v_n \geq i\}\right|^{2/\bar{q}}, \end{aligned}$$

where $\bar{q} = 2^* = 2N/(N-2)$ and $C = C(\Omega, N)$, if $N \geq 3$, $\bar{q} \in [2, +\infty)$ and $C = C(\Omega, \bar{q})$, if $N \leq 2$. Consequently,

$$\left|\{v_n \geq i\}\right|^{2/\bar{q}} \leq \frac{C_1 i}{i^2 C} = \frac{C_1}{iC},$$

which yields, $|\{v_n \geq i\}| \leq \left(\frac{C_1}{iC}\right)^{\bar{q}/2} = \frac{C_4}{i^{\bar{q}/2}}$. Since $u_n \geq 0$ in $\Omega$, $A_n(u_n) \geq A(u_n)$ in $\Omega$, $\{A(u_n) \geq i\} \subset \{v_n = A_n(u_n) \geq i\}$ and

$$|\{A(u_n) \geq i\}| \leq |\{v_n \geq i\}| \leq \frac{C_4}{i^{\bar{q}/2}},$$

hence

$$\left|\{u_n \geq A^{-1}(i)\}\right| \leq \frac{C_4}{i^{\bar{q}/2}},$$

this can be expressed as

$$|\{u_n \geq l\}| \leq \frac{C_4}{A(l)^{\bar{q}/2}}.$$

Therefore, thanks to (2) in (H.5) and (24), we have

$$\begin{aligned} \int_{\{u_n > M\}} \sigma(u_n)^{-p/p'} &\leq \sum_{i \geq M} \alpha(i+1)^{-p/p'} \frac{C_4}{A(i)^{\bar{q}/2}} \\ &\leq C_4 \int_{M-1}^{+\infty} \frac{\mathrm{ds}}{\alpha(s+1)^{p/p'} A(s)^{\bar{q}/2}} = C_5. \end{aligned}$$

This shows (23) and we deduce that

$$\int_\Omega |\nabla \varphi_n|^p \leq C_1^{p/2} C_2^{p'/2} = C_6, \tag{25}$$

which means that, $\varphi_n - \varphi_0$ is bounded in $W_0^{1,p}(\Omega)$. We then take a subsequence $(\varphi_m) \subset (\varphi_n)$ and $\varphi \in W^{1,p}(\Omega)$ such that

$$\varphi_m \rightharpoonup \varphi \text{ in } W^{1,p}(\Omega)\text{-weakly,} \tag{26}$$

$$\varphi_m \to \varphi \text{ in } L^{\bar{r}}(\Omega)\text{-strongly, for all } \bar{r} < p^* \text{ if } N \geq 2, \tag{27}$$

$$\varphi_m \to \varphi \text{ in } C(\bar{\Omega})\text{-strongly, if } N = 1, \tag{28}$$

$$\varphi_m \to \varphi \text{ a.e. in } \Omega. \tag{29}$$

From (H.5), $p > 2N/(N+2)$ which implies that $p^* = Np/(N-p) > 2$. In particular

$$\varphi_m \to \varphi \text{ in } L^2(\Omega)\text{-strongly.} \tag{30}$$

Thanks to (10) $\left(\sigma_n(u_n)^{1/2}\nabla\varphi_n\right)$ is bounded in $L^2(\Omega)^N$; and there exist a subsequence $\left(\sigma_m(u_m)^{1/2}\nabla\varphi_m\right) \subset \left(\sigma_n(u_n)^{1/2}\nabla\varphi_n\right)$ and $\Phi \in L^2(\Omega)^N$ such that

$$\sigma_m(u_m)^{1/2}\nabla\varphi_m \rightharpoonup \Phi \text{ in } L^2(\Omega)^N\text{-weakly.} \tag{31}$$

From (22) and (26) it is deduced that $\Phi = \sigma(u)^{1/2}\nabla\varphi \in L^2(\Omega)^N$. Moreover, taking into account (H.1), (22) and (31), we also have

$$\sigma_m(u_m)\nabla\varphi_m \rightharpoonup \sigma(u)\nabla\varphi \text{ in } L^2(\Omega)^N\text{-weakly.} \tag{32}$$

Consequently, $\nabla\cdot(\sigma(u)\nabla\varphi) \in H^{-1}(\Omega)$ and

$$\langle \nabla\cdot(\sigma(u)\nabla\varphi)\,,\phi\rangle = -\int_\Omega \sigma(u)\nabla\varphi\nabla\phi = 0, \text{ for all } \phi \in H_0^1(\Omega),$$

Going back to (9) and taking $\phi = \varphi_n\xi$, with $\xi \in \mathcal{D}(\Omega)$. Then

$$\begin{aligned} 0 &= \int_\Omega \sigma_n(u_n)\nabla\varphi_n\nabla(\varphi_n\xi) = \int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2\xi + \int_\Omega \sigma_n(u_n)\nabla\varphi_n\varphi_n\nabla\xi \\ &= \int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2\xi - \int_\Omega \nabla\cdot(\sigma_n(u_n)\varphi_n\nabla\varphi_n)\,\xi, \end{aligned}$$

and so,

$$\sigma_n(u_n)|\nabla\varphi_n|^2 = \nabla\cdot(\sigma_n(u_n)\varphi_n\nabla\varphi_n) \text{ en } \mathcal{D}'(\Omega); \tag{33}$$

from the equality

$$\int_\Omega \sigma_m(u_m)\varphi_m\nabla\varphi_m\nabla\xi = \int_\Omega \sigma_m(u_m)^{1/2}\varphi_m\sigma_m(u_m)^{1/2}\nabla\varphi_m\nabla\xi.$$

and by virtue of (21), (30) and (31), passing to the limit in $m \to \infty$, it yields

$$\int_\Omega \sigma(u)^{1/2}\varphi\sigma(u)^{1/2}\nabla\varphi\nabla\xi = \int_\Omega \sigma(u)\varphi\nabla\varphi\nabla\xi, \text{ for all } \xi \in \mathcal{D}(\Omega),$$

so, $\sigma_m(u_m)|\nabla\varphi_m|^2 = \nabla\cdot(\sigma_m(u_m)\varphi_m\nabla\varphi_m) \to \nabla\cdot(\sigma(u)\varphi\nabla\varphi)$ en $\mathcal{D}'(\Omega)$. Since $\sigma_m(u_m)|\nabla\varphi_m|^2 \geq 0$ is bounded in $L^1(\Omega)$, we conclude that $\nabla\cdot(\sigma(u)\varphi\nabla\varphi)$ is a positive Radon measure. This ends up the proof of theorem 1.

*Remark 2.* It is interesting to know if the equality $\nabla \cdot (\sigma(u)\varphi\nabla\varphi) = \sigma(u)|\nabla\varphi|^2$ holds in our setting. There are cases where this holds true (for instance in $N = 1$). In the general case and with the regularity deduced here for $u$ and $\varphi$, we do not know if this equality still holds ([10]).

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