EQUADIFF 10

Vladimír Ďurikovič; Monika Ďurikovičová

# Topological properties of nonlinear evolution equations

In: Jaromír Kuben and Jaromír Vosmanský (eds.): Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 2] Papers. Masaryk University, Brno, 2002. CD-ROM; a limited number of printed issues has been issued. pp. 159--168.

Persistent URL: http://dml.cz/dmlcz/700349

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Papers, pp. 159–168

# Topological Properties of Nonlinear Evolution Equations

Vladimír Ďurikovič[1] and Monika Ďurikovičová[2]

[1] Department of Applied Mathematics SS, Cyril and Methodius University
nám. J. Herdu 2, 917 00 Trnava, Slovak Republic
Email: vdurikovic@fmph.uniba.sk

[2] Department of Mathematics of Slovak Technical University,
nám. Slobody 17, 812 31 Bratislava, Slovak Republic
Email: durikovi@sjf.stuba.sk

**Abstract.** The generic properties of solutions of the second order ordinary differential equations were studied by L. Brüll and J. Mawhin in [2], J. Mawhin in [5] and by V. Šeda in [9]. Such questions were solved for nonlinear diffusional type problems with the Dirichlet, Neumann and Newton type conditions by V. Ďurikovič, Ma. Ďurikovičová in [4].
In the present paper we study the set structure of classic solutions, bifurcation points and the surjectivity of an associated operator to a general second order nonlinear evolution problem by the Fredholm operator theory.

**MSC 2000.** 35K20, 35K60, 47F05, 47A53, 47H30

**Keywords.** Initial-boundary value problem, Fredholm operator, Hölder space, bifurcation point, surjectivity

## 1 The formulation of problem and basic notations

Throughout this paper we assume that the set $\Omega \subset R^n$ for $n \in N$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$. The real number $T$ is positive and $Q := (0,T] \times \Omega, \Gamma := (0,T] \times \partial\Omega$.

We use the notation $D_t$ for $\partial/\partial t$ and $D_i$ for $\partial/\partial x_i$ and $D_{ij}$ for $\partial^2/\partial x_i \partial x_j$ where $i,j = 1,\dots,n$ and $D_0 u$ for $u$. The symbol $\operatorname{cl} M$ means the closure of a set M in $R^n$.

We consider the nonlinear differential equation possibly a non-parabolic type

$$D_t u - A(t,x,D_x)u + f(t,x,u,D_1 u,\dots,D_n u) = g(t,x) \tag{1.1}$$

*This is an overview article.*

for $(t,x) \in Q$, where the coefficients $a_{ij}, a_i, a_0$ for $i,j = 1,\dots,n$ of the second order linear operator

$$A(t,x,D_x)u = \sum_{i,j=1}^{n} a_{ij}(t,x)D_{ij}u + \sum_{i=1}^{n} a_i(t,x)D_i u + a_0(t,x)u$$

are continuous functions from the space $C(\mathrm{cl}\, Q, R)$. The function $f$ is from the space $C(\mathrm{cl}\, Q \times R^{n+1}, R)$ and $g \in C(\mathrm{cl}\, Q, R)$.

Together with the equation (1.1) we consider the following general homogeneous boundary condition

$$B_3(t,x,D_x)u|_\Gamma := \sum_{i=1}^{n} b_i(t,x)D_i u + b_0(t,x)u|_\Gamma = 0, \tag{1.2}$$

where the coefficients $b_i$ for $i = 1,\dots,n$ and $b_0$ are continuos functions from $C(\mathrm{cl}\, \Gamma, R)$.

Furthermore we require for the solution of (1.1) to satisfy the homogeneous initial condition

$$u|_{t=0} = 0 \quad \text{on} \quad \mathrm{cl}\, \Omega. \tag{1.3}$$

In the following definitions we shall use the notations

$$\langle u\rangle^{s}_{t,\mu,Q} := \sup_{\substack{(t,x),(s,x)\in \mathrm{cl}\, Q\\ t\neq s}} \frac{|u(t,x)-u(s,x)|}{|t-s|^{\mu}}, \tag{1.4}$$

$$\langle u\rangle^{y}_{x,\nu,Q} := \sup_{\substack{(t,x),(t,y)\in \mathrm{cl}\, Q\\ x\neq y}} \frac{|u(t,x)-u(t,y)|}{|x-y|^{\nu}}, \tag{1.5}$$

$$\langle f\rangle^{s,y,v}_{t,x,u} := |f(t,x,u_0,u_1,\dots,u_n) - f(s,y,v_0,v_1,\dots,v_n)|,$$

$$\langle f\rangle^{s,y,v(s,y)}_{t,x,u(t,x)} := |f[t,x,u(t,x),D_1u(t,x),\dots,D_nu(t,x)] - \\ - f[s,y,v(s,y),D_1v(s,y),\dots,D_nv(s,y)]|,$$

where $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ are from $R^n$, $\mu,\nu \in R$ and $|x-y| = \\ = \big[\sum_{i=1}^{n}(x_i-y_i)^2\big]^{\frac{1}{2}}$.

The concept of a domain with a locally smooth boundary is given in the following definition.

**Definition 1.1.** Let $r \in (1,\infty)$ and $\Omega \subset R^n$ be a bounded domain. We say that *the boundary $\partial\Omega$ belongs to the class $C^r, r \geq 1$* if:

(i) There exists a tangential space to $\partial\Omega$ at any point from the boundary $\partial\Omega$.

(ii) Assume $y \in \partial\Omega$ and let $(y; z_1,\dots,z_n)$ be a local orthonormal coordinate system with the center $y$ and with the axis $z_n$ oriented like the inner normal to $\partial\Omega$ at the point $y$. Then there exists a number $b > 0$ such that for every

$y \in \partial\Omega$ there exists a neighbourhood $O(y) \subset R^n$ of the point $y$ and a function $F \in C^r(\mathrm{cl}\, B, R)$ such that the part of boundary

$$\partial\Omega \cap O(y) = \{(z', F(z')) \in R^n, z' = (z_1, \dots, z_{n-1}) \in B\},$$

where $B = \{z' \in R^{n-1}; |z'| < b\}$.

Here $C^r(\mathrm{cl}\, B, R)$ is a vector space of the functions $u \in C^l(\mathrm{cl}\, B, R)$ for $l = [r]$ with the finite norm

$$||u||_{l+\alpha} = \sum_{0 \le k \le l} \sup_{x \in \mathrm{cl}\, B} \left| D_x^k u(x) \right| + \sum_{k=l} \left\langle D_x^k u \right\rangle_{x,\alpha,B}^{y} ,$$

whereby $\alpha = r - [r] \in [0, 1)$ and $r = l + \alpha$.

Further, we shall need the following Hölder spaces — see [3, p. 147].

**Definition 1.2.** Let $\alpha \in (0, 1)$.

1. By the symbol $C_{t,x}^{(1+\alpha)/2, 1+\alpha}(\mathrm{cl}\, Q, R)$ we denote the vector space of continuous functions $u\colon \mathrm{cl}\, Q \to R$ which have continuous derivatives $D_i u$ for $i = 1, \dots, n$ on $\mathrm{cl}\, Q$ and the norm

$$||u||_{(1+\alpha)/2, 1+\alpha, Q} := \sum_{i=0}^{n} \sup_{(t,x) \in \mathrm{cl}\, Q} |D_i u(t,x)| + < u >_{t,(1+\alpha)/2,Q}^{s} + \\ + \sum_{i=1}^{n} \langle D_i u \rangle_{t,\alpha/2,Q}^{s} + \sum_{i=1}^{n} \langle D_i u \rangle_{x,\alpha/2,Q}^{y} \quad (1.6)$$

is finite.

2. The symbol $C_{(t,x)}^{(2+\alpha)/2, 2+\alpha}(\mathrm{cl}\, Q, R)$ means the vector space of continuous functions $u\colon$ cl $Q \to R$ for which there exist continuous derivatives $D_t u, D_i u, D_{ij} u$ on $\mathrm{cl}\, Q$, $i, j = 1, \dots n$ and the norm

$$||u||_{(2+\alpha)/2, 2+\alpha, Q} := \sum_{i=0}^{n} \sup_{(t,x) \in \mathrm{cl}\, Q} |D_i u(t,x)| + \sup_{(t,x) \in \mathrm{cl}\, Q} |D_t u(t,x)| + \\ + \sum_{i,j=1}^{n} \sup_{(t,x) \in \mathrm{cl}\, Q} |D_{ij} u(t,x)| + \sum_{i=1}^{n} < D_i u >_{t,(1+\alpha)/2,Q}^{s} + < D_t u >_{t,\alpha/2,Q}^{s} + \\ + \sum_{i,j=1}^{n} < D_{ij} u >_{t,\alpha/2,Q}^{s} + < D_t u >_{x,\alpha,Q}^{y} + \sum_{i,j=1}^{n} < D_{ij} u >_{x,\alpha,Q}^{y} \quad (1.7)$$

is finite.

3. The symbol $C_{t,x}^{(3+\alpha)/2, 3+\alpha}(\mathrm{cl}\, Q, R)$ means the vector space of continuous functions $u\colon \mathrm{cl}\, Q \to R$ for which the derivatives $D_t, D_i u, D_t D_i u, D_{ij} u, D_{ijk} u, i, j, k = 1, \dots, n$ are continuous on $\mathrm{cl}\, Q$ and the norm

$$
\begin{aligned}
||u||_{(3+\alpha)/2,3+\alpha,Q} := &\sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\, Q} |D_i u(t,x)| + \sum_{i,j=1}^{n} \sup_{(t,x)\in \mathrm{cl}\, Q} |D_{ij} u(t,x)| + \\
&+ \sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\, Q} |D_t D_i u(t,x)| + \sum_{i,j,k=1}^{n} \sup_{(t,x)\in \mathrm{cl}\, Q} |D_{ijk} u(t,x)| + \\
&+ \langle D_t u\rangle^{s}_{t,(1+\alpha)/2,Q} + \sum_{i,j=1}^{n} \langle D_{ij} u\rangle^{s}_{t,(1+\alpha)/2,Q} + \\
&+ \sum_{i=1}^{n} \langle D_t D_i u\rangle^{s}_{t,\alpha/2,Q} + \sum_{i,j,k=1}^{n} \langle D_{ijk} u\rangle^{s}_{t,\alpha/2,Q} + \\
&+ \sum_{i=1}^{n} \langle D_t D_i u\rangle^{y}_{x,\alpha,Q} + \sum_{i,j,k=1}^{n} \langle D_{ijk} u\rangle^{y}_{x,\alpha,Q}
\end{aligned} \tag{1.8}
$$

is finite.

The above defined norm spaces are Banach ones and we call them Hölder spaces.

**Definition 1.3.** (The smoothness condition $(S_3^{1+\alpha})$.) Let $\alpha \in (0,1)$. We say that the differential operator $A(t,x,D_x)$ from (1.1) and $B_3(t,x,D_x)$ from (1.2), respectively satisfies *the smoothness condition* $(S_3^{1+\alpha})$ if

(i) the coefficients $a_{ij}, a_i, a_0$ from (1.1) for $i,j = 1,\dots,n$ belong to the space $C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\, Q, R)$ and $\partial\Omega \in C^{3+\alpha}$ and

(ii) the coefficients $b_i$ from (1.2) for $i = 1,\dots,n$ belong to the space $C_{t,x}^{(2+\alpha)/2,2+\alpha}(\mathrm{cl}\, \Gamma, R)$.

**Definition 1.4.** (The complementary condition $(C)$.) If at least one of the coefficients $b_i$ for $i = 1,\dots,n$ of the differential operator $B_3(t,x,D_x)$ in (1.2) is not zero we say that $B_3(t,x,D_x)$ satisfies the *complementary condition* $(C)$.

In the following part we shall reformulate the problem (1.1), (1.2), (1.3) to the operator equation

$$F_3 u = A_3 u + N_3 u = g$$

using several assumptions from

**Definition 1.5.**

1. Fredholm conditions

$(A_3.1)$ Consider the operator $A_3\colon X_3 \to Y_3$, where

$$A_3 u = D_t u - A(t,x,D_x)u,\;\; u \in X_3$$

and the operators $A(t,x,D_x)$ and $B_3(t,x,D_x)$ satisfy the smoothness condition $(S_3^{1+\alpha})$ for $\alpha \in (0,1)$ and the complementary condition $(C)$. Here we consider the vector spaces

$$D(A_3) := \{u \in C_{t,x}^{(3+\alpha)/2,3+\alpha}(\mathrm{cl}\, Q, R);$$
$$B_3(t,x,D_x)u|_{\Gamma} = 0,\ u|_{t=0}(x) = 0 \qquad \text{for } x \in \mathrm{cl}\, \Omega\}$$

and

$$H(A_3) := \{v \in C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\, Q, R);\ B_3(t,x,D_x)v(t,x)|_{t=0,x\in\partial\Omega} = 0\}$$

and Banach subspaces of the given Hölder spaces

$$X_3 = \big(D(A_3), ||.||_{(3+\alpha)/2,3+\alpha,Q}\big)$$

and

$$Y_3 = \big(H(A_3), ||.||_{(1+\alpha)/2,1+\alpha,Q}\big)\,.$$

$(A_3.2)$ There is a second order linear homeomorphism $C_3\colon X_3 \to Y_3$ with

$$C_3 u = D_t u - C(t,x,D_x)u,\ u \in X_3\ ,$$

where

$$C(t,x,D_x)u = \sum_{i,j=1}^{n} c_{ij}(t,x)D_{ij}u + \sum_{i=1}^{n} c_i(t,x)D_i u + c_0(t,x)u$$

satisfying the smoothness condition $(S_3^{1+\alpha})$. The operator $C_3$ is not necessarily parabolic one.

2. Local Hölder and compatibility conditions.
Let $f := f(t,x,u_0,u_1,\ldots,u_n)\colon \mathrm{cl}\, Q \times R^{n+1} \to R$, $\alpha \in (0,1)$ and let $p,q,p_r$ for $r = 0,1,\ldots,n$ be nonnegative constants. Here, $D$ represents any compact subset of $(\mathrm{cl}\, Q)\times R^{n+1}$. For $f$ we need the following assumptions:
$(N_3.1)$ Let $f \in C^1(\mathrm{cl}\, Q \times R^{n+1}, R)$ and let the first derivatives $\partial f/\partial x_i$, $\partial f/\partial u_j$ be locally Hölder continuous on $\mathrm{cl}\, Q \times R^{n+1}$ such that

$$\left.\begin{array}{l} \langle \partial f/\partial x_i\rangle_{t,x,u}^{s,y,v} \\ \langle \partial f/\partial u_j\rangle_{t,x,u}^{s,y,v} \end{array}\right\} \quad\le\quad p|t-s|^{\alpha/2} + q|x-y|^{\alpha} + \sum_{r=0}^{n} p_r|u_r - v_r|$$

for $i = 1,\ldots,n$ and $j = 0,1,\ldots,n$ and any $D$.
$(N_3.2)$ Let $f \in C^3(\mathrm{cl}\, Q \times R^{n+1}, R)$ and let the local growth conditions for the third derivatives of $f$ hold on any $D$:

$$\left.\begin{array}{l} \big\langle \partial^3 f/\partial\tau\partial x_i\partial u_j\big\rangle_{t,x,u}^{t,x,v} \\ \big\langle \partial^3 f/\partial\tau\partial u_j\partial u_k\big\rangle_{t,x,u}^{t,x,v} \\ \big\langle \partial^3 f/\partial x_i\partial x_l\partial u_j\big\rangle_{t,x,u}^{t,x,v} \\ \big\langle \partial^3 f/\partial x_i\partial u_j\partial u_k\big\rangle_{t,x,u}^{t,x,v} \\ \big\langle \partial^3 f/\partial u_j\partial u_k\partial u_r\big\rangle_{t,x,u}^{t,x,v} \end{array}\right\} \quad\le\quad \sum_{s=0}^{n} p_s|u_s - v_s|^{\beta_s}$$

where $\beta_s > 0$ for $s = 0, 1, \dots, n$ and $i, l = 1, \dots, n;\ j, k, r = 0, 1, \dots, n$.
$(N_3.3)$ The equality of compatibility

$$\sum_{i=1}^{n} b_i(t,x) D_i f(t,x,0,\dots,0) + b_0(t,x) f(t,x,0,\dots,0)|_{t=0,\ x\in S} = 0$$

holds.

3. Almost coercive condition.
Let for any bounded set $M_3 \subset Y_3$ there exist a number $K > 0$ such that for all solutions $u \in X_3$ of the problem (1.1), (1.2), (1.3) with the right hand side $g \in M_3$, the following alternative holds:
$(F_3.1)$ Either
$(\alpha_3)$ $\|u\|_{(1+\alpha)/2, 1+\alpha, Q} \le K,\ f := f(t,x,u_0)\colon \operatorname{cl} Q \times R \to R$ and the coefficients of the operators $A_3$ and $C_3$ (see (1.1) and $(A_3.2)$) satisfy the equations

$$a_{ij} = c_{ij},\ a_i = c_i \quad \text{for} \quad i, j = 1, \dots, n,\ a_0 \ne c_0 \quad \text{on} \quad \operatorname{cl} Q$$

or
$(\beta_3)$ $\|u\|_{(2+\alpha)/2, 2+\alpha, Q} \le K,\ f := f(t,x,u_0,u_1,\dots,u_n)\colon \operatorname{cl} Q \times R^{n+1} \to R$ and the coefficients of the operators $A_3$ and $C_3$ satisfy the relations

$$a_{ij} = c_{ij} \quad \text{for } i, j = 1, \dots, n \quad \text{and } a_i \ne c_i \text{ for at least one } i = 1, \dots, n$$

on $\operatorname{cl} Q$.

*Remark 1.6.*

1. Especially, the condition $(A_3.2)$ is satisfied for the diffusion operator

$$C_3 u = D_t u - \triangle u,\ u \in X_3$$

or for any uniformly parabolic operator $C_3$ with sufficiently smooth coefficients. However the operator $C_3$ is not necessarily uniform parabolic.

2. The local Hölder conditions in $(N_3.1)$ and $(N_3.2)$ admit sufficiently strong growths of $f$ in the last variables $u_0, u_1, \dots, u_n$. For example, they include exponential and power type growths.

**Definition 1.7.**

1. A couple $(u, g) \in X_3 \times Y_3$ will be called *the bifurcation point of the mixed problem* (1.1), (1.2), (1.3) if $u$ is a solution of that mixed problem and there exists a sequence $\{g_k\} \subset Y_3$ such that $g_k \to g$ in $Y_3$ as $k \to \infty$ and the problem (1.1), (1.2), (1.3) for $g = g_k$ has at least two different solutions $u_k, v_k$ for each $k \in N$ and $u_k \to u, v_k \to u$ in $X_3$ as $k \to \infty$.
2. The set of all solutions $u \in X_3$ of (1.1), (1.2), (1.3) (or the set of all functions $g \in Y_3$) such that $(u, g)$ is a bifurcation point of the problem (1.1), (1.2), (1.3) will be called *the domain of bifurcation (the bifurcation range)* of that problem.

Under the previous hypotheses we have proved the fundamental lemas:

**Lemma 1.8.** *The following implications are true:*

(1) ($A_3.1$), ($A_3.2$) *imply that the operator* $A_3 \colon X_3 \to Y_3$ *is a linear bounded Fredholm operator of the zero index.*

(2) ($N_3.1$), ($N_3.2$) *imply that the Nemitskij operator* $N_3 \colon X_3 \to Y_3$ *defined by*

$$(N_3 u)(t,x) = f[t, x, u(t,x), D_1 u(t,x), \ldots, D_n u(t,x)]$$

*for* $u \in X_3$ *and* $(t,x) \in \operatorname{cl} Q$ *is completely continuous.*

(3) ($A_3.1$), ($A_3.2$), ($N_3.1$), ($N_3.3$), ($F_3.1$) *imply that the operator* $F_3 = A_3 + N_3 \colon X_3 \to Y_3$ *is coercive.*

(4) ($N_3.2$), ($N_3.3$) *imply that* $N_3 \in C^1(X_3, Y_3)$ *and is completely continuous.*

**Lemma 1.9.** *Let* $A_3 \colon X_3 \to Y_3$ *be the linear operator satysfying* ($A_3.1$), ($A_3.2$) *and let* $N_3 \colon X_3 \to Y_3$ *be the Nemitskij operator satysfying* ($N_3.1$), ($N_3.3$) *and* $F_3 = A_3 + N_3 \colon X_3 \to Y_3$. *Then:*

(i) *The function* $u \in X_3$ *is a solution of the initial-boundary value problem* (1.1), (1.2), (1.3) *for* $g \in Y_3$ *if and only if* $F_3 u = g$.

(ii) *The couple* $(u, g) \in X_3 \times Y_3$ *is the bifurcation point of the initial-boundary value problem* (1.1), (1.2), (1.3) *if and only if* $F_3(u) = g$ *and* $u \in \Sigma$, *where* $\Sigma$ *means the set of all points of* $X_3$ *at which* $F_3$ *is not locally invertible.*

## 2 Generic properties for continuous operators

Aplying

**Theorem (Ambrosetti).** *Let* $F \in C(X, Y)$ *be a proper mapping. Then the cardinal number card* $F^{-1}(\{q\})$ *of the set* $F^{-1}(\{q\})$ *is constant and finite (it may be zero) for each* $q$ *taken from the same (connected) component of the set* $Y - F(\Sigma)$. *Here* $\Sigma$ *means the set of all points* $u \in X$ *for which* $F$ *is not locally invertible.*

and

**Theorem (S. Smale and F. Quinn).** *If* $F \colon X \to Y$ *is a Fredholm mapping of class* $C^q$, $q > \max(\operatorname{ind} F, 0)$ *and either*

*$X$ has a countable basis (Smale)*

*or*

*$F$ is $\sigma$-proper (Quinn),*

*then the set* $R_F$ *of all regular values of* $F$ *is residual in* $Y$. *If* $F$ *is proper, then* $R_F$ *is open and dense in* $Y$.

we can prove the main results for the nonlinear problem (1.1), (1.2), (1.3). Here $X$ and $Y$ are Banach spaces either both real or complex.

**Theorem 2.1.** *Under the assumptions* ($A_3$.1), ($A_3$.2) *and* ($N_3$.1), ($N_3$.3) *the following statements hold for the problem* (1.1), (1.2), (1.3):

(a) *The operator* $F_3 = A_3 + N_3 \colon X_3 \to Y_3$ *is continuous.*

(b) *For any compact set of the right hand sides* $g \in Y_3$ *from* (1.1), *the corresponding set of all solutions is a countable union of compact sets.*

(c) *For* $u_0 \in X_3$ *there exists a neighbourhood* $U(u_0)$ *of* $u_0$ *and* $U(F_3(u_0))$ *of* $F_3(u_0) \in Y_3$ *such that for each* $g \in U(F_3(u_0))$ *there is a unique solution of* (1.1), (1.2), (1.3) *if and only if the operator* $F_3$ *is locally injective at* $u_0$.

*Moreover, if* ($F_3$.1) *is assumed, then:*

(d) *For each compact set of* $Y_3$ *the corresponding set of all solutions is compact (possibly empty).*

**Theorem 2.2.** *If the hypotheses* ($A_3$.1), ($A_3$.2), ($N_3$.1), ($N_3$.3) *and* ($F_3$.1) *are satisfied, then for the initial-boundary value problem* (1.1), (1.2), (1.3) *the following statements hold:*

(e) *For each* $g \in Y_3$ *the set* $S_{3g}$ *of all solutions is compact (possibly empty).*

(f) *The set* $R(F_3) = \{g \in Y_3$; *there exists at least one solution of the given problem* $\}$ *is closed and connected in* $Y_3$.

(g) *The domain of bifurcation* $D_{3b}$ *is closed in* $X_3$ *and the bifurcation range* $R_{3b}$ *is closed in* $Y_3$. $F_3(X_3 - D_{3b})$ *is open in* $Y_3$.

(h) *If* $Y_3 - R_{3b} \neq \emptyset$, *then each component of* $Y_3 - R_{3b}$ *is a nonempty open set (i.e. a domain).*
*The number* $n_{3g}$ *of solutions is finite, constant (it may be zero) on each component of the set* $Y_3 - R_{3b}$, *i.e. for every* $g$ *belonging to the same component of* $Y_3 - R_{3b}$.

(i) *If* $R_{3b} = \emptyset$, *then the given problem has a unique solution* $u \in X_3$ *for each* $g \in Y_3$ *and this solution continuously depends on* $g$ *as a mapping from* $Y_3$ *onto* $X_3$.

(j) *If* $R_{3b} \neq \emptyset$, *then the boundary of the* $F_3$ *- image of the set of all points from* $X_3$ *in which the operator* $F_3$ *is locally invertible, is a subset of the* $F_3$ *- image of all points from* $X_3$ *in which* $F_3$ *is not locally invertible, i.e.*

$$\partial F_3(X_3 - D_{3b}) \subset F_3(D_{3b}) = R_{3b}$$

## 3 Generic properties for $C^1$-differentiable operator

In case the Nemitskij operator $N_3 \in C^1(X, Y)$, we get stronger results. Using the theorem on a local $C^1$-diffeomorphism

**Theorem (E. Zeidler).** *Let* $F \colon (U(u_0) \subset X) \to Y$ *be a* $C^1$*-mapping. Then* $F$ *is a local* $C^1$*-diffeomorphism at* $u_0$ *if and only if* $u_0$ *is a regular point of* $F$.

and

**Theorem (R. S. Sadyrchanov).** *Let* $\dim Y \geq 3$ *and let* $F\colon X \to Y$ *be a Fredholm mapping of the zero index. If* $u_0$ *is an isolated singular point of* $F$*, then the mapping* $F$ *is localy invertibly at* $u_0$*.*

we obtain main results for $C^1$-differentiable operators.

**Theorem 3.1.** *Assume that the hypotheses* ($A_3$.1), ($A_3$.2), ($N_3$.2), ($N_3$.3) *hold.*
*Then the open set* $Y_3 - R_{3b}$ *is dense in* $Y_3$ *and thus the range of bifurcation* $R_{3b}$ *of initial-boundary value problem* (1.1), (1.2), (1.3) *is nowhere dense in* $Y_3$*.*

Also we shall investigate the linear problem in $h \in X_3$ for some $u \in X_3$:

$$A_3 h(t,x) + \sum_{j=0}^{n} \frac{\partial f}{\partial u_j}\left[t, x, u(t,x), D_1 u(t,x), \ldots, D_n u(t,x)\right] D_j h(t,x) = g(t,x) \quad (3.1)$$

with the conditions (1.2), (1.3).

**Theorem 3.2.** *Assume that the hypotheses* ($A_3$.1), ($A_3$.2), ($N_3$.2), ($N_3$.3) *and* ($F_3$.1) *hold. Then*

(a) *The number of solutions of* (1.1), (1.2), (1.3) *is constant and finite (it may be zero) on each connected component of the open set* $Y_3 - F(S_3)$*, i.e. for any* $g$ *belonging to the same connected component of* $Y_3 - F_3(S_3)$*. Here* $S_3$ *means the set of all critical points of problem* (1.1), (1.2), (1.3)*.*

(b) *Let* $u_0 \in X_3$ *be a regular solution of* (1.1), (1.2), (1.3) *with the right hand side* $g_0 \in Y_3$*. Then there exists a neighbourhood* $U(g_0) \subset Y_3$ *of* $g_0$ *such that for any* $g \in U(g_0)$ *the initial-boundary value problem* (1.1), (1.2), (1.3) *has one and only one solution* $u \in X_3$*. This solution continuously depends on* $g$*.*
*The associated linear problem* (3.1), (1.2), (1.3) *for* $u = u_0$ *has a unique solution* $h \in X_3$ *for any* $g$ *from a neighbourhood* $U(g_0)$ *of* $g_0 = F_3(u_0)$*. This solution continuously depends on* $g$*.*

(c) *Denote by* $G_3$ *the set of all right hand sides* $g \in Y_3$ *of equation* (1.1) *for which the corresponding solutions* $u \in X_3$ *of the problem* (1.1), (1.2), (1.3) *are its critical solutions. Then* $G_3$ *is closed and nowhere dense in* $Y_3$*.*

(d) *If the singular points set of the initial-boundary value problem* (1.1), (1.2), (1.3) *is empty, then this problem has unique solution* $u \in X_3$ *for each* $g \in Y_3$*. It continuously depends of the right hand side* $g$*.*

**Corollary 3.3.** *Let the hypotheses of Theorem* 3.2 *hold and*

(i) *the linear homogeneous problem* (3.1), (1.2), (1.3) *(for* $g = 0$*) has only zero solution* $h = 0 \in X_3$ *for any* $u \in X_3$*.*

*Then the initial-boundary value nonlinear problem* (1.1), (1.2), (1.3) *has a unique solution* $u \in X_3$ *for any* $g \in Y_3$*. This solution* $u$ *continuously depends on* $g$*. Moreover linear problem* (3.1), (1.2), (1.3) *has a unique solution* $h \in X_3$ *for any* $u \in X_3$ *and for each right hand side* $g \in Y_3$ *of* (3.1) *and this solution continuously depends on* $g$*.*

**Theorem 3.4.** *Suppose that the hypotheses* ($A_3.1$), ($A_3.2$), ($N_3.2$), ($N_3.3$) *and* ($F_3.1$) *hold together with the condition*

(i) *Each point* $u \in X_3$ *is either a regular point or an isolated critical point of problem* (1.1), (1.2), (1.3).

*Then to each* $g \in Y_3$ *there exists one and only one solution* $u \in X_3$ *of the problem* (1.1), (1.2), (1.3) *and it continuously depends on* $g$.

## References


1. A. Ambrosetti, *Global inversion theorems and applications to nonlinear problems.* Conferenze del Seminario di Matematica dell' Università di Bari, Atti del 3° Seminario di Analisi Funzionale ed Applicazioni, in A Survey on the Theoretical and Numerical Trends in Nonlinear Analysis, Gius. Laterza et Figli, Bari 1976 pp. 211 to 232.
2. L. Brüll and J. Mawhin, *Finiteness of the set of solutions of some boundary value problems for ordinary differential equations.* Archivum Mathematicum (Brno) 24, 1988, pp. 163–172.
3. V. Ďurikovič, *An initial-boundary value problem for quasi–linear parabolic systems of higher order.* Ann. Polon. Math. XXX, 1974, pp. 145–164.
4. V. Ďurikovič and Ma. Ďurikovičová, *Some generic properties of nonlinear second order diffusional type problem.* Archivum Mathematicum (Brno) 35, 1999, pp. 229 to 244.
5. J. Mawhin, *Generic properties of nonlinear boundary value problems.* Differential Equations and Mathematical Physics, Academic Press Inc., New York 1992, pp. 217 to 234.
6. F. Quinn, *Transversal approximation on Banach manifolds.* In Proc. Sympos. Pure Math. (Global Analysis) 15 (1970), pp. 213–223.
7. R. S. Sadyrchanov, *Selected Questions of Nonlinear Functional Analysis.* Publishers ELM, Baku 1989 (in Russian).
8. S. Smale, *An infinite dimensional version of Sard's theorem.* Amer. J. Math. 87, 1965, pp. 861–866.
9. V. Šeda, *Fredholm mappings and the generalized boundary value problem.* Differential and Integral Equation 8, Nr.1, 1995, pp. 19–40.
10. E. Zeidler, *Nonlinear Functional Analysis and its Application I, Fixed-Point Theorems.* Springer-Verlag, Berlin, Heidelberg, Tokyo 1986.