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An existence criterion of positive solutions of *p*-type retarded functional differential equations

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Abstract. A criterion and conditions for existence of positive solutions of p-type retarded functional differential equations are presented.

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The function $p \in C[\mathbb{R} \times [-1,0], \mathbb{R}]$ is called a p-*function* if it has the following properties: $p(t,0) = t$, $p(t,-1)$ is a nondecreasing function of t and there exists a $\sigma \geq -\infty$ such that $p(t,\vartheta)$ is an increasing function in ϑ for each $t \in (\sigma, \infty)$ (see [2]). For $t \in [t_0, t_0 + A)$ with $A > 0$ we define $y_t(\vartheta) = y(p(t,\vartheta))$, $-1 \leq \vartheta \leq 0$. Consider the system

$$\dot{y}(t) = f(t, y_t) \tag{1}$$

where $f \in C[[t_0, t_0 + A) \times \mathcal{C}, \mathbb{R}^n]$ with $\mathcal{C} = [[-1,0], \mathbb{R}^n]$. This system is called the system of p-*type retarded functional differential equations* ([2]).
We say that the functional $g \in C(\Omega, \mathbb{R})$ is *strongly decreasing* (*increasing*) with respect to the second argument on $\Omega \subset \mathbb{R} \times \mathcal{C}$ if for each $(t,\varphi), (t,\psi) \in \Omega$ with $\varphi(p(t,\vartheta)) \ll \psi(p(t,\vartheta))$, $\vartheta \in [-1,0)$: $g(t,\varphi) - g(t,\psi) > 0$ (< 0). Let $k \gg 0$ and μ

This is the preliminary version of the paper.

be constant vectors, $\mu_i = -1$, $i = 1, \ldots, p$ and $\mu_i = 1$, $i = p+1, \ldots, n$. Let $\lambda(t)$ denote a real vector with continuous entries on $[p^*, \infty)$, $p^* = p(t^*, -1)$. Put

$$T(k, \lambda)(t) \equiv k\mathrm{e}^{\mu \int_{p^*}^{t} \lambda(s)ds} = \left(k_1 \mathrm{e}^{\mu_1 \int_{p^*}^{t} \lambda_1(s)ds}, \ldots, k_n \mathrm{e}^{\mu_n \int_{p^*}^{t} \lambda_n(s)ds}\right).$$

Theorem 1. *Suppose $\Omega = [t^*, \infty) \times \mathcal{C}$, $f \in C(\Omega, \mathbb{R}^n)$ is locally Lipschitzian with respect to the second argument and, moreover:*

(i) $f(t, 0) \equiv 0$ *if* $t \geq t^*$.

(ii) *The functional f_i is strongly decreasing if $i = 1, \ldots, p$ and strongly increasing if $i = p+1, \ldots, n$ with respect to the second argument on Ω.*

Then for the existence of a positive solution $y = y(t)$ on $[p^*, \infty)$ of the system (1) a necessary and sufficient condition is that there exists a vector $\lambda \in C([p^*, \infty), \mathbb{R}^n)$, such that $\lambda \gg 0$ on $[t^*, \infty)$, satisfying the system of integral inequalities

$$\lambda_i(t) \geq \frac{\mu_i}{k_i} \mathrm{e}^{-\mu_i \int_{p^*}^{t} \lambda_i(s)ds} \cdot f_i\left(t, T(k, \lambda)_t\right), \quad i = 1, \ldots, n$$

for $t \geq t^*$, with a positive constant vector k.

Consider the equation

$$\dot{y}(t) = -\int_{\tau(t)}^{t} K(t, s) y(s) ds, \tag{2}$$

where $K : [t^*, \infty) \times [p^*, \infty) \to \mathbb{R}^+$ is a continuous function, and $\tau : [t^*, \infty) \to [p^*, \infty)$ is a nondecreasing function with $\tau(t) < t$.

Theorem 2. *The equation (2) has a positive solution $y = y(t)$ on $[p^*, \infty)$ if and only if there exists a function $\lambda \in C([p^*, \infty), \mathbb{R})$, such that $\lambda(t) > 0$ for $t \geq t^*$ and*

$$\lambda(t) \geq \int_{\tau(t)}^{t} K(t, s) \mathrm{e}^{\int_s^t \lambda(u)du}\, ds$$

on the interval $[t^*, \infty)$.

Let us consider a partial case of Eq. (2) when $\tau(t) \equiv t - l$, $l \in \mathbb{R}^+$ and $K(t, s) \equiv c(t)$ for $t \in [t^*, \infty)$. Then Eq. (2) takes the form

$$\dot{y}(t) = -c(t) \int_{t-l}^{t} y(s)\, ds. \tag{3}$$

Theorem 3. *For the existence of a solution of Eq. (3), positive on $[t^* - l, \infty)$, the inequality*

$$c(t) \leq M, \quad t \in [t^*, \infty)$$

is sufficient for $M = \alpha(2-\alpha)/l^2 = \mathrm{const}$ with a constant α being the positive root of the equation $2 - \alpha = 2\mathrm{e}^{-\alpha}$. (The approximate values are $\alpha \doteq 1{,}5936$ and $M \doteq 0{,}6476/l^2$.)

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