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Positive and oscillating solutions of equation $\dot{x}(t) = -c(t)x(t-\tau)$

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Abstract. Positive and oscillating solutions of delayed equation

$$\dot{x}(t) = -c(t)x(t-\tau)$$

with $c \in C(I, \mathbb{R}^+)$, $I = [t_0, \infty)$, $\mathbb{R}^+ = (0, \infty)$ and $0 < \tau = \text{const}$ are studied.

MSC 2000. 34K15, 34K25

Keywords. Linear differential equations with delay, positive solution, oscillating solution

Let us consider the equation

$$\dot{x}(t) = -c(t)x(t-\tau) \tag{1}$$

where $c \in C(I, \mathbb{R}^+)$, $I = [t_0, \infty)$, $\mathbb{R}^+ = (0, \infty)$ and $0 < \tau = \text{const}$.

Define $\ln_k t = \ln(\ln_{k-1} t)$, $k \geq 1$ where $\ln_0 t \equiv t$ for $t > \exp_{k-2} 1$ where $\exp_k t \equiv (\exp(\exp_{k-1} t))$, $k \geq 1$, $\exp_0 t \equiv t$ and $\exp_{-1} t \equiv 0$. (Instead of expressions $\ln_0 t, \ln_1 t$ is only t and $\ln t$ written in the sequel.) Moreover, define so called critical functions for (1)

$$c_k(t) \equiv \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t \ldots \ln_k t)^2}$$

with $k \geq 0$.

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This is the preliminary version of the paper.

Theorem 1. [1]
A) *Let us assume that* $c(t) \le c_k(t)$ *for* $t \to \infty$ *and an integer* $k \ge 0$. *Then there is a positive solution* $x = x(t)$ *of Eq.* (1). *Moreover,*

$$x(t) < \nu_k(t) \equiv e^{-t/\tau} \sqrt{t \ln t \ln_2 t \dots \ln_k t}$$

as $t \to \infty$.
B) *Let us assume that*

$$c(t) > c_{k-1}(t) + \frac{\theta\tau}{8e(t \ln t \ln_2 t \dots \ln_k t)^2} \tag{2}$$

for $t \to \infty$, *an integer* $k \ge 1$ *and a constant* $\theta > 1$. *Then all solutions of Eq.* (1) *oscillate.*

Theorem 2. [1] *Assume that the inequality* (2) *holds for* $t \to \infty$, *an integer* $k \ge 1$ *and a constant* $\theta > 1$. *Then each solution of Eq.* (1) *has at least one zero on each interval* $(p - \tau, q)$ *for* $q = \exp_{k-2}(\ln_{k-2} p)^{\exp(\pi/\zeta)}, \zeta^2 < (\theta - 1)/4$, ($\zeta$ *is a positive constant*) *and* p *sufficiently large.*

Theorem 3. [2] *Let there exists a positive solution* $\tilde{x}$ *of* (1) *on* I. *Then there are positive solutions* x_1 *and* x_2 *of* (1) *on* I *satisfying the relation*

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = 0. \tag{3}$$

Moreover, every solution x *of* (1) *on* I *is represented by the formula*

$$x(t) = Kx_1(t) + O(x_2(t))$$

where $K \in \mathbb{R}$ *depends on* x.

Definition 4. [3] Let x_1 and x_2 be fixed positive solutions of the delayed equation (1) on I, with the property (3). Then (x_1, x_2) is called a *pair of dominant and subdominant* solutions on I.

Let us consider the equation (1) in the case when the coefficient c is equal to a critical function, i.e., in the case of equation

$$\dot{x}(t) = -c_k(t)x(t - \tau), \qquad k \ge 0;\ t \ge t_0 > \exp_{p-1} 1. \tag{4}$$

Theorem 5. [3] *Let* $k \ge 0$ *be fixed. Then for any fixed constants* $\delta_1 > 2$ *and* $\delta_2 < 0$ *there are a* t_0, *and a pair* (x_1, x_2) *of dominant and subdominant solutions of* (4) *on* I *satisfying the two-sided estimates*

$$e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1}^2 t} < x_1(t) < e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1}^{\delta_1} t}$$

and

$$e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1}^{\delta_2} t} < x_2(t) < e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t}$$

on I.

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