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Attractors of nonautonomous and stochastic differential inclusions

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Abstract. In this paper we study the existence of pullback global attractors for multivalued processes generated by differential inclusions. First, we define multivalued dynamical processes, prove abstract results on the existence of global attractors and study their topological properties (compactness, conectedness). Further, we apply the abstract results to nonautonomous differential inclusions of the reaction-diffusion type in which the forcing term can grow polynomially in time, and to stochastic differential inclusions as well.

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1 Introduction

In this paper we study the existence of pullback global attractors for multivalued processes generated by differential inclusions. The theory of pullback attractors has been developed for stochastic and nonautonomous systems in which the trajectories can be unbounded when times rises to infinite. In such systems the classical

This is the preliminary version of the paper.

theory of global attractors is not applicable. Hence, a different approach has been considered [4,5,6,9].

A new difficulty appears if the solution corresponding to each initial state can be non-unique. The classical results on attractors in the autonomous and nonautonomous cases are generalized to the multivalued case in [7] and [8], respectively, with applications to evolution inclusions.

In [1,2,3] the study of multivalued dynamical systems is extended to the stochastic case, generalizing in this way the results of [4,5].

In this paper we are mainly concerned with nonautonomous multivalued dynamical systems in which the trajectories can be unbounded in time and also with nonautonomous stochastic multivalued dynamical systems.

In the second section we define multivalued dynamical processes, prove abstract results on the existence global attractors and study their topological properties (compactness, conectedness). In the third section we apply the abstract results to nonautonomous differential inclusions of the reaction-diffusion type in which the forcing term can grow polynomially in time. In the fourth section we give applications to stochastic differential inclsuions with additive and multiplicative noises.

2 Attractors for multivalued processes

In this section we shall define multivalued dynamical processes in metric spaces. Maps of this kind appear in differential equations for which, although we are able to prove the existence of at least one global solution for each inital condition in some phase space, we do not know if it is unique or not. Hence, multivalued processes generalize the concept of processes, for which the uniqueness property holds.

Let X be a complete metric space with the metric denoted by ρ and let $P(X)$ $(\mathcal{B}(X), C_v(X))$ be the set of all non-empty (non-empty bounded, non-empty bounded closed and convex) subsets of X. Let us denote $\mathbb{R}_d = \left\{(t,s) \in \mathbb{R}^2 : t \geq s\right\}$, $dist\,(A,B) = \sup_{x\in A}\inf_{y\in B} \rho\,(x,y)$, $dist_H(A,B) = \max\{dist(A,B), dist(B,A)\}$, for any $A, B \subset X$.

Definition 1. The map $U : \mathbb{R}_d \times X \to P(X)$ is called a multivalued dynamical process (MDP) on X if:

1. $U\,(t,t,\cdot) = I$ is the identity map;
2. $U\,(t,s,x) \subset U\,(t,\tau,U\,(\tau,s,x))$, for all $x \in X, s \leq \tau \leq t$.

The MDP U is called strict if:

$$U\,(t,s,x) = U\,(t,\tau,U\,(\tau,s,x))\,, \textit{for all } x \in X, s \leq \tau \leq t.$$

Consider a parameter set $\Sigma = \Sigma_1 \times \Sigma_2$. If $\{U_\sigma : \sigma \in \Sigma\}$ is an arbitrary family of MDP, then for any $\sigma_2 \in \Sigma_2$ the map $U_{\Sigma_1,\sigma_2} : \mathbb{R}_d \times X \to P(X)$ defined by

$$U_{\Sigma_1,\sigma_2}\,(t,s,x) = \bigcup_{\sigma_1 \in \Sigma_1} U_{\sigma_1,\sigma_2}\,(t,s,x)\,.$$

is a MDP.

Suppose that we are given a one-parameter group $T(h) : \Sigma \to \Sigma$, where $\Sigma = \Sigma_1 \times \Sigma_2$, $h \in \mathbb{R}$ and $T(h) = (T_1(h), T_2(h)), T_i(h) : \Sigma_i \to \Sigma_i$, $i = 1, 2$. This is called the shift operator.

In the sequel we shall assume:

$(T1)$ For any $(t, s) \in \mathbb{R}_d$, $x \in X$, $\sigma \in \Sigma$, $h \in \mathbb{R}$ the following inclusion holds:

$$U_{\sigma_1,\sigma_2}(t, s, x) \subset U_{T_1(h)\sigma_1, T_2(h)\sigma_2}(t - h, s - h, x).$$

Lemma 2. *Condition* $(T1)$ *implies*

$$U_{\sigma_1,\sigma_2}(t, s, x) = U_{T_1(h)\sigma_1, T_2(h)\sigma_2}(t - h, s - h, x).$$

Lemma 3. $T(h)\Sigma = \Sigma$, *for all* $h \in \mathbb{R}$.

Definition 4. Let $(T1)$ hold. Then the family of sets $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2 \in \not\prec_2}$ is called a Σ_1-uniform global attractor of the MDP $\{U_\sigma\}$ if:

1. $\Theta_{\Sigma_1}(\sigma_2)$ is Σ_1-uniformly attracting at time 0 for any $\sigma_2 \in \Sigma_2$, that is,

$$\lim_{s \to -\infty} dist\left(U_{\Sigma_1,\sigma_2}(0, s, B), \Theta_{\Sigma_1}(\sigma_2)\right) = 0, \text{ for any } B \in \mathcal{B}(X). \tag{1}$$

2. It is semi-invariant, that is,

$$\Theta_{\Sigma_1}(T_2(t)\sigma_2) \subset U_{\Sigma_1,\sigma_2}(t, s, \Theta_{\Sigma_1}(T_2(s)\sigma_2)), \text{ for any } (t, s) \in \mathbb{R}_d, \sigma_2 \in \Sigma_2.$$

3. It is minimal, that is, for any $\sigma_2 \in \Sigma_2$ and any closed Σ_1-uniformly attracting set $Y(\sigma_2)$ at time 0, we have

$$\Theta_{\Sigma_1}(\sigma_2) \subset Y(\sigma_2).$$

Theorem 5. *Let X be a complete metric space in which every compact set is nowhere dense, $(T1)$ hold and let for any $B \in \mathcal{B}(X)$, $\sigma_2 \in \Sigma_2$ there exist a compact set $D(\sigma_2, B) \subset X$ such that*

$$\lim_{s \to -\infty} dist\left(U_{\Sigma_1,\sigma_2}(0, s, B), D(\sigma_2, B)\right) = 0. \tag{2}$$

Then the following statements hold:

1. *If for all $\tau \leq 0$ and $\sigma_2 \in \Sigma_2$ the graph of the map $x \mapsto U_{\Sigma_1,\sigma_2}(0, \tau, x) \in P(X)$ is closed, then there exists the Σ_1-uniform global attractor $\{\Theta_{\Sigma_1}(\sigma_2)\}$. Moreover,*

$$\Theta_{\Sigma_1}(\sigma_2) = \bigcup_{B \in \mathcal{B}(X)} \omega_{\Sigma_1}(0, \sigma_2, B) \neq X,$$

where $\omega_{\Sigma_1}(t, \sigma_2, B) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U_{\Sigma_1,\sigma_2}(t, \tau, B)}$, and further for each $\sigma_2 \in \Sigma_2$, $\Theta_{\Sigma_1}(\sigma_2)$ is a Lindelöf, normal space. It is locally compact in some topology $\tau_\oplus$, which is stronger than the topology induced by X in $\Theta_{\Sigma_1}(\sigma_2)$.

2. If, in addition, Σ_1 is a compact metric space, the map

$$\Sigma_1 \times X \ni (\sigma_1, x) \longmapsto U_{\sigma_1,\sigma_2}(0,\tau,x) \in P(X)$$

is upper semicontinuous for any $\tau \leq 0$, $\sigma_2 \in \Sigma_2$, U_σ has connected values for any $\sigma \in \Sigma$, $(0,\tau) \in \mathbb{R}_d$, $x \in X$, Σ_1 is a connected space and

$$\Theta_{\Sigma_1}(T_2(\tau)\sigma_2) \subset B_1(\sigma_2), \text{ for all } \tau \leq 0,$$

where $B_1(\sigma_2)$ is a bounded connected set for any $\sigma_2 \in \Sigma_2$, then the set $\Theta_{\Sigma_1}(\sigma_2)$ is connected for any $\sigma_2 \in \Sigma_2$.

Theorem 6. *Let us suppose that for all $(0,s) \in \mathbb{R}_d$ and $\sigma_2 \in \Sigma_2$ the graph of the map $x \mapsto U_{\Sigma_1,\sigma_2}(0,s,x) \in P(X)$ is closed. If, moreover, for any $\sigma_2 \in \Sigma_2$ there exists a compact set $D(\sigma_2)$, which is Σ_1-uniformly attracting at time 0, then the set*

$$\Theta_{\Sigma_1}(\sigma_2) = \overline{\bigcup_{B \in \mathcal{B}(X)} \omega_{\Sigma_1}(0,\sigma_2,B)}$$

is the Σ_1-uniform global attractor of U_σ. Moreover, the sets $\Theta_{\Sigma_1}(\sigma_2)$ are compact and, if the conditions of the second statement in Theorem 5 hold, then they are connected.

Proposition 7. *Let the MDP U_σ be strict, Σ_1 be a compact metric space and let the map*

$$\Sigma_1 \times X \ni (\sigma_1, x) \longmapsto U_{\sigma_1,\sigma_2}(0,\tau,x) \in P(X)$$

be lower semicontinuous. Then the global attractors obtained in Theorems 5 and 6 are invariant, that is, $\Theta_{\Sigma_1}(T_2(t)\sigma_2) = U_{\Sigma_1,\sigma_2}(t,\tau,\Theta_{\Sigma_1}(T_2(\tau)\sigma_2))$, for all $\tau \leq t$, $\sigma_2 \in \Sigma_2$.

3 Applications to nonautonomous evolution inclusions

Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary $\partial\Omega$. Consider the parabolic inclusion

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) \in f_1(t,u) + f_2(t,u) + g_1(t) + g_2(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times (\tau,T), \\ u\,|_{\partial\Omega} = 0, \\ u\,|_{t=\tau} = u_\tau, \end{cases} \tag{3}$$

where $\tau \in \mathbb{R}$, $p \geq 2$, $f_i : \mathbb{R} \times \mathbb{R} \to C_v(\mathbb{R})$, $i = 1,2$, $g_1 \in L_\infty(\mathbb{R}, L_2(\Omega))$, $g_2 \in L_2^{loc}(\mathbb{R}, L_2(\Omega))$ and the following conditions hold:

(F1) There exists $C \geq 0$ such that

$$dist_H(f_1(t,u), f_1(t,v)) \leq C|u-v|, \text{ for all } t \in \mathbb{R}, u,v \in \mathbb{R}.$$

(F2) For any $t, s \in \mathbb{R}$ and $u \in \mathbb{R}$, it holds

$$dist_H\left(f_1(t,u), f_1(s,u)\right) \leq l\left(|u|\right)\alpha\left(|t-s|\right),$$

where α is a continuous function such that $\alpha(t) \to 0$, as $t \to 0^+$, and l is a continuous nondecreasing function. Moreover, there exist $K_1, K_2 \geq 0$ such that

$$|l(u)| \leq K_1 + K_2|u|, \text{ for all } u \in \mathbb{R}.$$

(F3) There exist $D \in \mathbb{R}_+$, $v_0 \in \mathbb{R}$ for which

$$|f_1(t, v_0)|_+ \leq D, \text{ for all } t \in \mathbb{R},$$

where $|f_1(t,v_0)|_+ = \sup_{\zeta \in f_1(t,v_0)} |\zeta|$.

(F4) There exist $\alpha_1(t), \alpha_2(t) \geq 0,\ \alpha_1(\cdot), \alpha_2(\cdot) \in L_2^{loc}(-\infty,\infty)$, such that

$$\sup_{y \in f_2(t,u)} |y| \leq \alpha_1(t) + \alpha_2(t)|u|, \text{ for all } u, t \in \mathbb{R}.$$

(F5) For each $t \in \mathbb{R}$, the map $f_2(t,\cdot)$ is upper semicontinuous.

(F6) For each $s \in \mathbb{R}$, the map $f_2(\cdot, s)$ is measurable.

(F7) If $p = 2$, there exist $\epsilon > 0$ and $M \geq 0$ such that

$$yu \leq (\lambda_1 - \epsilon)u^2 + M, \text{ for all } u \in \mathbb{R}, t \in \mathbb{R}, y \in f_1(t,u) + f_2(t,u),$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

(F8) There exist $R_1, R_2, R_3 > 0$ such that

$$\|g_2(t)\|_{L_2(\Omega)} \leq R_1 + R_2|t|^{R_3}, \text{ for a.a. } t \in \mathbb{R}.$$

(F9) If $p > 2$, there exist $R_4, R_5, R_6 > 0$ such that

$$|\alpha_i(t)| \leq R_4 + R_5|t|^{R_6}, \text{ for a.a. } t \in \mathbb{R},\ i = 1, 2.$$

First let us construct the sets Σ_1, Σ_2.

Denote by W the space $C_v(\mathbb{R})$ endowed with the Hausdorff metric $\rho(x,y) = dist_H(x,y)$. The space W is complete. For any $\psi \in W$ let $|\psi|_+ = \max_{y\in\psi}|y|$. Define also the space

$$\mathcal{M} = \left\{\psi \in C(\mathbb{R}, W) : |\psi(v)|_+ \leq D_1 + D_2|v|\right\},$$

where the constants D_1, D_2 are such that $|y| \leq D_1 + D_2|u|$, for all $u \in \mathbb{R}$, $t \in \mathbb{R}$, $y \in f_1(t,u)$

The hull of (f_1, g_1) will be denoted by $\Sigma_1 = \mathcal{H}(f_1) \times \mathcal{H}(g_1)$, where $\mathcal{H}(f_1) = cl_{C(\mathbb{R},\mathcal{M})}\{f_1(\cdot + h) : h \in \mathbb{R}\}$, $\mathcal{H}(g_1) = cl_{L_{2,w}^{loc}(\mathbb{R},L_2(\Omega))}\{g_1(\cdot+h) : h \in \mathbb{R}\}$. The set Σ_1 is compact in the space $C(\mathbb{R},\mathcal{M}) \times L_{2,w}^{loc}(\mathbb{R}, L_2(\Omega))$, where $L_{2,w}^{loc}(\mathbb{R}, L_2(\Omega))$

is the space $L_2^{loc}(\mathbb{R}, L_2(\Omega))$ endowed with the weak topology. Then the set Σ_1 is a compact metric space and $T_1(h)\Sigma_1 = \Sigma_1$, for all $h \in \mathbb{R}$, where $T_1(h)\sigma_1(t) = \sigma_1(t+h)$.

For the set Σ_2 we put

$$\Sigma_2 = \bigcup_{h\in\mathbb{R}} (f_2(\cdot + h), g_2(\cdot + h)).$$

It is clear that $T_2(h)\Sigma_2 = \Sigma_2$, for all $h \in \mathbb{R}$, where $T_2(h)\sigma_2(t) = \sigma_2(t+h)$.

Now let $X = L_2(\Omega)$ with the norm $\|\cdot\|_X$ and the scalar product $(\cdot,\cdot)$. Consider the abstract evolution inclusion

$$\begin{cases} \frac{du}{dt}(t) \in A(u(t)) + F_\sigma(t, u(t)), \ t \in [\tau, \infty), \\ u(\tau) = u_\tau, \end{cases} \tag{4}$$

where $\sigma = (\sigma_1, \sigma_2) \in \Sigma$ and $A : D(A) \subset X \to 2^X$, $F_\sigma : \mathbb{R} \times X \to 2^X$, are multivalued maps defined as follows:

$$A(u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right), D(A) = \left\{u \in W_0^{1,p}(\Omega) : A(u) \in L_2(\Omega)\right\},$$

$$F_{\sigma_1}(t,u) = \{y \in X : y(x) \in f_{\sigma_1}(t, u(x)) + g_{\sigma_1}(t), \text{ a.e. on } \Omega\},$$

$$F_{\sigma_2}(t,u) = \{y \in X : y(x) \in f_{\sigma_2}(t, u(x)) + g_{\sigma_2}(t), \text{ a.e. on } \Omega\}$$

$$F_\sigma(t,u) = F_{\sigma_1}(t,u) + F_{\sigma_2}(t,u).$$

The operators A and F_σ satisfy the following properties:

(A1) The operator A is m-dissipative, i.e.

$$(\xi_1 - \xi_2, y_1 - y_2) \leq 0, \text{ for any } y_1, y_2 \in D(A), \xi_i \in A(y_i), i = 1, 2,$$

and $Im(A - \lambda I) = X$, for all $\lambda > 0$.

(A2) $\overline{D(A)} = L_2(\Omega)$ and A generates a compact semigroup S.

(S1) $F_\sigma : \mathbb{R} \times X \to C_v(X)$, *for all* $\sigma \in \Sigma$.

(S2) *For any fixed* $t \in \mathbb{R}$ *and* $\sigma \in \Sigma$ *the map* $u \longmapsto F_\sigma(t,u)$ *is w-upper semicontinuous*, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u - v\|_X < \delta$, then

$$dist(F_\sigma(t,u), F_\sigma(t,v)) < \varepsilon.$$

(S3) *For any* $\sigma \in \Sigma$ *there exist* $\beta_1, \beta_2 \geq 0$, $\beta_1, \beta_2 \in L_2^{loc}(-\infty, \infty)$ *(depending on* σ_2 *but not on* σ_1*), such that*

$$\|F_\sigma(t,u)\|_+ \leq \beta_1(t) + \beta_2(t)\|u\|_X, \textit{for all } u \in X \textit{ and a.a. } t \in \mathbb{R}.$$

(S4) *For any* $(T,\tau) \in \mathbb{R}_d$, $x \in X$, $\sigma \in \Sigma$, *the map* $t \mapsto F_\sigma(t,x)$ *has a measurable selection.*

Definition 8. The continuous function $u_\sigma(\cdot) \in C([\tau, T], X)$ is called an integral solution of (4) if $u_\sigma(\tau) = u_\tau$ and there exists $l(\cdot) \in L_1([\tau, T], X)$ such that $l(t) \in F_\sigma(t, u_\sigma(t))$, a.e. on (τ, T), and for any $\xi \in D(A)$, $v \in A(\xi)$ one has

$$\|u_\sigma(t) - \xi\|_X^2 \leq \|u_\sigma(s) - \xi\|_X^2 + 2\int_s^t (l(r) + v, u_\sigma(r) - \xi)\, dr,\ t \geq s. \tag{5}$$

It follows from $(A1) - (A2)$, $(S1) - (S4)$ that for any $u_\tau \in L_2(\Omega)$ there exists at least one integral solution u_σ to (4) for any $T > \tau$ [10, Theorem 2.1]. For a fixed $\sigma \in \Sigma$ let $\mathcal{D}_{\sigma,\tau}(x)$ be the set of all integral solutions corresponding to the initial condition $u(\tau) = x$. We shall define the map $U_\sigma : \mathbb{R}_d \times X \to P(X)$ by

$$U_\sigma(t, \tau, x) = \{z : \text{there exists } u(\cdot) \in \mathcal{D}_{\sigma,\tau}(x) \text{ such that } u(t) = z\}.$$

Proposition 9. *For each $\sigma \in \Sigma$, $h \in \mathbb{R}$, $\tau \leq s \leq t$, $x \in X$ we have*

$$U_\sigma(t, s, U_\sigma(s, \tau, x)) = U_\sigma(t, \tau, x),$$

$$U_{T(h)\sigma}(t, \tau, x) = U_\sigma(t + h, \tau + h, x).$$

Hence, U_σ is a multivalued process for each $\sigma \in \Sigma$ and condition $(T1)$ holds.

Theorem 10. *If $(F1)-(F9)$ hold and $g_1 \in L_\infty(\mathbb{R}, L_2(\Omega))$, $g_2 \in L_2^{loc}(\mathbb{R}, L_2(\Omega))$, then the family of MDP U_σ has the Σ_1- uniform global compact attractor $\Theta_{\Sigma_1}(\sigma_2)$.*

Let us consider now the connectivity of the global attractor.

Theorem 11. *In the conditions of Theorem 10, let $f_2 \equiv 0$ and let there exist a non-decreasing map $C(t)$ such that $\|g_2(t)\|_X \leq C(t)$, for a.a. $t \in \mathbb{R}$. Then the set $\Theta_{\Sigma_1}(\sigma_2)$ is connected in X for each $\sigma_2 \in \Sigma_2$.*

4 Stochastic non-autonomous evolution inclusions

4.1 Additive white noise case

Consider the following non-autonomous differential inclusion perturbed by an additive white noise

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \in f(t, u) + g_1(t) + g_2(t) + \sum_{i=1}^m \phi_i \frac{dw_i(t)}{dt}, \text{ on } D \times (\tau, T), \\ u \mid_{\partial D} = 0, \\ u \mid_{t=\tau} = u_\tau, \end{cases} \tag{6}$$

where $\tau \in \mathbb{R}$, $D \subset \mathbb{R}^n$ is an open bounded set with smooth boundary ∂D, $w_i(t)$ are independent two-sided, i.e. $t \in \mathbb{R}$, real Wiener processes with $w_i(0) = 0$, $\phi_i \in D(A)$ (where $A(u) = \Delta u$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega))$, $i = 1, ..., m$, f :

$\mathbb{R} \times \mathbb{R} \to C_v(\mathbb{R})$, $i = 1, 2$, $g_1 \in L_\infty(\mathbb{R}, L_2(D))$, $g_2 \in L_2^{loc}(\mathbb{R}, L_2(D))$. We write $\zeta(t) = \sum_{i=1}^m \phi_i w_i(t)$. Consider the Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \{\omega = (w_1(\cdot), ..., w_m(\cdot)) \in C(\mathbb{R}, \mathbb{R}^m) \mid \omega(0) = 0\},$$

equipped with the Borel $\sigma-$algebra $\mathcal{F}$ and the Wiener measure $\mathbb{P}$. Each $\omega \in \Omega$ generates a map $\zeta(\cdot) = \sum_{i=1}^m \phi_i w_i(\cdot) \in C(\mathbb{R}, L_2(D))$ such that $\zeta(0) = 0$.

Suppose that f satisfies (F1)–(F3), (F7), whereas g_2 satisfies (F8).

Firstly, let us construct the sets Σ_1, Σ_2. The set Σ_1 will be defined in the same way as in the previous section. For the set Σ_2 we write

$$\Sigma_2 = \tilde{\Sigma}_2 \times \Omega,\ \tilde{\Sigma}_2 = \underset{h\in\mathbb{R}}{\cup} g_2(\cdot + h).$$

We define the map $\theta_s : \Omega \to \Omega$ as follows

$$\theta_s\omega = (w_1(s + \cdot) - w_1(s), ..., w_m(s + \cdot) - w_m(s)) \in \Omega.$$

Then the function $\widetilde{\zeta}$ corresponding to $\theta_s\omega$ is defined by $\widetilde{\zeta}(\tau) = \zeta(s + \tau) - \zeta(s) = \sum_{i=1}^m \phi_i(w_i(s + \tau) - w_i(s))$.

The operator T_1 is defined as before. We define the shift operator $T_2 : \Sigma_2 \to \Sigma_2$ as

$$T_2(h)\sigma_2 = T_2(h)(\tilde{\sigma}_2, \omega) = (\tilde{\sigma}_2(\cdot + h), \theta_h\omega),\ \text{for all } \tilde{\sigma}_2 \in \tilde{\Sigma}_2,\ \omega \in \Omega.$$

Thus, $T_2(h)\Sigma_2 = \Sigma_2$, for all $h \in \mathbb{R}$.

To study (6), we make the change of variable $v(t) = u(t) - \zeta(t)$. Then inclusion (6) turns, for each $\omega \in \Omega$ fixed, into

$$\begin{cases} \frac{dv}{dt} \in \Delta v(t) + f(t, v(t) + \zeta(t)) + g_1(t) + g_2(t) + \sum_{i=1}^m \Delta\phi_i w_i(t), \\ v\mid_{\partial D} = 0,\ v(\tau) = v_\tau = u_\tau - \zeta(\tau). \end{cases} \tag{7}$$

Now let $X = L_2(\Omega)$. Consider the abstract evolution inclusion

$$\begin{cases} \frac{dv(t)}{dt} \in A(v(t)) + F_\sigma(t, v(t)),\ t \in [\tau, \infty), \\ v(\tau) = v_\tau = u_\tau - \zeta(\tau), \end{cases} \tag{8}$$

where $\sigma = (\sigma_1, \sigma_2) \in \Sigma$, A is defined as before and $F_\sigma : \mathbb{R} \times X \to 2^X$ is defined as follows:

$$F_\sigma(t, \omega, u) = g_{\sigma_2}(t) + F_{\sigma_1}(t, u + \zeta(t)) + A\zeta(t),$$

where F_{σ_1} is as in the previous section.

As before, the operators A, F_σ satisfy (A1)–(A2), (S1)–(S4), so that for any $v_\tau \in L_2(\Omega)$ there exists at least one integral solution to (8) for any $T > \tau$ [10, Theorem 2.1]. For a fixed $\sigma \in \Sigma$ let $\mathcal{D}_{\sigma,\tau}(x)$ be the set of all integral solutions corresponding to the initial condition $v(\tau) = x$. We define the map $U_\sigma : \mathbb{R}_d \times X \to P(X)$ by

$$U_\sigma(t, \tau, x) = \{z + \zeta(t) : \text{there exists } v(\cdot) \in \mathcal{D}_{\sigma,\tau}(x - \zeta(\tau)) \text{ such that } v(t) = z\}.$$

Theorem 12. *In the preceedings conditions, the family of MDP U_σ has the Σ_1-uniform global compact attractor $\Theta_{\Sigma_1}(\sigma_2)$.*

4.2 Multiplicative white noise case

Finally, consider the following non-autonomous differential inclusion perturbed by a linear multiplicative white noise in the Stratonovich sense

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \in f\left(t,u\right) + g_1\left(t\right) + g_2\left(t\right) + u \circ \frac{dw(t)}{dt}, \text{ on } D \times \left(\tau, T\right), \\ u \mid_{\partial D} = 0, \\ u \mid_{t=\tau} = u_\tau, \end{cases} \tag{9}$$

where $\tau \in \mathbb{R}$, $D \subset \mathbb{R}^n$ is and open bounded set with smooth boundar ∂D, $f : \mathbb{R} \times \mathbb{R} \to C_v\left(\mathbb{R}\right)$, $i = 1, 2$, $g_1 \in L_\infty\left(\mathbb{R}, L_2\left(D\right)\right)$, $g_2 \in L_2^{loc}\left(\mathbb{R}, L_2\left(D\right)\right)$. Consider the Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \left\{\omega = w\left(\cdot\right) \in C\left(\mathbb{R}, \mathbb{R}\right) \mid \omega\left(0\right) = 0\right\},$$

equipped with the Borel $\sigma-$algebra $\mathcal{F}$ and the Wiener measure $\mathbb{P}$.

Suppose again that f satisfies (F1)–(F3), (F7), whereas g_2 satisfies (F8).

We define $\Sigma = \Sigma_1 \times \Sigma_2 = \Sigma_1 \times \tilde{\Sigma}_2 \times \Omega$ and T_1, T_2 exactly as in the previous section, with $\theta_s : \Omega \to \Omega$

$$\theta_s \omega = \left(w\left(s + \cdot\right) - w\left(s\right)\right) \in \Omega.$$

To study (9), we make the change of variable $v\left(t\right) = \gamma(t) u\left(t\right)$, with $\gamma(t) = \gamma\left(\omega, t\right) = e^{-w(t)}$ (we shall omit ω). Then inclusion (9) turns into

$$\begin{cases} \frac{dv}{dt} \in \Delta v\left(t\right) + \gamma(t) f\left(t, \gamma^{-1}(t) v(t)\right) + \gamma(t)(g_1\left(t\right) + g_2\left(t\right)), \\ v \mid_{\partial D} = 0, \; v\left(\tau\right) = v_\tau = \gamma\left(\tau\right) u_\tau. \end{cases} \tag{10}$$

Now let $X = L_2\left(\Omega\right)$. Consider

$$\begin{cases} \frac{dv(t)}{dt} \in A\left(v\left(t\right)\right) + F_\sigma\left(t, v\left(t\right)\right), \; t \in \left[\tau, \infty\right), \\ v\left(\tau\right) = v_\tau, \end{cases} \tag{11}$$

where $\sigma = (\sigma_1, \sigma_2) \in \Sigma$, A is defined as before, and $F_\sigma : \mathbb{R} \times X \to 2^X$ is defined as

$$F_\sigma\left(t, \omega, u\right) = \gamma\left(t\right) g_{\sigma_2}(t) + \gamma\left(t\right) F_{\sigma_1}\left(t, \gamma^{-1}\left(t\right) u\right),$$

where F_{σ_1} is as in the previous section.

As before, the operators A, F_σ satisfy (A1)–(A2), (S1)–(S4). We define the map $U_\sigma : \mathbb{R}_d \times X \to P\left(X\right)$ by

$$U_\sigma\left(t, \tau, x\right) = \left\{\gamma^{-1}\left(t\right) z : \text{there exists } v\left(\cdot\right) \in \mathcal{D}_{\sigma,\tau}\left(\gamma\left(\tau\right) x\right) \text{ such that } v\left(t\right) = z\right\}.$$

Theorem 13. *In the preceedings conditions, the family of MDP U_σ has the Σ_1-uniform global compact attractor $\Theta_{\Sigma_1}\left(\sigma_2\right)$.*

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