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Extremality results for singular functional diffusion equations

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Abstract. We study the existence of extremal positive solutions for initial value problems of the type

$$((k \circ u)u')'(x) = f(x, u(x))u'(x) + g(x, u(x))u(x) \quad \text{a. e. in } I = [0, \alpha],$$
$$u(0) = 0, \quad \lim_{x \to 0^+} ((k \circ u)\, u')(x) = 0,$$

where k and g need not be continuous.

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1 Introduction

In [1] we consider the ordinary differential equation

$$((k \circ u)u')'(x) = f(x, u(x))u'(x) + g(x, u(x))u(x) \quad \text{a.e. in } I = [0, \alpha], \tag{1}$$

with initial data

$$u(0) = 0, \quad \lim_{x \to 0^+} ((k \circ u)\, u')(x) = 0, \tag{2}$$

for a fixed $\alpha > 0$. The function k is allowed to vanish for some values of u and, therefore, the equation becomes singular at those points. This type of differential operators naturally arises in diffusion processes [4].

This is an overview article.

From a purely mathematical point of view one of the main difficulties involved in these equations is a lack of a priori bounds on the derivatives, which puts a stop to a Bernstein–Nagumo approach. Even in our case, where the dependence on the derivatives is linear, we have to give up obtaining nontrivial $\mathcal{C}^1$ solutions.

We define a solution of (1) – (2) as any element of the set

$$\mathcal{S} = \left\{ u \in W^{1,1}(I) : u(x) > 0 \text{ for } x \in (0,\alpha] \text{ and } (k \circ u)u' \in W^{1,1}(I) \right\},$$

that satisfies the relations (1) – (2). Following the standard notation, $W^{1,1}(I)$ stands for the Sobolev space of L^1 functions whose generalized derivative belongs to $L^1(I)$.

We note that for every element of $W^{1,1}(I)$ there exists a unique absolutely continuous function on I that equals it almost everywhere on I. We mean that $u \in \mathcal{S}$ satisfies (2) when the continuous functions that equal u and $(k \circ u)u'$ a.e. fulfill the respective relations.

On the other hand, we say that a solution $u_* \in \mathcal{S}$ of (1)–(2) is the minimal solution in $\mathcal{S}$ if $u_* \le u$ for any other solution $u \in \mathcal{S}$. Analogously we can define the concept of maximal solution. When both the minimal and the maximal solutions of (1)–(2) in $\mathcal{S}$ exist, we call them extremal solutions.

Existence of extremal nontrivial solutions for (1)–(2) is proven in [1] under the following assumptions on the functions k, f and g:

(k0) $k \in L^1_{loc}[0,\infty)$, $k(\xi) > 0$ a.e. $\xi \in [0,\infty)$, $\displaystyle\int_0^1 \frac{k(\xi)}{\xi} d\xi < \infty$ and $\displaystyle\int_1^\infty \frac{k(\xi)}{\xi} d\xi = \infty$.

(f0) $f : [0,\alpha] \times [0,\infty) \to \mathbb{R}$ is continuous on $[0,\alpha] \times [0,\infty)$; for a.e. $\xi \in [0,\infty)$, $f(\cdot,\xi)$ is absolutely continuous on $[0,\alpha]$ and

$$|D_1 f(x,\xi)| \le B(\xi) \quad \text{for a.e. } x \in [0,\alpha],$$

where $B \in L^1_{loc}[0,\infty)$; there exists a null set $N \subset [0,\alpha]$ such that for each fixed $x \in [0,\alpha] \backslash N$ the derivative $D_1 f(x,\xi)$ exists for a.e. $\xi \in [0,\infty)$.

(f1) $f(x,\xi) > 0$ on $[0,\alpha] \times [0,\infty)$ and for all $x \in [0,\alpha] \backslash N$ we have that $D_1 f(x,\xi) \le 0$ for a.e. $\xi \in [0,\infty)$.

(f2) $f(x,\cdot)$ is nondecreasing in $[0,\infty)$ for a.e. $x \in [0,\alpha]$.

(h0) There exists $h_1 \in L^1[0,\alpha]$ such that $f(x,\xi) \le h_1(x)$ for a.e. $x \in [0,\alpha]$ and all $\xi \in [0,\infty)$.

(h1) There exists $h_2 \in L^1[0,\alpha]$ such that for all $x \in [0,\alpha] \backslash N$ we have $D_1 f(x,\xi) \ge h_2(x)$ for a.e. $\xi \in [0,\infty)$ (Notice that h_2 has to be nonpositive a.e. so that this assumption and (f1) are consistent).

(g0) $g : [0,\alpha] \times [0,\infty) \to \mathbb{R}$ is such that $g(\cdot, h(\cdot))$ is a measurable function on $[0,\alpha]$ whenever h is continuous on $[0,\alpha]$.

(g1) For a.e. $x \in [0,\alpha]$, $g(x,\xi)\xi$ is nondecreasing with respect to $\xi \in [0,\infty)$.

(g2) There exists $\psi \in L^1[0,\alpha]$ such that $0 \le g(x,\xi) \le \psi(x)$ for a.e. $x \in [0,\alpha]$ and all $\xi \in [0,\infty)$.

The main difficulty that we have to overcome in [1] concerns how to turn problem (1) – (2) into a problem of finding fixed points of an equivalent integral operator. Such an operator is constructed by using the formula

$$\int_a^b f(s,h(s))h'(s)ds = \int_{h(a)}^{h(b)} f(a,\xi)d\xi - \int_a^b \int_{h(b)}^{h(s)} D_1 f(s,\xi)d\xi ds,$$

which extends both the formula of change of variables and that of integration by parts.

From the assumed conditions, the so constructed operator is, in general, not continuous. Despite this, we deduce existence of extremal fixed points by using the results proved in [3] for this type of operators.

The previous existence result can be extended to cover a wider class of functional equations. In fact, we consider in [2] the solvability of functional equations of the form

$$\frac{d}{dx}(k(u(x),u)u'(x)) = f(x,u(x),u)u'(x) + \ g(x,u(x),u)u(x) \text{ a.e. on } [0,\alpha], \quad (3)$$

together with the initial conditions $u(0) = 0$, $\lim_{x\to 0^+} k(u(x),u)\, u'(x) = 0$.

Roughly speaking, the main idea can be sketched as follows: we require the equation

$$\frac{d}{dx}(k(u(x),v)u'(x)) = f(x,u(x),v)u'(x) + \ g(x,u(x),v)u(x) \text{ a.e. on } [0,\alpha], \quad (4)$$

to satisfy the conditions $(k0) - (g2)$ in an uniform way for v in a suitable set of functions. As a consequence we have, by virtue of the main result in [1], that the corresponding problem has extremal solutions. Subsequently we consider an iteration scheme based on the previous type of problems that leads to our existence result. The key assumption on the functionals $k(u,\cdot)$, $f(x,u,\cdot)$ and $g(x,u,\cdot)$ is monotonicity (the reader is referred to [2] for the details).

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