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Rectifiability and perimeter in step 2 Groups

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Abstract. We study finite perimeter sets in step 2 Carnot groups. This way we extend the classical De Giorgi's theory, developed in Euclidean spaces ([2], [3]), as well as its generalization, considered by the authors, in Heisenberg groups ([7]). A structure theorem for sets of finite perimeter is obtained and consequently a divergence theorem. Full proofs of these results, comments and an exhaustive bibliography can be found in [9].

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1 Definitions

1.1 Carnot Groups

We recall the definition of *Carnot group of step 2* and some of its properties (see [5], [15], [12] and [16]). Let $\mathbb{G}$ be a connected, simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a *step 2 stratification*, i.e. there exist linear subspaces V_1, V_2 such that

$$\mathfrak{g} = V_1 \oplus V_2, \qquad [V_1, V_1] = V_2, \quad [V_1, V_2] = 0, \tag{1.1}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. A base $e_1, \ldots, e_n$ of $\mathfrak{g}$ is adapted to the stratification if $e_1, \ldots, e_m$ is a base of V_1 and $e_{m+1}, \ldots, e_n$ is a base of V_2. Let $X = \{X_1, \ldots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$. Given (1.1), the vector fields $X_1, \ldots, X_m$ together with their commutators of length 2 generate all $\mathfrak{g}$; we will refer to $X_1, \ldots, X_m$ as a family of *generating vector fields* of the group.

The exponential map exp is a one to one map from $\mathfrak{g}$ to $\mathbb{G}$. Hence any $p \in \mathbb{G}$ can be written, in a unique way, as $p = \exp(p_1 X_1 + \cdots + p_n X_n)$. Using these *exponential coordinates*, we identify p with the n-tuple $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and we identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the new product in $\mathbb{R}^n$ is such that $\exp(p \cdot q) = \exp(p)\exp(q)$. The identification $\mathbb{G} \simeq (\mathbb{R}^n, \cdot)$ is used from now on, without being mentioned anymore.
The n-dimensional Lebesgue measure $\mathcal{L}^n$, is the Haar measure of the group $\mathbb{G}$.

As a consequence of the stratification (1.1), a natural family of automorphisms of $\mathbb{G}$ are the so called intrinsic dilations. For any $x \in \mathbb{G}$ and $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda x_1, ..., \lambda x_m, \lambda^2 x_{m+1}, \ldots, \lambda^2 x_n). \tag{1.2}$$

The subbundle, of the tangent bundle $T\mathbb{G}$, spanned by the first m vector fields $X_1, \ldots, X_m$ is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span}\,\{X_1(x), \ldots, X_m(x)\}, \qquad x \in \mathbb{G}.$$

Sections of $H\mathbb{G}$ are called *horizontal sections* and vectors of $H\mathbb{G}_x$ are *horizontal vectors*. Each horizontal section ϕ is identified by its coordinates $(\phi_1, \ldots, \phi_m)$ with respect to the moving frame $X_1(x), \ldots, X_m(x)$. That is horizontal sections are functions $\mathbb{R}^n \to \mathbb{R}^m$.

A subriemannian structure is defined on $\mathbb{G}$, endowing each fiber of $H\mathbb{G}$ with a scalar product making the basis $X_1(x), \ldots, X_m(x)$ an orthonormal basis. That is if $v = (v_1, \ldots, v_m)$ and $w = (w_1, \ldots, w_m)$ are in $H\mathbb{G}_x$, $\langle v, w\rangle_x := \sum_{j=1}^m v_j w_j$ and $|v|_x^2 := \langle v, v\rangle_x$. It will simplify our notations to extend the scalar product $\langle v, w\rangle_x$ also to $v, w \in T\mathbb{G}_x$, keeping the same definition: $\langle v, w\rangle_x := \sum_{j=1}^m v_j w_j$.

Given a sub-riemannian structure there is standard procedure to introduce a natural distance, i.e. the Carnot-Carathéodory distance, on $\mathbb{G}$ (see e.g. [11]). Consider the family of the so-called sub-unit curves in $\mathbb{G}$: an absolutely continuous curve $\gamma : [0,T] \to \mathbb{G}$ is a *sub-unit curve* with respect to $X_1, \ldots, X_m$ if for a.e. $t \in [0,T]$,

$$\dot\gamma(t) \in H\mathbb{G}_{\gamma(t)}, \quad \text{and} \quad |\dot\gamma(t)|_{\gamma(t)} \le 1.$$

Definition 1.1 (Carnot-Carathéodory distance). If $p, q \in \mathbb{G}$, their cc-distance is defined by

$$d_c(p,q) = \inf\left\{T > 0 :\ \gamma : [0,T] \to \mathbb{G} \text{ is sub-unit } \gamma(0) = p,\ \gamma(T) = q\right\}.$$

It is a classical result in control theory, usually known as Chow's theorem, that, under assumption (1.1), the set of sub-unit curves joining p and q is not empty. Hence $d_c(p,q)$ is never infinity and d_c is a distance on $\mathbb{G}$ inducing the same topology as the standard Euclidean distance.
The Carnot-Carathéodory distance is usually difficult to compute and sometimes it is more convenient to deal with distances, equivalent with d_c, but that can be explicitly evaluated. Several ones have been used in the literature, here we choose the following

$$d_\infty(x,y) = d_\infty(y^{-1} \cdot x, 0),$$

where

$$d_\infty(p,0) = \max\{|p_1|, \ldots, |p_m|, \varepsilon|p_{m+1}|^{1/2}, \ldots, \varepsilon|p_n|^{1/2}\}. \tag{1.3}$$

Here $\varepsilon \in (0,1)$ is a suitable positive constant.
Finally, we denote $U_c(p,r)$ and $U_\infty(p,r)$ the open balls associated, respectively, with d_c and d_∞.

Related with these distances, different Hausdorff measures can be constructed, following Carathéodory's construction as in [4], Section 2.10.2.

Definition 1.2. For $\alpha > 0$ denote by $\mathcal{H}^\alpha$ the α-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^n \simeq \mathbb{G}$, by $\mathcal{H}_c^\alpha$ the one obtained from d_c in $\mathbb{G}$, and by $\mathcal{H}_\infty^\alpha$ the one obtained from d_∞ in $\mathbb{G}$. Analogously, $\mathcal{S}^\alpha$, $\mathcal{S}_c^\alpha$, and $\mathcal{S}_\infty^\alpha$ denote the corresponding spherical Hausdorff measures.

The *homogeneous dimension* of $\mathbb{G}$ is the integer $Q := \text{dim}V_1 + 2\text{dim}V_2 = m + 2(n-m)$ that is the Hausdorff dimension of $\mathbb{G}$ with respect to the cc-distance d_c (see [14]).

1.2 $\mathbb{G}$-regular functions and surfaces

The following definitions and result about intrinsic differentiability are due to Pansu ([16]), or are inspired by his ideas.
A map $L :\ \mathbb{G} \to \mathbb{R}$ is $\mathbb{G}$-*linear* if it is a homomorphism from $\mathbb{G} \equiv (\mathbb{R}^n, \cdot)$ to $(\mathbb{R}, +)$ and if it is positively homogeneous of degree 1 with respect to the dilations of $\mathbb{G}$, that is $L(\delta_\lambda x) = \lambda Lx$ for $\lambda > 0$ and $x \in \mathbb{G}$. It is easy to see that L is $\mathbb{G}$-linear if and only if there is $a \in \mathbb{R}^m$ such that $Lx = \sum_{j=1}^m a_j v_j$, for all $x \in \mathbb{G}$.

Given $f : \mathbb{G} \to \mathbb{R}$ such that $X_1 f, \dots, X_m f$ exist, we denote by $\nabla_{\mathbb{G}} f$ the horizontal section defined as

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^{m} (X_i f) X_i.$$

whose coordinates are $(X_1 f, ..., X_m f)$. Moreover, if $\phi = (\phi_1, \dots, \phi_m)$ is an horizontal section such that $X_j \phi_j$ exist for $j = 1, \dots, m$, we define $\mathrm{div}_{\mathbb{G}}\, \phi$ as the real valued function

$$\mathrm{div}_{\mathbb{G}}\, (\phi) := \sum_{j=1}^{m} X_j \phi_j.$$

Definition 1.3. $f : \mathbb{G} \to \mathbb{R}$ is *Pansu-differentiable* or $\mathbb{G}$-differentiable (see [16] and [13]) at x_0 if there is a $\mathbb{G}$-linear map $d_{\mathbb{G}} f_{x_0}$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - d_{\mathbb{G}} f_{x_0}(x_0^{-1} \cdot x)}{d_c(x, x_0)} = 0.$$

Notice that if f is $\mathbb{G}$-differentiable in x_0 then $X_j f(x_0)$ exist for $j = 1, \dots, m$ and

$$d_{\mathbb{G}} f_{x_0}(v) = \langle \nabla_{\mathbb{G}} f, v \rangle_{x_0} = \sum_{j=1}^{m} v_j X_j f(x_0).$$

Conversely, if, for $j = 1, \dots, m$ all of $X_j f(x)$ are continuous in an open set Ω, then f is differentiable in each point of Ω. We denote by $\mathbb{C}^1_{\mathbb{G}}(\Omega)$ the set of continuous real functions in Ω such that $X_j f(x)$ are continuous in Ω for $j = 1, \dots, m$. Moreover, we shall denote by $\mathbb{C}^1_{\mathbb{G}}(\Omega, H\mathbb{G})$ the set of all sections ϕ of $H\mathbb{G}$ all whose canonical coordinates $\phi_j \in \mathbb{C}^1_{\mathbb{G}}(\Omega)$. The corresponding spaces of Euclidean differentiable functions are denoted as $\mathbf{C}^1(\Omega)$, $\mathbf{C}^1(\Omega, H\mathbb{G})$; $\mathbf{C}^1_0(\Omega, H\mathbb{G})$ is the space of smooth, compactly supported sections of $H\mathbb{G}$.

Remark 1.4. We recall that $\mathbf{C}^1(\Omega) \subset \mathbb{C}^1_{\mathbb{G}}(\Omega)$ and that the inclusion may be strict, indeed functions in $\mathbb{C}^1_{\mathbb{G}}(\Omega)$ are, a priori, only Hölder continuous functions with respect to the Euclidean metric. An example is provided in Remark 6 of [7].

Following [8], we define $\mathbb{G}$-regular hypersurfaces in a Carnot group $\mathbb{G}$ as non critical level sets of functions in $\mathbb{C}^1_{\mathbb{G}}(\mathbb{G})$.

Definition 1.5 ($\mathbb{G}$-regular hypersurfaces). $S \subset \mathbb{G}$ is a $\mathbb{G}$-*regular hypersurface* if for every $x \in S$ there exist a neighborhood $\mathcal{U}$ of x and $f \in \mathbb{C}^1_{\mathbb{G}}(\mathcal{U})$ such that

$$S \cap \mathcal{U} = \{y \in \mathcal{U} : f(y) = 0\}; \qquad (i)$$

$$\nabla_{\mathbb{G}} f(y) \neq 0 \quad \text{for } y \in \mathcal{U}. \qquad (ii)$$

Notice that the d_c, Hausdorff dimension of a $\mathbb{G}$-regular hypersurface, is always $Q - 1$ (see [8]).

Definition 1.6 (Tangent group). If $S = \{x \in \mathbb{G} : f(x) = 0\}$ is a $\mathbb{G}$-regular hypersurface, the *tangent group* $T^g_{\mathbb{G}}S(x_0)$ to S at x_0 is

$$T^g_{\mathbb{G}}S(x_0) := \{v \in \mathbb{G} :\ d_{\mathbb{G}} f_{x_0}(v) = 0\}.$$

$T^g_{\mathbb{G}}S(x_0)$ is a proper subgroup of $\mathbb{G}$. We define the *tangent plane* to S at x_0 as

$$T_{\mathbb{G}}S(x_0) := x_0 \cdot T^g_{\mathbb{G}}S(x_0).$$

The above definition is a good one: indeed the tangent group does not depend on the particular function f defining the surface S because of point (iii) of Implicit Function Theorem below that yields

$$T^g_{\mathbb{G}}S(x) = \{v \in \mathbb{G} :\ \langle \nu_E(x), v\rangle_x = 0\}$$

where ν_E, the inward unit normal is defined in (1.7) and depends only on the set S.

Remark 1.7. The class of $\mathbb{G}$-regular hypersurfaces is strongly different from the class of Euclidean $\mathbf{C}^1$ embedded surfaces in $\mathbb{R}^n$. From one side, Euclidean $\mathbf{C}^1$-surfaces are not $\mathbb{G}$-regular at points x where the Euclidean tangent space $T_xS \supset H\mathbb{G}_x$. On the other side, as one can guess from Remark 1.4, $\mathbb{G}$-regular surfaces can be very irregular as subsets of Euclidean $\mathbb{R}^n$. It is less obvious that they could even have Euclidean Hausdorff dimension larger than $n-1$. It is rather amazing that, even for such surfaces, the notion of tangent plane and related properties are utterly natural.

1.3 $BV_{\mathbb{G}}$-functions and finite perimeter sets

The definition of BV functions in a group follows closely the one in Euclidean $\mathbb{R}^n$; simply the horizontal vector fields X_j, $j = 1, \dots, m$ take the place of the partial derivatives $\frac{\partial}{\partial x_i}$, for $i = 1, \dots, n$ (see e.g. [10]).

If Ω is an open subset of $\mathbb{G}$, the space $BV_{\mathbb{G}}(\Omega)$ is the set of, functions $f \in L^1(\Omega)$ such that

$$\|\nabla_{\mathbb{G}} f\|(\Omega) := \sup\left\{\int_\Omega f(x)\mathrm{div}_{\mathbb{G}}\,\phi(x)\,dx : \phi \in \mathbf{C}^1_0(\Omega, H\mathbb{G}),\ |\phi| \le 1\right\} < \infty. \quad (1.4)$$

The space $BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ is the set of functions belonging to $BV_{\mathbb{G}}(U)$ for each open set $U \subset\subset \Omega$.

By the Riesz representation theorem we have

Theorem 1.8 (Structure of $BV_{\mathbb{G}}$ functions). *If $f \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ then $\|\nabla_{\mathbb{G}} f\|$ is a Radon measure on Ω, there exists a $\|\nabla_{\mathbb{G}} f\|$-measurable horizontal section $\sigma_f : \Omega \to H\mathbb{G}$ such that $|\sigma_f(x)| = 1$ for $\|\nabla_{\mathbb{G}} f\|$-a.e. $x \in \Omega$, and, for all $\phi \in \mathbf{C}^1_0(\Omega, H\mathbb{G})$,*

$$\int_\Omega f(x)\mathrm{div}_{\mathbb{G}}\,\phi(x)\,d\mathcal{L}^n = \int_\Omega \langle \phi, \sigma_f\rangle\, d\|\nabla_{\mathbb{G}} f\|. \quad (1.5)$$

Following De Giorgi, we define sets with finite perimeter as

Definition 1.9 ($\mathbb{G}$-Caccioppoli sets). A measurable set $E \subset \mathbb{G}$ is of *finite* $\mathbb{G}$-*perimeter* (respectively of *locally finite* $\mathbb{G}$-*perimeter* or a $\mathbb{G}$-*Caccioppoli set*) in Ω if the characteristic function $\mathbf{1}_E \in BV(\Omega)$ (respectively $\mathbf{1}_E \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$). We call *perimeter of* E the measure

$$|\partial E|_{\mathbb{G}} := \|\nabla_{\mathbb{G}} \mathbf{1}_E\| \tag{1.6}$$

and *(generalized inward)* $\mathbb{G}$-*normal* to ∂E the horizontal vector

$$\nu_E(x) := -\sigma_{\mathbf{1}_E}(x). \tag{1.7}$$

It is interesting to observe that (1.5), when applied to the characteristic function of a finite perimeter set E, reads as an abstract divergence theorem

$$\int_E \mathrm{div}\,_{\mathbb{G}}\phi \, d\mathcal{L}^n = -\int_{\mathbb{G}} \langle \phi(x), \nu_E(x)\rangle_x \, d|\partial E|_{\mathbb{G}}, \tag{1.8}$$

giving more geometric substance to (1.8) is one of the main results here presented.

Notice that for $\mathbb{G}$-Caccioppoli sets whose boundary is an Euclidean regular surface, the perimeter measure coincides with the natural definition of surface area in Carnot groups.

Proposition 1.10. *If E is a $\mathbb{G}$-Caccioppoli set with Euclidean C^1 boundary, then there is an explicit representation of the $\mathbb{G}$-perimeter in terms of the Euclidean $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$*

$$|\partial E|_{\mathbb{G}}(\Omega) = \int_{\partial E \cap \Omega} \left(\sum_{j=1}^{m} \langle X_j, n\rangle^2_{\mathbb{R}^n} \right)^{1/2} d\mathcal{H}^{n-1},$$

where $n = n(x)$ is the Euclidean unit outward normal to ∂E.

The topological boundary of a finite perimeter set can be really bad, and it can even have positive $\mathcal{L}^n$-measure. One of the main achievements of De Giorgi's theory is proving the existence of a subset of the topological boundary, the so called reduced boundary, that carries all the perimeter measure (the $|\partial E|_{\mathbb{G}}$ measure in our case) and that is reasonably regular: i.e. it is a rectifiable set. So, following once more De Giorgi, we define the *reduced boundary* $\partial^*_{\mathbb{G}} E$ of a $\mathbb{G}$-Caccioppoli set $E \subset \mathbb{G}$ as

Definition 1.11 (Reduced boundary). Let E be a $\mathbb{G}$-Caccioppoli set; we say that $x \in \partial^*_{\mathbb{G}} E$ if

$$|\partial E|_{\mathbb{G}}(U_c(x,r)) > 0 \qquad \text{for any } r > 0; \tag{i}$$

$$\text{there exists} \quad \lim_{r\to 0} \fint_{U_c(x,r)} \nu_E \, d|\partial E|_{\mathbb{G}}; \tag{ii}$$

$$\left| \lim_{r\to 0} \fint_{U_c(x,r)} \nu_E \, d|\partial E|_{\mathbb{G}} \right|_{\mathbb{R}^{m_1}} = 1. \tag{iii}$$

The limits in Definition 1.11 should be understood as a convergence of the averages of the coordinates of ν_E.

2 Main Results

The main results of the present paper are

1. At each point of the reduced boundary of a $\mathbb{G}$-Caccioppoli set there is a (generalized) tangent group;
2. The reduced boundary is a $(Q-1)$-dimensional $\mathbb{G}$-rectifiable sets;
3. $|\partial E|_{\mathbb{G}} = c\,\mathcal{S}_{\infty}^{Q-1} \llcorner \partial^* E$, i.e. the perimeter measure equals a constant times the spherical $(Q-1)$-dimensional Hausdorff measure restricted to the reduced boundary.
4. An intrinsic divergence theorem holds for $\mathbf{C}^1_{\mathbb{G}}(\mathbb{G}, H\mathbb{G})$ vector fields in $\mathbb{G}$-Caccioppoli sets.

We discuss now briefly each of these points.

First of all we recall a result of independent interest. An Implicit Function Theorem holds in $\mathbb{G}$, stating that any $\mathbb{G}$-regular hypersurface $S = \{y \in \mathbb{G} : f(y) = 0\}$ is locally the graph, along the integral curves of an horizontal vector field, of a function of $n-1$ variables. Moreover, the $\mathbb{G}$-perimeter of E can be written explicitly in terms of the associated parameterization and of $\nabla_{\mathbb{G}} f$.

Theorem 2.1 (Implicit Function Theorem). *Let Ω be an open set in $\mathbb{R}^n$, $0 \in \Omega$, and let $f \in \mathbb{C}^1_{\mathbb{G}}(\Omega)$ be such that $f(0) = 0$ and $X_1 f(0) > 0$. Define $E = \{x \in \Omega : f(x) < 0\}$, $S = \{x \in \Omega : f(x) = 0\}$, and, for $\delta > 0$, $h > 0$, $I_\delta = \{\xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}, |\xi_j| \le \delta\}$, $J_h = [-h, h]$. If $\xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$ and $t \in J_h$, denote by $\gamma(t, \xi)$ the integral curve of the vector field X_1 at the time t issued from $(0, \xi) = (0, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, i.e.*

$$\gamma(t, \xi) = \exp(tX_1)(0, \xi).$$

Then there exist $\delta, h > 0$ such that the map $(t, \xi) \to \gamma(t, \xi)$ is a homeomorphism of a neighborhood of $J_h \times I_\delta$ onto an open subset of $\mathbb{R}^n$, and, if we denote by $\mathcal{U} \subset\subset \Omega$ the image of $\mathrm{Int}(J_h \times I_\delta)$ through this map, we have

$$E \text{ has finite } \mathbb{G}\text{-perimeter in } \mathcal{U}; \tag{i}$$

$$\partial E \cap \Omega = S \cap U; \tag{ii}$$

$$\nu_E(x) = -\frac{\nabla_{\mathbb{G}} f(x)}{|\nabla_{\mathbb{G}} f(x)|_x} \text{ for all } x \in S \cap \Omega, \tag{iii}$$

where ν_E is the generalized inner unit normal defined by (1.7). Moreover, there exists a unique continuous function $\phi = \phi(\xi) : I_\delta \to J_h$ such that the following parameterization holds: if $\xi \in I_\delta$, put $\Phi(\xi) = \gamma(\phi(\xi), \xi)$, then

$$S \cap \tilde{\Omega} = \{x \in \tilde{\mathcal{U}} : x = \Phi(\xi), \xi \in I_\delta\}; \tag{iv}$$

the $\mathbb{G}$-perimeter has an integral representation

$$|\partial E|_{\mathbb{G}}(\tilde{\mathcal{U}}) = \int_{I_\delta} \Big(\sum_{j=1}^m |X_j f(\Phi(\xi))|^2\Big)^{1/2} \big(X_1 f(\Phi(\xi))\big)^{-1}\, d\xi. \tag{v}$$

2.1 The generalized tangent plane

For any set $E \subset \mathbb{G}$, $x_0 \in \mathbb{G}$ and $r > 0$ we consider the translated and dilated sets E_{r,x_0} defined as

$$E_{r,x_0} = \{x : x_0 \cdot \delta_r(x) \in E\} = \delta_{\frac{1}{r}}(x_0^{-1} \cdot E).$$

If $v \in H\mathbb{G}_{x_0}$ the halfspace $S^+_{\mathbb{G}}(v)$ is $\{x : \langle x, v\rangle_0 \geq 0\}$, and its topological boundary is the subgroup $T^g_{\mathbb{G}}(v)$ of $\mathbb{G}$ defined as $\{x : \langle x, v\rangle_0 = 0\}$. We say that E has a *generalized tangent plane* at a point x_0 if the sets E_{r,x_0} converge to $S^+_{\mathbb{G}}(\nu_E(x_0))$ as $r \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{G})$. The following blow-up theorem states that at each point of $\partial^*_{\mathbb{G}}E$ there is a generalized tangent plane. Besides its intrinsic interest, it provides one of the key tools for our structure theorem.

Theorem 2.2 (Blow-up Theorem). *If E is a $\mathbb{G}$-Caccioppoli set, $x_0 \in \partial^*_{\mathbb{G}}E$ and $\nu_E(x_0) \in H\mathbb{G}_{x_0}$ is the inward normal as defined in (1.7) then*

$$\lim_{r\to 0} \mathbf{1}_{E_{r,x_0}} = \mathbf{1}_{S^+_{\mathbb{G}}(\nu_E(x_0))} \qquad \text{in } L^1_{loc}(\mathbb{G}) \tag{2.3}$$

and for all $R > 0$

$$\begin{aligned}\lim_{r\to 0} |\partial E_{r,x_0}|_{\mathbb{G}}(U_c(0,R)) &= |\partial S^+_{\mathbb{G}}(\nu_E(x_0))|_{\mathbb{G}}(U_c(0,R)) \\ &= \mathcal{H}^{n-1}(T^g_{\mathbb{G}}(\nu_E(0)) \cap U_c(0,R)).\end{aligned}$$

The proof of the above theorem relies on careful asymptotic estimates, and on the following lemma, that is far from being trivial as the corresponding statement in the Euclidean space, and relies on the structure of step 2 groups.

Lemma 2.3. *Let $\mathbb{G}$ be a step 2 group and let $Y_1, \ldots, Y_m$ be left invariant orthonormal (horizontal) sections of $H\mathbb{G}$. Assume that $g : \mathbb{G} \to \mathbb{R}$ satisfies*

$$Y_1 g \geq 0 \quad \text{and} \quad Y_j(g) = 0 \quad \text{if} \quad j = 2, \ldots, m. \tag{2.4}$$

Then the level lines of g are "vertical hyperplanes orthogonal to Y_1" that is sets that are group translations of

$$S(Y_1) := \{p : \langle p, Y_1\rangle_0 = 0\}.$$

Notice that for more complicated groups, as are groups of step 3 or larger, the above statement is false and also Theorem 2.2 fails; indeed there are examples of point of the reduced boundary where no tangent group exists, even in our generalized sense.

The existence of a generalized tangent at each point of the reduced boundary, together with a suitable Whitney type extention theorem, yields, through a fairly standard procedure in geometric measure theory, the rectifiability of the reduced boundary as stated in the following structure theorem.

2.2 Structure of $\mathbb{G}$-Caccioppoli sets and Divergence Theorem

The following differentiation lemma plays a key role in the present paper, showing that in fact the perimeter measure is concentrated on the reduced boundary. In the Euclidean setting it is a simple consequence of Lebesgue-Besicovitch differentiation lemma, while in Carnot groups (where such Lemma fails to hold: see [13], [17]) it relies on a deep asymptotic estimate proved by Ambrosio in [1].

Lemma 2.4. *Assume E is a $\mathbb{G}$-Caccioppoli set, then*

$$\lim_{r\to 0} \fint_{U_c(x,r)} \nu_E \, d|\partial E|_{\mathbb{G}} = \nu_E(x), \qquad \text{for } |\partial E|_{\mathbb{G}}\text{-a.e. } x,$$

that is $|\partial E|_{\mathbb{G}}$-a.e. $x \in \mathbb{G}$ belongs to the reduced boundary $\partial^*_{\mathbb{G}} E$.

We can state now our main results.

Theorem 2.5 (Structure of $\mathbb{G}$-Caccioppoli sets). *If $E \subseteq \mathbb{G}$ is a $\mathbb{G}$-Caccioppoli set, then*

$$\partial^*_{\mathbb{G}} E \text{ is } (Q-1)\text{-dimensional } \mathbb{G}\text{-rectifiable}, \qquad (i)$$

that is $\partial^*_{\mathbb{G}} E = N \cup \bigcup_{h=1}^{\infty} K_h$, where $\mathcal{H}_c^{Q-1}(N) = 0$ and K_h is a compact subset of a $\mathbb{G}$-regular hypersurface S_h;

$$\nu_E(p) \text{ is } \mathbb{G}\text{-normal to } S_h \text{ at } p, \text{ for all } p \in K_h, \qquad (ii)$$

that is $\nu_E(p) \in H\mathbb{G}_p$ and $\langle \nu_E(p), v\rangle_p = 0$ for all $v \in T_{\mathbb{G}} S_h(p)$;

$$|\partial E|_{\mathbb{G}} = \theta_c \mathcal{S}_c^{Q-1} \llcorner \partial^*_{\mathbb{G}} E, \text{ where } \theta_c(x) = \mathcal{H}^{n-1}(\partial S^+_{\mathbb{G}}(\nu_E(x)) \cap U_c(0,1)). \qquad (iii)$$

If we replace the $\mathcal{H}_c$-measure by the $\mathcal{H}_\infty$-measure, the corresponding density θ_∞ turns out to be a constant. More precisely

$$|\partial E|_{\mathbb{G}} = \theta_\infty \, \mathcal{S}_\infty^{Q-1} \llcorner \partial^*_{\mathbb{G}} E, \qquad (iv)$$

where (ε is the one in (1.3)) $\theta_\infty = \frac{\omega_{m-1}\omega_{n-m}\varepsilon^{2(m-n)}}{\omega_{Q-1}}$. Notice $\omega_{m-1}\omega_{n-m}\varepsilon^{2(m-n)} = \mathcal{H}^{n-1}(\partial S^+_{\mathbb{G}}(\nu_E(0)) \cap U_\infty(0,1))$ is independent of $\nu_E(0)$.

Theorem 2.6 (Divergence Theorem). *If E is a $\mathbb{G}$-Caccioppoli set, then*

$$|\partial E|_{\mathbb{G}} = \theta_\infty \, \mathcal{S}_\infty^{Q-1} \llcorner \partial^*_{\mathbb{G}} E, \qquad (i)$$

and the following version of the divergence theorem holds

$$-\int_E div_{\mathbb{G}} \phi \, d\mathcal{L}^n = \theta_\infty \int_{\partial^*_{\mathbb{G}} E} \langle \nu_E, \phi\rangle \, d\mathcal{S}_\infty^{Q-1}, \qquad \forall \phi \in \mathbf{C}^1_0(\mathbb{G}, H\mathbb{G}). \qquad (ii)$$

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