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Dense Sets and Far-field Patterns for the Vector Thermoelastic Equation

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Abstract. We study the set of far-field patterns which are generated by entire incident thermoelastic fields scattered by a boundary nopenetrable obstacle. Necessary and sufficient conditions are given for the set to be dense in the set of all square integrable vector fields defined on the unit sphere. The method of Herglotz thermoelastic function is utilized to prove the dense properties of the asymptotic fields.

AMS Subject Classification. 35L, 73D, 47A

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1 Introduction

A basic problem in inverse scattering theory is the classification of far field patterns corresponding to the scattering of a time harmonic thermoelastic incident wave by a bounded, connected obstacle. Indeed, if $\mathcal{T}$ denotes the operator mapping the incident field and scattering obstacle onto the far field patterns, then the inverse scattering problem is to construct $\mathcal{T}^{-1}$ defined on the range of $\mathcal{T}$, and the determination of this range is nothing more than the description of the class of far field patterns. It is easily verifiable that the class of functions that can be far field patterns is a subset of the class of the entire functions for each positive fixed value of the wave numbers. The crucial point is the question if the far field patterns for a fixed obstacle and all incident plane wave are complete in a production of $\mathbf{L}^2(\Omega)$. Colton, Kress and Kirsch [2,3] gave an answer of this question for some acoustic and electromagnetic scattering problems. Dassios [4,5] has investigated the case of elastic rigid scattering problems where the situation becomes much harder since in elasticity, besides the vectorial (displacement, surface traction) as well as the tensorial (stress, strain) characteristics of the fields, there are two separate wave solutions propagating at different phase velocities. The purpose of this work is to extend the mentioned results to coupled thermoelasticity. In this case, there are five types of waves present, two of which

This is the preliminary version of the paper.

are longitudinal elastic, one is transverse elastic, and two are thermal waves [7]. Consequently, there are four complex dimensionless parameters by means of which the different wave numbers are connected. The situation is much more complicated.
In particular we shall show that the set of thermoelastic far field patterns corresponding to the scattering of the entire incident fields by a bounded thermoelastic rigid at zero temperature obstacle is dense in the production space of square integrable function on $\partial\Omega$ if and only if does not exist a eigenfunction of an eigenvalue problem which is a *thermoelastic Herglotz function*. This result will be established by first constructing an appropriate complete set of the function defined on the boundary of the scattering obstacle and then establishing an integral representation for the displacement part of the thermoelastic Herglotz function. In order to avoid the difficulties come from the existence of polarization of the transverse displacement wave, we have to raise the rank of the tensorial character of the fields involved by one. This idea is used by Twersky in electromagnetic scattering and after by Dassios in elastic case.

2 Scattering Problems

The direct scattering problem asks: given an open domain $V \subset \mathbb{R}^3$ with connected $\mathbf{C}^2$ boundary S and $V^e = \mathbb{R}^3 \setminus \overline{V}$, given a plane incident wave of time harmonic dependence $e^{-i\omega t}$

$$\mathbf{U}^i(\mathbf{r}, \hat{\mathbf{k}}) : V^e \to \mathbb{R}^4 \tag{1}$$

($\hat{\mathbf{k}}$ the direction of propagation), determine in V^e a solution

$$\mathbf{U}(\mathbf{r}, \hat{\mathbf{k}}) = \mathbf{U}^i(\mathbf{r}, \hat{\mathbf{k}}) + \mathbf{U}^s(\mathbf{r}, \hat{\mathbf{k}}) \tag{2}$$

of the equation

$$\tilde{\mathbf{L}}(\partial_r)\mathbf{U}(\mathbf{r}, \hat{\mathbf{k}}) = \begin{pmatrix} (\mu\Delta + \rho\omega^2)\tilde{\mathbf{I}}_3 + (\lambda + \mu)\nabla\nabla\cdot & -\gamma\nabla \\ q\kappa\eta\nabla\cdot & \Delta + q \end{pmatrix} \begin{pmatrix} \mathbf{u}(\mathbf{r}, \hat{\mathbf{k}}) \\ \Theta(\mathbf{r}, \hat{\mathbf{k}}) \end{pmatrix} = 0 \tag{3}$$

such that

$$\tilde{\mathbf{B}}_k(\partial_r, \hat{\mathbf{n}})\mathbf{U}(\mathbf{r}, \hat{\mathbf{k}}) = 0 \tag{4}$$

on S, where the boundary differential operator $\tilde{\mathbf{B}}_k(\partial_r, \hat{\mathbf{n}})$ is expressed via the thermoelastic surface traction operator.
More over $\mathbf{U}(\mathbf{r}, \hat{\mathbf{k}})$ has to satisfy the asymptotic Kupradze condition as $r \to \infty$ [7].
In the theory of direct problems in thermoelasticity [1,4], it is shown how the solution of a boundary value problem and related far-field corresponding to an incident field and to a given obstacle can be calculated. We call the set of vectors

$$\begin{aligned} &\left\{\mathbf{P}_0^j : \Omega \to \mathbb{C}^3; j = 1, 2, s\right\} = \\ &\qquad = \left\{P_{r0}^1(\hat{\mathbf{r}}, \hat{\mathbf{k}})\hat{\mathbf{r}}, P_{r0}^2(\hat{\mathbf{r}}, \hat{\mathbf{k}})\hat{\mathbf{r}}, P_{\theta 0}^s(\hat{\mathbf{r}}, \hat{\mathbf{k}})\hat{\theta} + P_{\phi 0}^s(\hat{\mathbf{r}}, \hat{\mathbf{k}})\hat{\phi}\right\} \end{aligned} \tag{5}$$

defined on the unite sphere Ω, as *set of thermoelastic far-field patterns* corresponding to the thermoelastic radiation solution $\mathbf{U}(\mathbf{r})$, propagating in the direction $\hat{\mathbf{k}}$. The thermal component is cancelled in this definition because of linear dependence between the module of asymptotic displacement fields $P^1_{r0}(\hat{\mathbf{r}},\hat{\mathbf{k}})$, $P^2_{r0}(\hat{\mathbf{r}},\hat{\mathbf{k}})$ and the asymptotic thermal fields $t^1_0(\hat{\mathbf{r}},\hat{\mathbf{k}}), t^2_0(\hat{\mathbf{r}},\hat{\mathbf{k}})$ [1,4] with known coefficient. Let us do the following symbolic notation $\mathbf{U}_\infty = \left\{\mathbf{P}^j_0, j=1,2,s\right\} \in [\mathbf{L}^2(\Omega)]^3$. In terms of the mapping

$$\mathcal{F}:\mathbf{U}\to\mathbf{U}_\infty \tag{6}$$

we want to solve the equation $\mathcal{F}\mathbf{U}=\mathbf{U}_\infty$
We have proved in [1] by means of Atkinson expansion theorem the following result known as ***the correspondence theorem***.

There exists one to one correspondence between:
elastothermal far-field pattern P^1_{r0} *and elastothermal 4-dimensional part of radiation solution* $\mathbf{U}^1$ *;*
thermoelastic far-field pattern P^2_{r0} *and thermoelastic 4-dimensional part of radiation solution* $\mathbf{U}^2$ *;*
transverse far-field patterns $P^s_{\theta 0}, P^s_{\phi 0}$ *and elastothermal 3-dimensional part of radiation solution* $\mathbf{U}^s=(\mathbf{u}^s,0)$ *.*
In other words, the mapping (6) is an one to one in its range.

Theorem 1. *The regular solution of thermoelastic equation (3) allows the following representation of the displacement part and the temperature part*

$$\mathbf{u}(\mathbf{r}) = \nabla\Big[-\frac{\lambda+2\mu}{\rho\omega^2}\Phi_1+\frac{\gamma}{\rho\omega^2}\Theta_1-\frac{\lambda+2\mu}{\rho\omega^2}\Phi_2+\frac{\gamma}{\rho\omega^2}\Theta_2\Big] + \nabla\times\left[\mathbf{r}\Psi(\mathbf{r})+\frac{1}{k_s}\nabla\times(\mathbf{r}\chi(\mathbf{r}))\right], \tag{7}$$

$$\Theta(\mathbf{r}) = \Theta_1(\mathbf{r})+\Theta_2(\mathbf{r})\,, \tag{8}$$

where the potentials $\Phi_1,\Phi_2,\Psi,\chi,\Theta_1,\Theta_2$ *solve the scalar Helmholtz equation*

$$(\Delta+k_1^2)\Phi_1 = (\Delta+k_1^2)\Theta_1 = 0, \tag{9}$$

$$(\Delta+k_2^2)\Phi_2 = (\Delta+k_2^2)\Theta_2 = 0, \tag{10}$$

$$(\Delta+k_s^2)\Psi = (\Delta+k_s^2)\chi = 0, \tag{11}$$

where k_1,k_2,k_3 *are wave numbers.*

The prove is a consequence of Kupradze decomposition [7] by straightforward calculation. We write again the above relations as $U=U_1+U_2+U_s$

$$\mathbf{U}_1 = \begin{pmatrix} \nabla[-\frac{\lambda+2\mu}{\rho\omega^2}\Phi_1+\frac{\gamma}{\rho\omega^2}\Theta_1] \\ \Theta_1 \end{pmatrix}, \tag{12}$$

$$\mathbf{U}_2 = \begin{pmatrix} \nabla[-\frac{\lambda+2\mu}{\rho\omega^2}\Phi_2+\frac{\gamma}{\rho\omega^2}\Theta_2] \\ \Theta_2 \end{pmatrix}, \tag{13}$$

$$\mathbf{U}_s = \begin{pmatrix} \nabla\times[\mathbf{r}\Psi(\mathbf{r})+\frac{1}{k_s}\nabla\times(\mathbf{r}\chi(\mathbf{r}))] \\ 0 \end{pmatrix}, \tag{14}$$

where the irrotational and solenoidal part of the displacement fields are distinguished.

3 Herglotz thermoelastic function

We restrict so far to the rigid at zero temperature scattering thermoelastic problem.

Definition 2. An *thermoelastic Herglotz function* is defined to be a classical solution of the thermoelastic equation (3) $\mathbf{U}(\mathbf{r})$ in all of $\mathbb{R}^3$, which satisfies the growth condition

$$\lim_{r\to\infty} sup \frac{1}{r}\int_{B(o,r)} \|\mathbf{U}(\mathbf{r}')\|^2 dv(\mathbf{r}') < \infty \,. \tag{15}$$

Using orthogonality of the vector spherical harmonics we can easily verify

Proposition 3.

$$[L^2(\Omega)]^3 = [L^2_r(\Omega)]^3 \oplus [L^2_t(\Omega)]^3 \,, \tag{16}$$

where $[L^2_r(\Omega)]^3$ *is the subspace spanned by the set* $\{\mathbf{P}_n^m\}$ *and* $[L^2_t(\Omega)]^3$ *is the subspace spanned by the set* $\{\mathbf{B}_n^m\} \cup \{\mathbf{C}_n^m\}$.

This implies that for every $\mathbf{f} \in [L^2_r(\Omega)]^3$ we have the unique expansion in L^2 sense

$$\mathbf{f}(\hat{\mathbf{r}}) = \mathbf{f}_r(\hat{\mathbf{r}}) + \mathbf{f}_t(\hat{\mathbf{r}}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} [\alpha_n^m \mathbf{P}_n^m(\hat{\mathbf{r}})] + [\beta_n^m \mathbf{B}_n^m(\hat{\mathbf{r}}) + \gamma_n^m \mathbf{C}_n^m(\hat{\mathbf{r}})] \,. \tag{17}$$

Moreover $\mathbf{U}(\mathbf{r}) = f_n(kr)\mathbf{Y}_n(\hat{\mathbf{r}})$ where $f_n(r)$ denotes any spherical Bessel function, is a solution of the Helmholtz equation $(\Delta + k^2)\mathbf{U}(\mathbf{r}) = 0$. Obviously, from these considerations we get

Proposition 4. *Every solution of thermoelastic equation in a regular domain satisfies the following unique decomposition.*

$$\begin{aligned}\mathbf{u}(\mathbf{r}) = \mathbf{u}_1(\mathbf{r}) + \mathbf{u}_2(\mathbf{r}) + \mathbf{u}_s(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} [\alpha_{1n}^m \mathbf{L}_{1n}^m(\mathbf{r}) +& \\ + \alpha_{2n}^m \mathbf{L}_{2n}^m(\mathbf{r}) + \beta_n^m \mathbf{M}_n^m(\mathbf{r}) + \gamma_n^m \mathbf{N}_n^m(\mathbf{r})] \,,&\end{aligned} \tag{18}$$

$$\begin{aligned}\Theta(\mathbf{r}) = \Theta_1(\mathbf{r}) + \Theta_2(\mathbf{r}) =& \\ = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \delta_{1n}^m f_n(k_1 r)\mathbf{Y}_n(\hat{\mathbf{r}}) + \delta_{2n}^m f_n(k_2 r)\mathbf{Y}_n(\hat{\mathbf{r}}) \,.&\end{aligned} \tag{19}$$

The convergence is considered to be in the L^2 *sense.*

The Navier elastostatic eigenvectors are given by

$$\mathbf{L}_{(1,2)n}^m(\mathbf{r}) = \frac{1}{k_{(1,2)}}\nabla\big[f_n(k_{(1,2)}r)\mathbf{Y}_n(\hat{\mathbf{r}})\big] = \\ = \Big[\frac{d}{d(k_{(1,2)}r)}f_n(k_{(1,2)}r)\Big]\mathbf{P}_n^m(\hat{\mathbf{r}}) + \\ + \sqrt{n(n+1)}\,\frac{f_n(k_{(1,2)}r)}{k_{(1,2)}r}\mathbf{B}_n^m(\hat{\mathbf{r}})\,, \qquad (20)$$

$$\mathbf{M}_n^m(\mathbf{r}) = \nabla\times\big[\mathbf{r}f_n(k_sr)\mathbf{Y}_n(\hat{\mathbf{r}})\big] = \sqrt{n(n+1)}f_n(k_sr)\mathbf{C}_n^m(\hat{\mathbf{r}})\,, \qquad (21)$$

$$\mathbf{N}_n^m(\mathbf{r}) = \frac{1}{k_s}\nabla\times\nabla\times\big[\mathbf{r}f_n(k_sr)\mathbf{Y}_n(\hat{\mathbf{r}})\big] = \\ = n(n+1)\frac{f_n(k_sr)}{k_sr}\mathbf{P}_n^m(\hat{\mathbf{r}}) + \\ + \sqrt{n(n+1)}\frac{1}{k_sr}\Big[\frac{d}{d(k_sr)}(k_srf_n(k_sr))\Big]\mathbf{B}_n^m(\hat{\mathbf{r}}) \qquad (22)$$

and satisfy the vector Helmholtz equation

$$(\varDelta + k_{(1,2)}^2)\mathbf{L}_{(1,2)n}^m = (\varDelta + k_s^2)\mathbf{M}_n^m = (\varDelta + k_s^2)\mathbf{N}_n^m = \mathbf{0}\,. \qquad (23)$$

We recall the result from [5] that there is a one to one correspondence between vector spherical harmonics and elastostatic eigenvectors. In the terminology that will be introduced latter, spherical harmonics $\mathbf{P}_n^m, \mathbf{C}_n^m, \mathbf{B}_n^m$ are the vector Herglotz kernels of the elastostatic eigenvectors $\mathbf{L}_n^m, \mathbf{M}_n^m, \mathbf{N}_n^m$ with $f_n = j_n$ respectively. In other words, if the entire elastostatic eigenvectors are to be decomposed in plane waves propagating in all directions, then the corresponding vector spherical harmonics provide the distribution of the amplitudes over directions. The same between the solid harmonics and spherical harmonics.
The behavior of Herglotz solution of thermoelastic equation (3) and in particular the connection of the its displacement part with the far-field patterns they generate are the main subject of the following.
Let $\mathbf{U}:\mathbb{R}^3\to\mathbb{C}^4$ an Herglotz function that satisfies the thermoelastic equation in the classical sense. Then, by the completeness of the elastostatic eigenvectors and spherical harmonics and the general theory of eigenfunction expansions we obtain for the displacement and temperature part the expansions (18), (19) respectively. The asymptotic analysis of the above expansions as $r\to\infty$ gives for the far-field patterns generated by the Herglotz solution the following formulas

$$\mathbf{L_1}(\hat{\mathbf{r}}) = \frac{ik_1}{2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^{-n}\alpha_{1n}^m\mathbf{P}_n^m(\hat{\mathbf{r}})\,, \qquad (24)$$

$$\mathbf{L_2}(\hat{\mathbf{r}}) = \frac{ik_2}{2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^{-n}\alpha_{2n}^m\mathbf{P}_n^m(\hat{\mathbf{r}})\,, \qquad (25)$$

$$\mathbf{T}(\hat{\mathbf{r}}) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n\sqrt{n(n+1)}[\beta_n^m \mathbf{C}_n^m(\hat{\mathbf{r}}) + i\gamma_n^m \mathbf{B}_n^m(\hat{\mathbf{r}})] \tag{26}$$

and for the temperature part

$$l_1(\hat{\mathbf{r}}) = \frac{ik_1}{2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^{-n}\delta_{1n}^m Y_n^m(\hat{\mathbf{r}}), l_2(\hat{\mathbf{r}}) = \frac{ik_2}{2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^{-n}\delta_{2n}^m Y_n^m(\hat{\mathbf{r}}). \tag{27}$$

By virtue of the Riesz-Fisher theorem and the relations involving the Fourier coefficients of (18), (19) we claim that *the far field patterns* $\mathbf{L}_{(1,2)}$, $\mathbf{T}$ *are well defined in the* L^2*-sense.* The coefficient of the linear dependence between the displacement far fields and temperature far fields is $\frac{q\kappa\eta ik_{(1,2)}}{k_{(1,2)}^2-q}$ see [1].
The most important result in the theory of Herglotz functions is given by the following representation theorem.

Theorem 5 (Representation). *If* $\mathbf{U}$ *is an Herglotz solution of the thermoelastic equation* (3)*, then there are functions* $\mathbf{L}_1, \mathbf{L}_2, \mathbf{T} : \Omega \to \mathbb{C}^4$ *which belongs to* $L^2(\Omega)$ *(i.e. the corresponding far field patterns), such that*

$$\mathbf{U}(\mathbf{r}) = \frac{1}{2\pi}\int_{\Omega}\mathbf{L}_1(\hat{\mathbf{k}})e^{k_1 i\hat{\mathbf{k}}\cdot\mathbf{r}}ds(\hat{\mathbf{k}}) + \frac{1}{2\pi}\int_{\Omega}\mathbf{L}_2(\hat{\mathbf{k}})e^{k_2 i\hat{\mathbf{k}}\cdot\mathbf{r}}ds(\hat{\mathbf{k}}) + \\ + \frac{1}{2\pi}\int_{\Omega}\mathbf{T}(\hat{\mathbf{k}})e^{k_s i\hat{\mathbf{k}}\cdot\mathbf{r}}ds(\hat{\mathbf{k}}). \tag{28}$$

Conversely, if $\mathbf{U}$ *is given by* $\mathbf{L}_1, \mathbf{L}_2, \mathbf{T}$ *in* $L^2(\Omega)$*, then it is a thermoelastic Herglotz function.*

The L^2 functions $\mathbf{L}_1, \mathbf{L}_2, \mathbf{T}$ are known as the Herglotz kernels. The proof argument of the first part is the interpretation of the series (24), (25), (26), (27) as the corresponding Herglotz kernels. Using orthogonality arguments of the considered eigenfunctions and the uniform convergence we obtain the growth condition of the Herglotz function provided the L^2 Herglotz kernels exist. Theorem of the representation furnishes a proof that $\mathbf{U}$ is uniquely determined by $\mathbf{L}_1, \mathbf{L}_2, \mathbf{T}$. This result is also obtainable from the unique determination of the Fourier coefficients of the expansions for $\mathbf{U}$ and $\mathbf{L}_1, \mathbf{L}_2, \mathbf{T}$ in the appropriate eigenvectors for both components, displacement and temperature.

4 Dense properties of the far field patterns

Let us turn back to the functional equation (6). The correspondence theorem provides the uniqueness of its solution. As we proved the thermoelastic far field patterns (which are defined as the displacement amplitudes) must satisfy the expantion (24), (25), (26). By using the technique of Colton, Kress [2] (theorem 2.15) and the fact that the elastostatic eigenvectors are expressed via spherical wave functions we easily verify that the existence of a solution requires a kind of growth condition of the Fourier coefficients (it is analytically complicated that is

why we do not present it here)to be satisfied, for a given function $\mathbf{U}_\infty \in [L^2(\Omega)]^3$. So, the solution of equation (6) will, in general, not exist. The argument of [2, p. 36] is valid here to claim more over that, if a solution $\mathbf{U}$ does exist it will not depend continuously on $\mathbf{U}_\infty$ in any reasonable norm. *That is that the equation* (3) *is ill-posed.* The image of the linear operator $\mathcal{F}$ is not equal to $[L^2(\Omega)]^3$.
But, is the far field patterns for a fixed rigid obstacle at zero temperature and all incident thermoelastic plane waves complete in $[L^2(\Omega)]^3$*?*
In order to have an answer of this question we need the following dense result. To avoid long repetitions of requirements upon fields, we introduce the following spaces:

- the space of incident Herglotz-type field

$$\mathcal{H}(\mathbb{R}^3) = \Big\{\Phi : \mathbb{R}^3 \to \mathbb{R}^4 \;\Big|\; \Phi(\mathbf{r}) = \\ = \int_\Omega \mathbf{L}_1(\hat{\mathbf{k}})e^{ik_1\hat{\mathbf{k}}\cdot\mathbf{r}} + \mathbf{L}_2(\hat{\mathbf{k}})e^{ik_2\hat{\mathbf{k}}\cdot\mathbf{r}} + \mathbf{T}(\hat{\mathbf{k}})e^{ik_s\hat{\mathbf{k}}\cdot\mathbf{r}}; \mathbf{L}_1, \mathbf{L}_2, \mathbf{T} \in L^2(\Omega)\Big\} \quad (29)$$

- the space of scattered fields

$$\mathcal{S}(V^e) = \big\{\mathbf{U} : V^e \to \mathbb{R}^4 \big| \; \mathbf{U} \in C^2(V^e) \cap C(\overline{V}^e) \text{ s.t. } \mathbf{U} \\ \text{satisfies the equation (3) and the Kupradze condition at } \infty\big\} \quad (30)$$

- the space of rigid at zero temperature solutions

$$\mathcal{P}(V^e) = \{\Psi = \mathbf{U} + \Phi : V^e \to \mathbb{R}^4 \mid \mathbf{U} \in \mathcal{S}, \Phi \in \mathcal{H}, \mathbf{r} \in \partial V \Psi(\mathbf{r}) = 0\} \quad (31)$$

- the space of traction traces

$$\tilde{\mathbf{R}}\mathcal{P}(\partial V) = \big\{\tilde{\mathbf{R}}\Psi : \partial V \to \mathbb{R}^4; \Psi \in \mathcal{P}\big\} \quad (32)$$

Theorem 6. *The space* $\tilde{\mathbf{R}}\mathcal{P}(\partial V)$ *is dense in* $L^2(\partial V)$

Proof. From [6, Theorem 2] we have that if $\mathbf{g} \in L^2(\partial V)$ and

$$(\tilde{T}\Psi, \overline{\mathbf{g}}) = \int_{\partial V} \overline{\mathbf{g}} \cdot \tilde{\mathbf{T}}(\partial_r, \hat{\mathbf{n}})\Psi(\mathbf{r})ds(\mathbf{r}) = 0$$

for every $\tilde{T}\Psi \in \tilde{\mathbf{R}}\mathcal{P}(\partial V) \cdot \tilde{\mathbf{I}}_3$, then $\mathbf{g} = 0$ almost everywhere on ∂V. The same for the operator ∂_n. Now, let us consider a 4-dimensional vector $\mathbf{G} \in L^2(\partial V)$. The shape of operator $\tilde{\mathbf{R}}(\partial_r, \hat{\mathbf{n}}) = \begin{pmatrix} \tilde{\mathbf{R}}(\partial_r, \hat{\mathbf{n}}) & -\gamma\hat{\mathbf{n}} \\ 0 & \partial_n \end{pmatrix}$ implies that $\mathbf{G} = 0$ if for every $\tilde{R}\Psi \in \tilde{\mathbf{R}}\mathcal{P}(\partial V)$,

$$(\tilde{R}\Psi, \overline{\mathbf{G}}) = \int_{\partial V} \overline{\mathbf{G}} \cdot \tilde{\mathbf{R}}(\partial_r, \hat{\mathbf{n}})\Psi(\mathbf{r})ds(\mathbf{r}) = 0$$

which ends the proof of the theorem.

It is well known that there are three types of plane displacement fields and tow types of plane temperature fields that can propagate in a thermoelastic medium. The displacement waves depend on two orthogonal vectors which are $\hat{\mathbf{k}}$ and $\hat{\mathbf{p}} \perp \hat{\mathbf{k}}$ respectively for two types of longitudinal waves and one type of transversal wave. These reflects the same vectorial nature of the far field patterns (5). The existence of two perpendicular vector complicates the study of the far field patterns. To avoid this difficulty we raise the rank of the tensorial character of the fields involved by one. Then the incident displacement fields depend only on one vector, the direction of propagation, while the transverse polarization vector is now replaced by 2-dimensional complement of the direction of propagation.
Our tensorial thermoelastic model is as following.
The scatterer is exited by a tensorial 4×3 thermoelastic time harmonic wave

$$\begin{aligned}\tilde{\mathbf{U}}^i(\mathbf{r};\hat{\mathbf{k}}) = A^1(\hat{\mathbf{k}} \otimes \hat{\mathbf{k}}, \beta_1 \hat{\mathbf{k}}) e^{ik_1 \hat{\mathbf{k}}\cdot\mathbf{r}} + A^2(\beta_2 \hat{\mathbf{k}} \otimes \hat{\mathbf{k}}, \hat{\mathbf{k}}) e^{ik_2 \hat{\mathbf{k}}\cdot\mathbf{r}} + \\ + A^s(\tilde{\mathbf{I}}_3 - \hat{\mathbf{k}} \otimes \hat{\mathbf{k}}, 0) e^{ik_s \hat{\mathbf{k}}\cdot\mathbf{r}} \,. \end{aligned} \tag{33}$$

The tensorial incident field can be interpreted as a tensor superposition of three vector fields which appear as the first vectors of tensors, while the second vectors are provided by the incident orthogonal base $\{\hat{\mathbf{k}}, \hat{\theta}_k, \hat{\varphi}_k\}$. This tensorial character of the incident field is inherited in the scattered field $\tilde{\mathbf{U}}^s = (\mathbf{u}^s, \Theta^s)$ it generates

$$\begin{aligned}\tilde{\mathbf{u}}^s(\mathbf{r};\hat{\mathbf{k}}) &= \mathbf{u}_1^s(\mathbf{r};\hat{\mathbf{k}},\hat{\mathbf{k}}) \otimes \hat{\mathbf{k}} + \mathbf{u}_2^s(\mathbf{r};\hat{\mathbf{k}},\hat{\mathbf{k}}) \otimes \hat{\mathbf{k}} + \\ &\quad + \mathbf{u}_s^s(\mathbf{r};\hat{\mathbf{k}},\hat{\theta}_k) \otimes \hat{\theta}_k + \mathbf{u}_s^s(\mathbf{r};\hat{\mathbf{k}},\hat{\varphi}_k) \otimes \hat{\varphi}_k \,, \\ \Theta^s(\mathbf{r};\hat{\mathbf{k}}) &= \Theta_1^s(\mathbf{r};\hat{\mathbf{k}})\hat{\mathbf{k}} + \Theta_2^s(\mathbf{r};\hat{\mathbf{k}})\hat{\mathbf{k}} \,. \end{aligned} \tag{34}$$

Then the total tensorial field

$$\tilde{\mathbf{U}}(\mathbf{r};\hat{\mathbf{k}}) = \tilde{\mathbf{U}}^i(\mathbf{r};\hat{\mathbf{k}}) + \tilde{\mathbf{U}}^s(\mathbf{r};\hat{\mathbf{k}}) \tag{35}$$

solve the tensorial thermoelastic coupled system

$$\mu\Delta\tilde{\mathbf{u}}(\mathbf{r};\hat{\mathbf{k}}) + (\lambda+\mu)\nabla \otimes \nabla \cdot \tilde{\mathbf{u}}(\mathbf{r};\hat{\mathbf{k}}) = \gamma \nabla \otimes \Theta(\mathbf{r};\hat{\mathbf{k}}) \,, \tag{36}$$

$$\Delta\Theta(\mathbf{r};\hat{\mathbf{k}}) + \frac{i\omega}{\kappa}\Theta(\mathbf{r};\hat{\mathbf{k}}) = -i\omega\nabla \cdot \tilde{\mathbf{u}}(\mathbf{r};\hat{\mathbf{k}}) \tag{37}$$

and the same Kupradze asymptotic conditions as $r \to \infty$. More over $\tilde{\mathbf{U}}(\mathbf{r};\hat{\mathbf{k}}) = \tilde{\mathbf{0}}$ on S.
The asymptotic analysis uniform over Ω leads to the tensorial shape of far field patterns

$$\tilde{\mathbf{U}}_\infty = \left\{\tilde{\mathbf{P}}_0^j, j = 1,2,s\right\} \in [\mathbf{L}^2(\Omega)]^9 \,. \tag{38}$$

Note that in accord with known results the radial patterns $\tilde{\mathbf{P}}_0^1, \tilde{\mathbf{P}}_0^2$ are the longitudinal wave of displacement part propagating along $\hat{\mathbf{r}}$ and $\tilde{\mathbf{P}}_0^s$ is transversal spherical wave propagating along $\hat{\mathbf{r}}$ and polarized orthogonally to $\hat{\mathbf{r}}$. Obviously,

the definition and the results of thermoelastic Herglotz function may be translated to the terms of tensors. Everything remains true provided the vectors are replaced by tensors.
An answer of the question whether is the set of far field patterns complete in $[L^2(\Omega)]^3$ is given by the following theorem

Theorem 7. *Let* $(\hat{\mathbf{k}}_n)$ *be a sequence of unit vectors that is dense on* Ω*.*

$$[\mathbf{L}^2(\Omega)]^9 = \overline{span\tilde{\mathbf{U}}_\infty(\hat{\mathbf{r}}, \hat{\mathbf{k}}_n)} \tag{39}$$

if and only if there does not exist a Herglotz thermoelastic function

$$\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \Theta \end{pmatrix} = \begin{pmatrix} \mathbf{u}^1 + \mathbf{u}^2 + \mathbf{u}^s \\ \Theta^1 + \Theta^2 \end{pmatrix} \tag{40}$$

such that

$$\tilde{\mathbf{u}} = \mathbf{u}^1 \otimes \hat{\mathbf{k}} + \mathbf{u}^2 \otimes \hat{\mathbf{k}} + \mathbf{u}^s \otimes \hat{\theta} + \mathbf{u}^s \otimes \hat{\varphi} \tag{41}$$

is an eigenfunction of the interior eigenvalue problem

$$\begin{array}{c} \left\{ \begin{array}{c} (\Delta^* + \omega^2)\tilde{\mathbf{u}} = 0 \\ \nabla \cdot \tilde{\mathbf{u}} = 0 \end{array} \right. \quad in\ V \\ \tilde{\mathbf{u}} = 0 \quad on\ S \end{array} \tag{42}$$

where Δ^* *is elastostatic operator.*

Proof. Recall that W is complete in the Hilbert space X if and only if $(w, \varphi) = 0$ for all $w \in W$ implies that $\varphi = 0$.
Let us write the dual relation in the space $[L^2(\Omega)]^9$ for every $\tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, \tilde{\mathbf{t}} \in [L^2(\Omega)]^9$

$$\int_\Omega \left[\tilde{\mathbf{P}}^1_{r0}(\hat{\mathbf{r}}; \hat{\mathbf{k}}_n) + \tilde{\mathbf{P}}^2_{r0}(\hat{\mathbf{r}}; \hat{\mathbf{k}}_n) + \tilde{\mathbf{P}}^s_{t0}(\hat{\mathbf{r}}; \hat{\mathbf{k}}_n)\right] : \\ : \left[\tilde{\mathbf{l}}_1(\hat{\mathbf{r}}; \hat{\mathbf{k}}_n) + \tilde{\mathbf{l}}_2(\hat{\mathbf{r}}; \hat{\mathbf{k}}_n) + \tilde{\mathbf{t}}(\hat{\mathbf{r}}; \hat{\mathbf{k}}_n)\right] ds(\hat{\mathbf{r}}) = \tilde{\mathbf{0}}\,, \tag{43}$$

which in a simpler symbolic way should be written

$$\int_\Omega \tilde{U}_\infty(\hat{\mathbf{r}}; \hat{\mathbf{k}}_n)\tilde{\mathbf{K}}(\hat{\mathbf{r}}; \hat{\mathbf{k}}_n) ds(\hat{\mathbf{r}}) = \tilde{\mathbf{0}}\,, \tag{44}$$

where $\hat{\mathbf{r}} \in \Omega$ and $n = 1, 2, \ldots$ The relation (43) implies that exists a nontrivial thermoelastic Herglotz function, which we take as incident wave $\tilde{\mathbf{U}}^i$, (with kernels for displacement part $\tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, \tilde{\mathbf{t}} \in [L^2(\Omega)]^9$) for which the far field patterns of the corresponding scattered wave $\tilde{\mathbf{U}}^s$ is $\tilde{\mathbf{U}}^\infty = 0$. By the one to one correspondence between the radial solution and corresponding far field patterns we have that the vanishing far field $\tilde{\mathbf{U}}^\infty = 0$ on Ω is equivalent to $\tilde{\mathbf{U}}^s = 0$ in V^e. By the boundary condition $\tilde{\mathbf{U}}^s + \tilde{\mathbf{U}}^i = 0$ on S and the uniqueness Dirichlet thermoelastic eigenvector problem [7], this is equivalent to $\tilde{\mathbf{U}}^i = 0$ on S. Again by Kupradze

[7], the first thermoelastic eigenvalue problem is equivalent with the eigenvalue problem (42). The above result gives the basic tool to construct a Colton-Monk type algorithm for the thermoelastic inverse scattering problem corresponding to a rigid scatterer at zero temperature of the reconstruction the structures. This inverse frame means: given the thermoelastic incident field $\mathbf{U}^i \in \mathcal{H}$ and by the knowledge of the far field patterns $\mathbf{U}_\infty$ correspond to the scattered field $\mathbf{U}^s \in \mathcal{S}(V^e)$, Dirichlet boundary condition $\Psi = 0$, $\Psi \in \mathcal{P}(V^e)$ and the equations (3) which govern the phenomena, one determines the geometrical shape of the boundary ∂V. Apriori conditions are given about the boundary, for example to be star shape.

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