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Numerical Analysis of High-Temperature Strains and Stresses in Superalloys

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Abstract. In [10] a new micromechanical approach to the prediction of creep flow in composites with perfect matrix/particle interfaces, making use of the nonlinear Maxwell viscoelastic model, taking into account a finite number of discrete slip systems in the matrix, has been suggested; high-temperature creep in such composites is conditioned by the dynamic recovery of the dislocation structure due to slip/climb motion of dislocations along the matrix/particle interfaces. In this contribution the proper formulation of the system of PDE's generated by this model is presented together with the overview of existence and uniqueness results, based on the properties of Rothe sequences, applied in the original non-commercial FEM software CDS; complete proofs will be published in [15].

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1 Introduction

The creep resistance of composites (metal reinforced by hard particles) depends strongly on the active diffusion processes on the interface between matrix and particles. Since in standard software packages all such processes are neglected, the conventional time-dependent strain and stress analysis at high temperature (e.g. making use of some "user supplied material description" in the ABAQUS finite element code as in [8]) gives no satisfactory results. To overcome this difficulty, in 1992–97 at the Institute of Physics of Materials of the Academy of Sciences of the Czech Republic in Brno the non-commercial PC software CDS (abbreviation for "creep with diffusion and sliding") in C++ language for the study of high-temperature phenomena in materials consisting of several phases has been developed; its functions have been demonstrated e.g. in [12]. One interesting application of this software to the design of metal matrix composites (where the particles are added into the matrix during solidification) has been

This is the final form of the paper.

published in [11]. In this contribution we pay attention to the case of superalloys (where the particles precipitate in the matrix during the heat treatment of the material).

The physical background of the processes of elastic deformation, slip in the matrix, deposition of dislocation on matrix/particle interfaces and recovery of dislocation structure is discussed in details in [10] (making use of some ideas from [9]), including the confrontation with other approaches to similar problems in the literature and software. The access of [10] leads to the formulation of 5 types of equations (to these types we shall refer later):

A) equations of the principle of virtual displacement rates for materials consisting of two phases,
B) constitutive equations for stress components, making use of the serial Maxwell model with one linear elastic and one nonlinear viscous parts,
C) compatibility conditions for geometrical configuration on the matrix/particle interfaces,
D) equations for the kinetics of the displacement between the matrix and the particle,
E) equations for the evolution of the dislocation density.

Our aim will be to formulate correctly this system of PDE's of evolution and to verify its solvability and convergence of sequences of Rothe functions, using the method of discretization in time.

The limited extent of this CD ROM text file (documenting the author's communication at the 9$^{\text{th}}$ Equadiff Conference in Brno with the same title — see its abstract [13]) does not admit to present neither detailed physical analysis nor complete mathematical proofs of lemmas and theorems. For more information about physical processes taken into consideration we must refer to [10] and [14]; proofs mentioned here can be found in [15].

2 Virtual configurations and stresses

To make the resulting system of equations as simple as possible, we shall formulate in an explicit way only the equations of type A) and B), containing two types of unknown abstract functions — actual stresses σ and actual displacement rates v mapping some closed time interval Θ with zero starting point into corresponding function spaces V of virtual displacement rates and S of virtual stresses in a deformable body, respectively. The spaces V and S must be defined carefully to include the equations of type C) — in other words, to respect both support boundary conditions and the special slip structure of deformation with discontinuities in matrix/particle interface configuration.

In the 3-dimensional Euclidean space R^3 let us consider an open set Ω, occupied by a deformable body, containing a finite number $N+1$ of open sets Ω_0 (creeping matrix) and $\Omega_1,\dots,\Omega_N$ (mutually separated elastic particles) such that all intersections (bars denote closures in R^3) of $\bar{\Omega}_i$, $\bar{\Omega}_j$ are empty for any $i,j \in \{1,\dots,N\}$ and $\bar{\Omega} = \bar{\Omega}_0 \cup \ldots \cup \bar{\Omega}_N$ holds and for each $i \in \{0,\dots,N\}$ the following requirements are fulfilled:

- the imbedding of $W_2^1(\Omega_i, R)$ into $L_2(\Omega_i, R)$ is compact ("Rellich theorem", see [6], p.204),
- if $i > 0$ then the imbedding of $W_2^1(\Omega_i, R)$ into $L_2(\Psi_i, R)$ is compact ("trace theorem", following [4] see [6], p.222),
- the condition of coerciveness of strains on Ω_i in sense of [7], p.75, is preserved;

the decomposition of a boundary $\partial\Omega_i = \Gamma_i \cup \Psi_i$ for $\Psi_i = \partial\Omega_i \setminus \partial\Omega$ and $\Gamma_i = \partial\Omega_i \setminus \Psi_i$ is used for simplicity. In the whole article we shall apply the standard notation of Sobolev spaces from [6] and of spaces of Bochner integrable abstract functions from [1].

Following the physical considerations from [10] we shall assume that creep in Ω_0 is active only in a finite number M of slip systems, characterized by slip directions $a^k \in R^3$ and normals to slip planes $c^k \in R^3$ for $k \in \{1, \ldots, M\}$. (It will be useful to introduce tensors b^k of order 2 such that $b^k_{ij} = \frac{1}{2}(a^k_i c^k_j + a^k_j c^k_i)$ for every $i, j \in \{1, 2, 3\}$ too.) Let U_{ik} be the set of all boundaries of non-empty 2-dimensional cuts of Ω_i by parallel planes $x \cdot c^k = \xi$ (centered dot symbols for scalar products in R^3 are applied) for any $\xi \in R$, $x \in R^3$ and certain $i \in \{1, \ldots, N\}$, $k \in \{1, \ldots, M\}$. Let

$$\Lambda \in \prod_{j=0}^{N} L_\infty(\Gamma_j, R^{3\times3})$$

(is sense of Cartesian products) be given (not variable in time) support characteristics (in extremal cases locally represented by a regular matrix for a Dirichlet boundary condition or by a zero matrix for a Neumann one). The evolution of deformation of Ω can be quantified using the the rate of displacement between the corresponding points of Ω in actual and reference (initial) configurations. In this way, making use of the Sobolev space

$$\breve{V} = \prod_{j=0}^{N} W_2^1(\Omega_j, R^3)\,,$$

we can define the **space of virtual displacement rates** V as the completion of the set of all

$$v \in \prod_{j=0}^{N} C^\infty(\bar{\Omega}_j, R^3)$$

with the properties

$$\forall\, i \in \{1, \ldots, N\}\ \forall\, k \in \{1, \ldots, M\}\ \forall\, \Upsilon \in U_{ik}\ \forall\, x, y \in \Upsilon \mid (\mathrm{D}v(x) - \mathrm{D}v(y)).a^k = 0$$

and (as a matrix multiplied by a vector in R^3)

$$\forall\, x \in \partial\Omega \ \mid\ \Lambda(x)v(x) \text{ is a zero point of } R^3$$

in the norm of $\breve{V}$; $\mathrm{D}v$ here has to be understood in the following sense: let x be a point of Ψ_i for some $i \in \{1, \ldots, N\}$, then $\mathrm{D}v(x)$ means the difference $v^i(x) - v^0(x)$

in sense of traces of functions $v^0 \in W_2^1(\Omega_0, R^3)$, $v^i \in W_2^1(\Omega_i, R^3)$ (see assumption (ii)); similarly Λ and v on $\partial\Omega$ can be represented by corresponding traces of v^0 or v^i.

The configuration changes, forced by external loads (that will be specified in the proceeding section), are closely connected with the redistribution of stresses in time. Unfortunately, the constitutive relations between strains (and strain rates) and stresses, based on the elastic deformation and on the creep flow in the matrix and modified by the matrix/particle interface diffusion, is not trivial; therefore (unlike the pure theory of elasticity) we have to introduce the **space of virtual stresses**

$$S = \prod_{j=0}^{N} L_2(\Omega_j, R^{3\times 3}_{\mathrm{sym}}) \,.$$

Evidently, $\check{V}$, V, and S are Hilbert spaces.

3 Equations of type A)

We shall assume that the strain and stress development in a deformable body in time is the consequence of superposition of the following 4 types of forces:

a) surface loads

$$\gamma \in W_2^2(\Theta, B) \quad \text{for} \quad B = \prod_{j=0}^{N} L_2(\Gamma_j, R^3) \,,$$

b) volume loads (body forces)

$$\varphi \in L_2(\Theta, H) \quad \text{for} \quad H = \prod_{j=0}^{N} L_2(\Omega_j, R^3) \,,$$

c) prescribed strains (e.g. generated by temperature changes or support motions)

$$\vartheta \in W_2^1(\Theta, S) \,,$$

d) forces caused by length changes of dislocation loops (for details see [10]) for particular slip systems

$$\eta^1, \ldots, \eta^M \in W_2^2(\Theta, Q) \quad \text{for} \quad Q = \prod_{j=1}^{N} L_2(\Psi_j, R) \,,$$

recalling that $W_2^1(\Theta, X)$ is a space of all abstract functions $v \in L_2(\Theta, X)$ mapping the time interval Θ to certain Hilbert space X such that $\dot{v} \in L_2(\Theta, X)$ too (dotted symbols are used for time derivatives). In this classification a), b) and d) will be incorporated into the principle of virtual displacement rates, c) will appear in the constitutive equations in the next section.

In special Hilbert spaces we shall apply the brief notation for scalar products

$$(.,.) \text{ in } H\,, \qquad \langle .,. \rangle \text{ in } B\,, \qquad [.,.] \text{ in } S\,, \qquad \lfloor .,. \rfloor \text{ in } Q$$

and for norms

$$|.|_{\mathrm{H}} \text{ in } H\,, \qquad |.|_{\mathrm{B}} \text{ in } B\,, \qquad |.|_{\mathrm{S}} \text{ in } S\,, \qquad |.|_{\mathrm{Q}} \text{ in } Q\,, \qquad \|.\| \text{ in } V$$

(as usually, $|.|$ are absolute values in R). For every $k \in \{0, \ldots, M\}$ let

$$q^k \in \prod_{j=1}^{N} L_2(\Psi_j, R)$$

be the magnitudes of (a priori unknown) contact loads in a^k directions on matrix/particle interfaces. Moreover, let

$$\rho \in \prod_{j=0}^{N} L_\infty(\Omega_j, R_+)$$

be the material density (needed for inertia forces $\rho\dot{v}$). Then we are ready to formulate the variational **principle of virtual displacement rates** in any time $t \in \Theta$ (for the sake of brevity the time variable t is not emphasized explicitly if there is no danger of misunderstanding)

$$\forall\, \tilde{v} \in V \;\mid\; (\tilde{v}, \rho\dot{v}) + [\varepsilon(\tilde{v}), \sigma] + \left\lfloor \mathrm{D}\tilde{v}.a^k, q^k \right\rfloor = (\tilde{v}, \varphi) + \langle \tilde{v}, \gamma \rangle + \left\lfloor \mathrm{D}\tilde{v}.a^k, \eta^k \right\rfloor \tag{1}$$

where Einstein summation rule for indices k is applied (for k not repeated if will be explicitly underlined in the same sense) and the symbol $\varepsilon(.)$ is reserved for the well-known small deformation tensor from the linearized theory of elasticity (see e.g. [7], p.33).

Now it is necessary to involve the equations of types D) and E). Let us define

$$S_1 = \prod_{j=1}^{N} L_2(\Omega_j, R)\,, \qquad T = \prod_{j=1}^{N} L_\infty(\Psi_j, R_+)\,.$$

Let us select an arbitrary k-th slip system for $k \in \{1, \ldots, M\}$. Let ρ^k (unlike the material density ρ) be the dislocation density on Ψ_k corresponding to such system. If in time $t = 0$ some dislocation density $\rho_0^k \in Q$ is a priori known then by [10] the values $\rho^k(t)$ in every time $t \in \Theta$ can be computed from the relation

$$\rho^k(t) = \rho_0^k + \int_0^t G(f(\sigma(t') : b^k), \mathrm{D}v(t').a^k)\, \mathrm{d}t'$$

(a binary operator : here means the sum of products of all corresponding elements of 2 tensors of order 2) where

$$G : Q \times Q \to Q$$

is some bounded mapping continuous in both variables and

$$f : S_1 \to Q$$

some compact and weakly continuous mapping; this is the equation of type E). (In general we cannot replace f by the standard trace operator, as no traces to some elements from $L_2(\Omega_j, R)$ with $j \in \{1, \dots, N\}$ may exist.) Finally, for some mapping

$$\beta : Q \to T$$

such that $g_1 \leq \beta(\tilde{\rho}) \leq g_2$ for any $\tilde{\rho} \in Q$ with certain $g_1, g_2 \in R_+$ ($g_1 \leq g_2$) Lipschitz continuous with a constant $\kappa \in R_+$ in sense

$$\forall\, \tilde{\rho}, \hat{\rho} \in Q \mid |\beta(\tilde{\rho}) - \beta(\hat{\rho})|_{\mathrm{T}} \leq \kappa\, |\tilde{\rho} - \hat{\rho}|_{\mathrm{Q}}$$

the equations of type D) can be considered in form

$$q^k = \beta(\rho^k)\mathrm{D}v.a^k\,.$$

Substituting these expressions into (1) we receive

$$\begin{aligned}\forall\, \tilde{v} \in V \ \mid\ & (\tilde{v}, \rho\dot{v}) + [\varepsilon(\tilde{v}), \sigma] + \\ & + \left\lfloor \mathrm{D}\tilde{v}.a^k, \beta\left(\rho_0^k + \int_0^t G(f(\sigma(t') : b^k), \mathrm{D}v(t').a^k)\ \mathrm{d}t'\right) \mathrm{D}v.a^k \right\rfloor = \\ & = (\tilde{v}, \varphi) + \langle \tilde{v}, \gamma \rangle + \left\lfloor \mathrm{D}\tilde{v}.a^k, \eta^k \right\rfloor\,. \end{aligned} \tag{2}$$

For the complete analysis of strain and stress distribution in time we need to know initial values $v_0 \in V$, $\sigma_0 \in S$ of abstract functions v and σ (their required properties will be precised in the next section) and initial values $\dot{v}_0$, $\dot{\sigma}_0$ of their derivatives too. It is easy to see that they cannot be chosen in an arbitrary way. Let us suppose that some $\varphi_\star \in H$ exists that

$$\forall\, \tilde{v} \in V \mid (\tilde{v}, \varphi_\star) = [\varepsilon(\tilde{v}), \sigma_0] + \langle \tilde{v}, \gamma(0) \rangle + \left\lfloor \mathrm{D}\tilde{v}.a^k, \beta(\rho_0^k)\mathrm{D}v_0.a^k - \eta^k(0) \right\rfloor$$

holds. Then it is possible to set

$$\dot{v}_0 = \frac{\varphi(0) - \varphi_\star}{\varrho}\,.$$

The analogy of the resulting relation

$$\begin{aligned}\forall\, \tilde{v} \in V \mid (\tilde{v}, \rho\dot{v}_0) + [\varepsilon(\tilde{v}), \sigma_0] + \left\lfloor \mathrm{D}\tilde{v}.a^k, \beta(\rho_0^k)\mathrm{D}v_0.a^k \right\rfloor = \\ = (\tilde{v}, \varphi(0)) + \langle \tilde{v}, \gamma(0) \rangle + \left\lfloor \mathrm{D}\tilde{v}.a^k, \eta^k(0) \right\rfloor \end{aligned} \tag{3}$$

with (2) is obvious.

4 Equations of type B)

The principle of virtual displacement rates from the preceding section assumes no special constitutive strain-stress relations (it is evident that σ has to be calculated using $\varepsilon(v)$, but no such algorithm is available). Therefore it is necessary to formulate them separately in the equations of type B). Equations presented in this and following sections without additional comment should be valid on Ω_i for any $i \in \{0, \dots, N\}$; this will be not repeated.

We shall come out from the Maxwell viscoelastic model (for the classification of rheological models see [5], p.143, variational principles are discussed in [16], p.593). For simplicity we shall suppose that there is no viscous component for $i = 0$. Then for any time $t \in \Theta$ we have

$$\varepsilon(v) = A\dot{\sigma} - \vartheta \tag{4}$$

where the a priori strains ϑ have been mentioned in the preceding section (as type c) loads) and

$$A \in \prod_{j=0}^{N} L_\infty(\Omega_j, R^{(3\times3)\times(3\times3)})$$

are conventional stiffness characteristics from the Hooke law that are symmetrical in sense

$$\forall\, \tilde{\sigma}, \hat{\sigma} \in S \;\mid\; [\tilde{\sigma}, A\hat{\sigma}] = [\hat{\sigma}, A\tilde{\sigma}]$$

and, moreover, positive definite in sense

$$\forall\, \tilde{\sigma} \in S \;\mid\; [\tilde{\sigma}, A\tilde{\sigma}] \geq \alpha\, |\tilde{\sigma}|_{\mathrm{S}}^2$$

where a constant $\alpha \in R_+$ must exist independently on the choice of $\tilde{\sigma} \in S$.

For $i \neq 0$ the dominant creep flow cannot be ignored. The resulting creep strain rate can be studied as a superposition of the strain rates corresponding to particular slip systems; one its form of practical importance (based on the Norton power-law relation with the initial Orowan stress barrier) has been suggested in [10] and [15]. In general we shall assume the existence of some weakly continuous mapping

$$F : \prod_{j=0}^{N} L_2(\Omega_j, R^{3\times3}_{\mathrm{sym}}) \to \prod_{j=0}^{N} L_2(\Omega_j, R^{3\times3}_{\mathrm{sym}})$$

such that a zero point of $R^{3\times3}_{\mathrm{sym}}$ is mapped to itself and there exists such $\zeta \in R_+$ that for all $\tilde{\sigma}, \hat{\sigma} \in S$

$$|F(\tilde{\sigma}) - F(\hat{\sigma})|_{\mathrm{S}} \leq \zeta\, |\tilde{\sigma} - \hat{\sigma}|_{\mathrm{S}}$$

holds. Using the mapping F we can write (as the generalization of (4)) for every time $t \in \Theta$ the **constitutive equations**

$$\varepsilon(v) = A\dot{\sigma} + F(\sigma) - \vartheta\,. \tag{5}$$

Obviously the validity of (5) must be preserved also in case $t = 0$. This influences the choice of admissible initial status. Since we have accepted no assumption concerning $\dot{\sigma}_0$ yet, we can consider

$$\dot{\sigma}_0 = A^{-1}\varepsilon(v_0) + A^{-1}\vartheta(0) - F(\sigma_0)$$

(A^{-1} is an inverse operator to A) which is nothing else the analogy of (5) in time $t = 0$

$$\varepsilon(v_0) = A\dot{\sigma}_0 + F(\sigma_0) - \vartheta(0)\,. \tag{6}$$

Let us remark that (5) and (6) can be easily transformed into the form similar to (2) and (3)

$$\forall\,\tilde{\sigma} \in S \;\mid\; [\tilde{\sigma}, \varepsilon(v)] - [\tilde{\sigma}, A\dot{\sigma}] - [\tilde{\sigma}, F(\sigma)] = -\,[\tilde{\sigma}, \vartheta]$$

for every $t \in \Theta$ and

$$\forall\,\tilde{\sigma} \in S \;\mid\; [\tilde{\sigma}, \varepsilon(v_0)] - [\tilde{\sigma}, A\dot{\sigma}_0] - [\tilde{\sigma}, F(\sigma_0)] = -\,[\tilde{\sigma}, \vartheta(0)]\;.$$

We can summarize that we have reached the following **main problem** (L indices are used to emphasize the Lipschitz continuity):

Problem 1. For given $v_0 \in V$, $\sigma_0 \in S$ and $\dot{v}_0 \in V$, $\dot{\sigma}_0 \in S$ preserving (3) and (6) to find such $v \in C_{\mathrm{L}}(\Theta, H) \cap L_\infty(\Theta, V)$ and $\sigma \in C_{\mathrm{L}}(\Theta, S)$ that $\dot{v} \in L_\infty(\Theta, H)$ and $\dot{\sigma} \in L_\infty(\Theta, S)$ and (2) and (5) are satisfied.

5 Existence and convergence results

Problem 1 can be studied using the standard technique of discretization in time. Let m be certain positive integer. In addition to a closed interval Θ from 0 to some real constant (final time) τ we shall consider also partial time intervals Θ_s from $(s-1)h$ (open) to sh (closed) for $h = \tau/m$ and $s \in \{1, \ldots, m\}$. Let X be a Hilbert space (supplied with a norm $|.|_{\mathrm{X}}$). To arbitrary elements $a_0, \ldots, a_m \in X$ for every $t \in \Theta$ we are able to set $\bar{a}^m(t) = \breve{a}^m(t) = a^m(t) = a_0$ and for every non-zero $t \in \Theta$ and $s \in \{1, \ldots, m\}$ also $\bar{a}^m(t) = a_s$, $\breve{a}^m(t) = a_{s-1}$ and $a^m(t) = a_s + (sh - t)\delta a_s$; the notation $\delta a_s = (a_s - a_{s-1})/h$ is used here for simplicity. If some $a \in L_2(\Theta, X)$ exists, it is natural similarly to $\bar{a}^m(t)$ to consider $\hat{a}^m(0) = a(0)$ and for every non-zero $t \in \Theta$ and $s \in \{1, \ldots, m\}$ also $\hat{a}^m(t) = a(sh)$; (in all situations with no danger of misunderstanding we shall write a_0 and a_s instead of $a(0)$ and $a(sh)$ too).

Now we shall try to substitute (2) by

$$\begin{aligned}\forall\,\tilde{v} \in V \;\;\mid\; &(\tilde{v}, \rho\dot{v}^m) + [\varepsilon(\tilde{v}), \bar{\sigma}^m] + \\ &+ \left\lfloor \mathrm{D}\tilde{v}.a^k, \beta\left(\int_0^t G(f(\breve{\sigma}^m(t') : b^k), \mathrm{D}\breve{v}^m(t').a^k)\;\mathrm{d}t'\right)\mathrm{D}\bar{v}^m.a^k \right\rfloor = \\ &= (\tilde{v}, \hat{\varphi}^m) + \langle \tilde{v}, \hat{\gamma}^m \rangle + \lfloor \mathrm{D}\tilde{v}.a^k, \hat{\eta}^{km} \rfloor\end{aligned} \tag{7}$$

and (5) by

$$\varepsilon(\bar{v}^m) = A\dot{\sigma}^m + F(\bar{\sigma}^m) - \hat{\vartheta}^m \,. \tag{8}$$

This trick leads to the decomposition of an original system of PDE's of evolution, step-by-step for $s \in \{1, \dots, m\}$, into m systems without any time variables — (7) yields

$$\begin{aligned} \forall\, \tilde{v} \in V \quad | \ (\tilde{v}, \rho\delta v_s) + [\varepsilon(\tilde{v}), \sigma_s] + \lfloor \mathrm{D}\tilde{v}.a^k, \beta(\rho^k_{s-1})\mathrm{D}v_s.a^k \rfloor = \\ = (\tilde{v}, \varphi_s) + \langle \tilde{v}, \gamma_s \rangle + \lfloor \mathrm{D}\tilde{v}.a^k, \eta^k_s \rfloor \end{aligned} \tag{9}$$

with the notation

$$\rho^k_s = \rho^k_0 + h \sum_{r=1}^{s} G(f(\sigma_r : b^k), \mathrm{D}v_r.a^k)$$

and (8) similarly

$$\varepsilon(v_s) = A\delta\sigma_s + F(\sigma_s) - \vartheta_s \,. \tag{10}$$

In this way we obtain a **discretized problem**:

Problem 2. For given $v_0, \dots, v_{s-1} \in V$ and $\sigma_0, \dots, \sigma_{s-1} \in S$ to find such $v_s \in V$ and $\sigma_s \in S$ that (9) and (10) are satisfied.

We need to study certain limit procedure from (7) and (8) to (2) and (5). Using a special technique of approximation of nonlinear operators by sequences of linear ones (compatible with the algorithms incorporated in CDS software) and standard Lax-Milgram and Eberlein-Shmul'yan theorems (see [17], pp.134, 201) we receive:

Theorem 3. *Problem 2 has always a solution.*

The following "discrete version" (similar to [3], p.29) of Gronwall lemma is needed frequently:

Lemma 4. *Let m be an arbitrary positive integer and $a_0, a_1, \dots, a_m$ some positive real numbers. There exist a positive real number $a_\star$ and a positive integer $m_\star$ such that if $m > m_\star$ then the relation*

$$\forall\, s \in \{1, \dots, m\} \ | \ a_s \le a_0 + h(a_1 + \ldots + a_s)$$

implies $a_s \le a_\star$ for any $s \in \{1, \dots, m\}$.

Let us select positive integers $m \in \{1, 2, \dots\}$ and $s \in \{1, \dots, m\}$ again; the constants c_0, c_1, c_2, c in following Lemmas 5, 6, 7 and Theorem 8 do not depend on this choice. The (nearly trivial) consequence of (10) (often applied together with the condition of coerciveness of strains) is:

Lemma 5. *There exists* $c_0 \in R_+$ *such that*

$$|\varepsilon(v_s)|_{\mathrm{S}}^2 \le c_0 \left(1 + |\sigma_s|_{\mathrm{S}}^2 + |\delta\sigma_s|_{\mathrm{S}}^2\right) .$$

The alternative setting $v = v_s$, $\sigma = \sigma_s$ and $v = \delta v_s$, $\sigma = \delta\sigma_s$ in (9) and (10) (with respect to (3) and (6) in the second case) leads to a couple of a priori estimates:

Lemma 6. *There exists* $c_1 \in R_+$ *such that*

$$|v_s|_{\mathrm{H}}^2 + |\sigma_s|_{\mathrm{S}}^2 \le c_1\left(1 + h\sum_{r=1}^{s}|v_r|_{\mathrm{H}}^2 + h\sum_{r=1}^{s}|\sigma_r|_{\mathrm{S}}^2 + h\sum_{r=1}^{s}|\delta\sigma_r|_{\mathrm{S}}^2\right).$$

Lemma 7. *If* (3) *and* (6) *are preserved then there exists* $c_2 \in R_+$ *such that*

$$\begin{aligned}&|\delta v_s|_{\mathrm{H}}^2 + |\delta\sigma_s|_{\mathrm{S}}^2 \le \\ &\qquad \le c_2\left(1 + |v_s|_{\mathrm{H}}^2 + |\sigma_s|_{\mathrm{S}}^2 + h\sum_{r=1}^{s}|v_r|_{\mathrm{H}}^2 + h\sum_{r=1}^{s}|\sigma_r|_{\mathrm{S}}^2 + h\sum_{r=1}^{s}|\delta\sigma_r|_{\mathrm{S}}^2\right).\end{aligned}$$

Lemmas 4, 5, 6, 7 together with Theorem 3 guarantee:

Theorem 8. *If* (3) *and* (6) *are preserved then there exists* $c \in R_+$ *such that*

$$\max\left(\|v_s\|, |\sigma_s|_{\mathrm{S}}, |\delta v_s|_{\mathrm{H}}, |\delta\sigma_s|_{\mathrm{S}}\right) \le c\,.$$

Let us construct all Rothe sequences needed in (7) and (8). Theorem 8 makes it possible to analyze their boundedness and equicontinuity properties. From the theory of Bochner integral (see [1], p.124, and [3], p.24), the Arzelà-Ascoli theorem (see [17], p.125, and [3], p.24) and some other facts from the functional analysis (e.g. Eberlein-Shmul'yan theorem) we are able to prove the (weak or strong) convergence of these sequences (or at least of their appropriate subsequences) to certain limits in spaces of abstract functions mapping Θ into corresponding Sobolev spaces. (For more precise formulations see [15].) It can be verified that these limit coincide with v and σ from (2) and (5) and their time derivatives. This yields:

Theorem 9. *Problem* 1 *has always a solution.*

The study of uniqueness of solution of Problem 1 is based on the following lemma, derived from (2) and (5) with help of the usual "continuous version" of Gronwall lemma (see [2], p.52, and [3], p.28):

Lemma 10. *All couples (v, σ), $(\bar{v}, \bar{\sigma})$ of solutions of Problem 1 satisfy the equation*

$$\frac{1}{2}(\Delta v, \rho \Delta v) + \frac{1}{2}[\Delta\sigma, A\Delta\sigma] + \int_0^t [\Delta\sigma(t'), F(\sigma(t')) - F(\bar{\sigma}(t'))]\,\mathrm{d}t' + \\ + \int_0^t \lfloor \mathrm{D}\Delta v(t').a^k, \beta(\bar{\rho}^k(t'))\mathrm{D}\Delta v.a^k \rfloor\,\mathrm{d}t' = \\ = \int_0^t \lfloor \mathrm{D}\Delta v(t').a^k, \left(\beta(\bar{\rho}^k(t')) - \beta(\rho^k(t'))\right)\mathrm{D}v(t').a^k \rfloor\,\mathrm{d}t'$$

in every time $t \in \Theta$ for $\Delta v = v - \bar{v}$ and $\Delta\sigma = \sigma - \bar{\sigma}$.

(The new dislocation density $\bar{\rho}^k$ here is supposed to be defined analogously to ρ^k with the same ρ_0^k for every $k \in \{1, \dots, M\}$). Let us introduce an additional growth condition

$$\forall\, \tilde{\varsigma}, \hat{\varsigma} \in S_1\, \forall\, \tilde{q}, \hat{q} \in Q \;\mid\; |G(f(\tilde{\varsigma}), \tilde{q}) - G(f(\hat{\varsigma}), \hat{q})|_{\mathrm{Q}} \leq \\ \leq \varpi\left(|\tilde{\varsigma} - \hat{\varsigma}|_{\mathrm{S}_1} + |\tilde{q} - \hat{q}|_{\mathrm{Q}}\right). \quad (11)$$

Then Lemma 10 yields the uniqueness result:

Theorem 11. *If (11) is valid then Problem 1 has exactly one solution.*

We can summarize: we have verified the solvability of problem of time evolution of strain and stress distributions in superalloy composites under some physically motivated assumptions and simplifications. This conclusion corresponds with the "reasonable" numerical results obtained for special material structures from CDS software; some of them have been published in [10] and [14].

References

[1] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie Verlag, Berlin, 1974.

[2] M. Greguš, M. Švec M., V. Šeda, *Obyčajné diferenciálne rovnice*, Alfa, Bratislava, 1985.

[3] J. Kačur, *Method of Rothe in evolution equations*, Teubner Verlag, Leipzig, 1985.

[4] J. Kačur, *Solution to strongly nonlinear parabolic problem by a linear approximation scheme*, lectures at the Department of Mathematics of the Faculty of Mechanical Engineering, Technical University in Brno, 1996.

[5] V. Kolář, *Nelineární mechanika*, ČSVTS, Ostrava, 1985.

[6] V. G. Maz'ya, *Prostranstva S. L. Soboleva*, Izdatel'stvo Leningradskogo Universiteta, Leningrad (St. Petersburg), 1985.

[7] J. Nečas, I. Hlaváček, *Úvod do teorie pružných a pružně plastických těles*, SNTL, Prague, 1983.

[8] H. E. Petterman, H. J. Böhm, F. G. Rammerstorfer, *An elasto-plastic constitutive law for composite materials*, in: Modelling in materials science and processing (M. Rappaz, M. Kedro, eds.), European Commission (COST 512 Action Management Committee), Brussels, 1996, 384–392.

[9] J. Svoboda, P. Lukáš, *Modelling of kinetics of directional coarsening in Ni-superalloys*, Acta Materialia, **44** (1996), 2557–2565.

[10] J. Svoboda, J. Vala, *Micromodelling of creep in composites with perfect matrix/particle interfaces*, Modeling in Materials Science and Engineering, to appear.

[11] J. Vala, J. Svoboda, V. Kozák, J. Čadek, *Modelling discontinuous metal matrix composite behavior under creep conditions: effect of diffusional matter transport and interface sliding*, Scripta Metallurgica et Materialia, **30** (1994), 1201–1206.

[12] J. Vala, *Programový systém CDS pro analýzu napjatosti a deformace vícefázových materiálů*, Proceedings of the Summer School Programs and Algorithms of Numerical Mathematics **8** in Janov nad Nisou (1996), 199–206.

[13] J. Vala, *Numerical analysis of high-temperature strains and stresses in superalloys*, Abstracts of the Equadiff **9** Conference in Brno (1997), 86.

[14] J. Vala, *Micromechanical considerations in modelling of superalloy creep flow*, Abstracts of the Conference Numerical Modelling in Continuum Mechanics **3** in Prague (1997), 69, and Proceedings, in print.

[15] J. Vala, *On one mathematical model of creep in superalloys*, Applications of Mathematics, in print.

[16] K. C. Valanis, *A gradient theory of finite viscoelasticity*, Archives of Mechanics, **49** (1997), 589–609.

[17] K. Yosida, *Funkcional'nyj analiz*, Mir, Moscow, 1967.