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Mathematical Models of Suspension Bridges: Existence of Unique Solutions

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Abstract. In this paper, we try to explain two mathematical models describing a dynamical behaviour of suspension bridges such as Tacoma Narrows Bridge. Our attention is concentrated on their analysis concerning especially the existence of a unique solution. Finally, we include an interpretation of particular parameters and a discussion of known and obtained results. This paper is based on our diploma thesis which deals with a qualitative study of dynamical structures of this type.

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1 Introduction and historical review

One of the most problematic and not fully explained areas of mathematical modelling involves nonlinear dynamical systems, especially systems with so called jumping nonlinearity. It can be seen that its presence brings into the whole problem unexpected difficulties and very often it is a cause of multiple solutions.

An example of such a dynamical system can be a suspension bridge. The nonlinear aspect is caused by the presence of supporting cable stays which restrain the movement of the center span of the bridge in a downward direction, but have no influence on its behaviour in the opposite direction.

Our paper sets a goal to develop a simple model describing the behaviour of the suspension bridge, to make its analysis which means to determine under what conditions the existence of a unique stable solution is guaranteed, and to find out safe parameters of the bridge constructions.

We do not try to model the bridge in its full complexity, but on the other hand, we would like to avoid some over-simplifications. That is why we consider only one dimensional model and neglect the torsional motion, but we do not simplify the problem even more — e.g. by eliminating the space variable at all.

This is the final form of the paper.

As a result of this effort, we describe the behaviour of the suspension bridge by one beam equation, or by a system of two coupled equations of "string-beam" type, respectively.

As a motivation of our interest, we can mention the event which changed radically the common view of these nonlinear dynamical systems.

On July 1, 1940, the Tacoma Narrows bridge in the state of Washington was completed and opened to traffic. From the day of its opening the bridge began to undergo vertical oscillations, and it was soon nicknamed "Galloping Gertie". As a result of its novel behaviour, traffic on the bridge increased tremendously. People came from hundreds of miles to enjoy riding over a galloping, rolling bridge. For four months, everything was all right, and the authorities in charge became more and more confident of the safety of the bridge. They were even planning to cancel the insurance policy on the bridge.

At about 7:00 a.m. of November 7, 1940, the bridge began to undulate persistently for three hours. Segments of the span were heaving periodically up and down as much as three feet. At about 10:00 a.m., the bridge started suddenly oscillating more wildly. At one moment, one edge of the roadway was twenty-eight feet higher than the other; the next moment it was twenty-eight feet lower than the other edge. At 10:30 a.m. the bridge began cracking, and finally, at 11:00 a.m. the entire structure fell down into the river.

The federal report on the failure of the Tacoma Narrows suspension bridge points out that the essentially new feature of this bridge was its extreme flexibility. Already, the Golden Gate bridge exhibited travelling waves, or in the Bronx Whitestone Bridge, large amplitude oscillations were observed of such a magnitude to make a traveller seasick. But due to a combination of damping and readjusted stays, they were not considered threatening to the structure.

As soon as the more flexible Tacoma Narrows bridge was built, it began to exhibit complex oscillatory motion with an order of magnitude higher than that of earlier mentioned bridges. This resulted in a pronounced tendency to oscillate vertically, under widely differing wind conditions. The bridge might be quiet in winds of forty miles per hour, and might oscillate with large amplitude in winds as low as three or four miles per hour. These vertical oscillations were standing waves of different nodal types. They were not considered to be dangerous, and it was expected that the bridge would be stabilized by a combination of the same devices as in case of the Bronx Whitestone bridge.

The second type of oscillation was observed just before the collapse of the bridge. It was a pronounced torsional mode with some of the cables alternately loosening and tightening. Sometimes the oscillations even preferred one end of the bridge to the other. These phenomenons caused that a large portion of the center span fell into the river.

Subsequently, the entire structure was destroyed, and a new, much more expensive bridge of more conventional and less flexible design was built in its place.

The first standard explanation (see e.g. M. Braun [4]) claims that the frequency of a periodic force caused by *alternating trailing vortices* just happened to be very close to the natural frequency of the bridge, and caused the linear *resonance*. Thus, even though the magnitude of the forcing term was small, this could explain the large oscillations and eventual collapse of the bridge.

However, the federal report includes the following paragraph:

> "It is very improbable that resonance with alternating vortices plays an important role in the oscillations of suspension bridges. First, it was found that there is no sharp correlation between wind velocity and oscillation frequency, as is required in the case of resonance with vortices whose frequency depends on the wind velocity. Second there is no evidence for the formation of alternating vortices at a cross section similar to that used in the Tacoma bridge ... It seems that it is more correct to say that the vortex formation and frequency is determined by the oscillation of the structure than that the oscillatory motion is induced by the vortex formation."

But the precise cause of the large-scale oscillations of suspension bridges has not been satisfactorily explained yet.

The aspect which distinguishes the suspension bridges is their fundamental nonlinearity. As we have mentioned above, it is caused by the presence of supporting cable stays which restrain the movement of the center span in a downward direction, but have no influence on its behaviour in the opposite direction.

This type of nonlinearity, often called *jumping* or *asymmetric*, has given rise to the following principle:

> Systems with asymmetry and large uni-directional loading tend to have multiple oscillatory solutions: the greater the asymmetry, the larger the number of oscillatory solutions, the greater the loading, the larger the amplitude of the oscillations.

As we mentioned above, our paper tries to analyze such nonlinear dynamical systems and to bring some new pieces of information into this area.

First of all, we present two possibilities how to model suspension bridges — by a single beam and by a beam coupled with a vibrating string by nonlinear cables — and give a brief survey of known facts in this field.

Then we introduce our own results concerning existence and uniqueness of time-periodic solutions of two chosen models. We use two different attitudes. The first one is based on the Banach contraction theorem which needs some restrictions on the bridge parameters. The second one works in relatively greater generality but with an additional assumption of sufficiently small external forces.

In the end, we summarize our intention and results and make a short discussion where we compare our foundations with known facts.

We would like to emphasis that this paper is a short abstract of our diploma thesis [21] and that is why it does not contain proofs of the assertions stated here.

2 Mathematical models and known results

One of the easiest ways how to model a behaviour of a suspension bridge is to consider only one dimension. We do not have to take into account the other two dimensions because proportions of the bridge in these dimensions are very small in comparison with its length and so can be omitted (see Fig. 1). If we also neglect the influence of the towers and side parts, we can use a model of a simply supported one-dimensional beam.

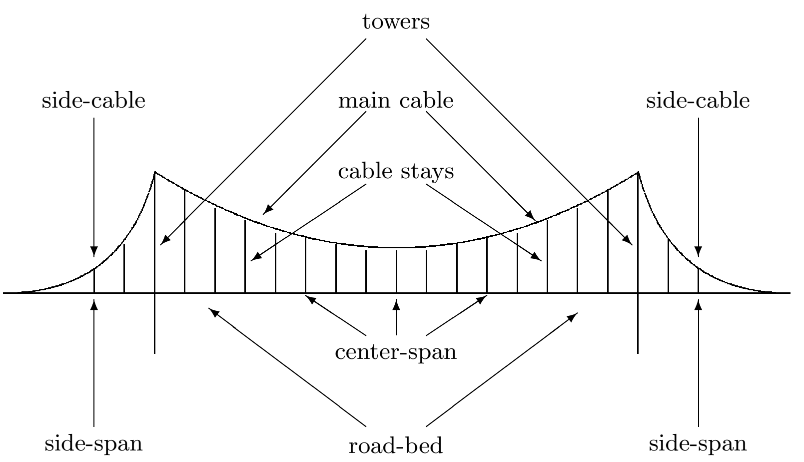

Fig. 1. *The main ingredients in a model of a one-dimensional suspension bridge.*

2.1 Single beam equation

In the first idealization, the construction holding the cable stays can be taken as a solid and immovable object. Then we can describe the behaviour of the suspension bridge by a vibrating beam with simply supported ends. It is subjected to the gravitation force, to the external periodic force (e.g. due to the wind) and in an opposite direction to the restoring force of the cable stays hanging on the solid construction. Our system is illustrated on Fig. 2.

The displacement $u(x,t)$ of this beam is described by nonlinear partial differential equation:

$$m\frac{\partial^2 u(x,t)}{\partial t^2} + EI\frac{\partial^4 u(x,t)}{\partial x^4} + b\frac{\partial u(x,t)}{\partial t} = -\kappa u^+(x,t) + W(x) + \varepsilon f(x,t), \tag{1}$$

with the boundary conditions

$$\begin{aligned} &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \\ &u(x,t+2\pi) = u(x,t), \quad -\infty < t < \infty, \ x \in (0,L). \end{aligned} \tag{2}$$

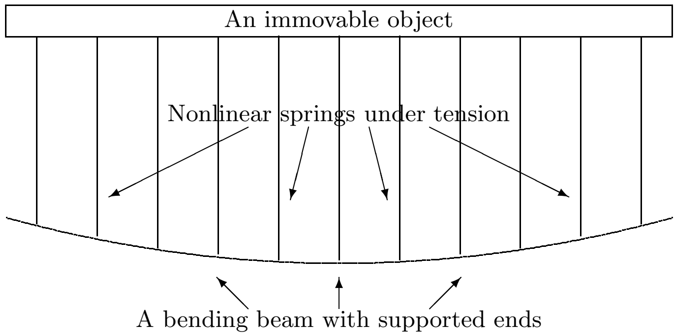

Fig. 2. *The simplest model of a suspension bridge — the bending beam with simply supported ends, held by nonlinear cables, which are fixed on an immovable construction.*

The meaning of particular parameters used in the equation is the following:

m	mass per unit length of the bridge,
E	Young's modulus,
I	moment of inertia of the cross section,
b	damping coefficient,
κ	stiffness of the cables (spring constant),
W	weight per unit length of the bridge,
εf	external time-periodic forcing term (due to the wind),
L	length of the center-span of the bridge.

As we can see from the equation (1) and the boundary conditions (2), we are describing vibrations of a beam of length L, with simply supported ends. Its deflection $u(x,t)$ at the point x and at time t is measured in the downward direction. The first term in the equation represents an inertial force, the second term is an elastic force and the last term on the left hand side describes a viscous damping. On the right hand side, we have the influence of the cable stays, the gravitation force and the external force due to the wind (we assume it to be time-periodic). The cable stays can be taken as one-sided springs, obeying Hooke's law, with a restoring force proportional to the displacement if they are stretched, and with no restoring force if they are compressed. This fact is described by the expression κu^+, where $u^+ = \max\{0, u\}$ and κ is a coefficient, which characterizes the stiffness of the cable stays.

We have not considered the inertial effects of the rotation motion (in a plane xu) in the equation since they are usually omitted.

This model was introduced e.g. in a paper [16] by P. J. McKenna and A. C. Lazer and is used as a starting point for study of suspension bridges in the most of cited works by the other authors. It does not describe exactly the behaviour of a suspension bridge but on the other hand it is reasonably simple and applicable.

For further considerations, it would be useful to transform the equation (by making a change of the scale of the variable x and dividing by the mass m) to the following form:

$$\begin{aligned} &u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + k u^+ = W(x) + \varepsilon f(x,t), \\ &u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \\ &u(x,t+2\pi) = u(x,t), \ \ -\infty < t < \infty, \ x \in (0,\pi), \end{aligned} \tag{3}$$

where $\alpha^2 = \frac{EI}{m}\left(\frac{\pi}{L}\right)^4 \neq 0$ and $\beta = \frac{b}{m} > 0$. (We use the same symbols for rescaled W, ε and f.)

As for as the results, which are known for this model, we can mention the theorem proved (by the degree theory) in [5] by P. Drábek.

It says that the problem (3) has at least one solution for an arbitrary right hand side. Further, there is proved that in case that there is no external force (it means no wind), the bridge achieves a unique position (called the equilibrium) determined only by its weight $W(x)$. Under some special assumptions on $W(x)$, the paper [5] shows that in case of small external disturbances, there is always a solution "near" to the equilibrium. If we assume that $W(x) = W_0 \sin x$ and a periodic function $f(x,t)$ is of a special form then there is another solution which is in a certain sense "far" from this position.

Another known result concerns the case when the damping term is equal to zero. This was studied by W. Walter and P. J. McKenna in paper [19]. Under an additional assumption $\alpha = 1$ they proved the theorem which says that if $W(x) \equiv W_0$ (positive constant) and $f(x,t)$ is even and π-periodic in the time variable t and symmetric in the space variable x about $\frac{\pi}{2}$, then, if $0 < k < 3$, the equation (3) has a unique periodic solution of the period π, which corresponds to small oscillations about the equilibrium. If $3 < k < 15$, the equation has in addition another periodic solution with a large amplitude.

In other words, this theorem says that strengthening the stays, which means increasing the coefficient k, can paradoxically lead to the destruction of the bridge.

The similar result can be proved for the system of ordinary differential equations which we obtain from the equation (3) using the spatial discretization. The theorem proved in [1] by J. M. Alonso and R. Ortega says that if the condition

$$k < \beta^2 + 2\alpha\beta$$

holds then there exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$ then the discretization of a suspension bridge equation has a unique bounded solution that is exponentially asymptotically stable in the large.

This result has a similar sense as the previous one — the more flexible the cable stays are, then the better the situation is and oscillations of the bridge cannot be too high.

2.2 "String-beam" system

Another possible but a little more complicated process is not to consider the construction holding the cable stays as an immovable object, but to treat it as a vibrating string, coupled with the beam of the roadbed by nonlinear cable stays (see Fig. 3).

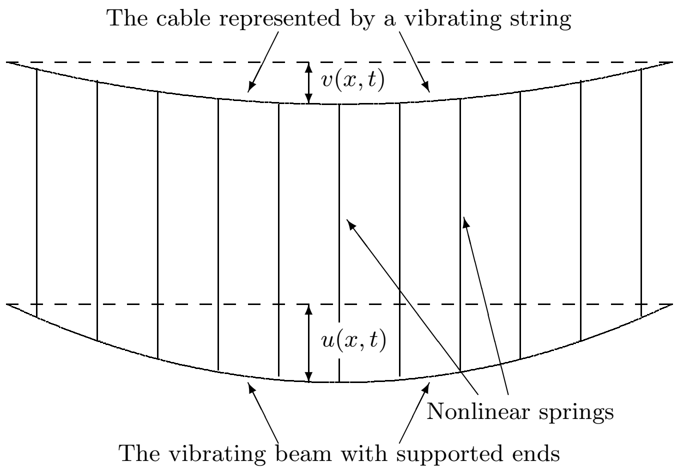

Fig. 3. *A more complicated model of a one-dimensional suspension bridge — the coupling of the main cable (a vibrating string) and the roadbed (a vibrating beam) by the stays, treated as nonlinear springs.*

Instead of one equation, we have now a system of two connected equations in the following form:

$$\begin{aligned} m_1 v_{tt} - T v_{xx} + b_1 v_t - \kappa (u-v)^+ &= W_1 + \varepsilon f_1(x,t), \\ m_2 u_{tt} + EI u_{xxxx} + b_2 u_t + \kappa (u-v)^+ &= W_2 + \varepsilon f_2(x,t), \end{aligned} \tag{4}$$

with boundary conditions

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = v(0,t) = v(L,t) = 0,$$

where $v(x,t)$ measures the displacement of the vibrating string representing the main cable and $u(x,t)$ means — as in the previous section — the displacement of the bending beam standing for the roadbed of the bridge. Both functions are considered to be periodic in the time variable. The nonlinear stays connecting the beam and the string pull the cable down, hence we have the minus sign in front of $k(u-v)^+$ in the first equation, and hold the roadbed up, therefore we consider the plus sign in front of the same term in the second equation.

We can transform both equations into a simpler form in the same way as in the previous section. It means that we divide by the mass m_1, and m_2 respectively, and change the scale of the space variable x. Then we obtain

$$\begin{aligned} &v_{tt} - \alpha_1^2 v_{xx} + \beta_1 v_t - k_1(u-v)^+ = W_1 + \varepsilon f_1(x,t), \\ &u_{tt} + \alpha_2^2 u_{xxxx} + \beta_2 u_t + k_2(u-v)^+ = W_2 + \varepsilon f_2(x,t), \\ &u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = v(0,t) = v(\pi,t) = 0, \\ &-\infty < t < \infty, x \in (0,\pi), \end{aligned} \tag{5}$$

where $\alpha_1^2 = \frac{T}{m_1}\left(\frac{\pi}{L}\right)^2$, $\alpha_2^2 = \frac{EI}{m_2}\left(\frac{\pi}{L}\right)^4$, $k_1 = \frac{\kappa}{m_1}$, $k_2 = \frac{\kappa}{m_2}$, $\beta_1 = \frac{b_1}{m_1}$ and $\beta_2 = \frac{b_2}{m_2}$. We use the same symbols as in the previous equations for the other transformed parameters.

We can find a description of this model again in A. C. Lazer and P. J. McKenna [16], but these authors consider the right hand sides in a rather purer form. In the first equation, they neglect the weight of the string W_1, and on the other hand, in the second equation, they ignore the external force $\varepsilon f_2(x,t)$. However, nobody (as far as we know) has treated this model in detail yet.

3 Application of Banach contraction principle

As we can see from the previous survey of known results, one of the problems is to prove the existence of the solutions of particular models and find out the conditions, under which the solution is unique and stable. In particular, it means that we are looking for conditions which guarantee that the bridge cannot exhibit large-scale oscillations and cannot be destructed by any wind of an arbitrary power. We have tried to clear up these problems with use of Banach contraction principle for both one-dimensional models — the first one considers the bridge as a single beam supported by nonlinear springs, and the second one describes the bridge as a beam coupled with a string by nonlinear cables.

3.1 The first case — a single beam

As we stated above, we model the suspension bridge as a one-dimensional beam with simply supported ends, which is held by nonlinear springs hanging on an immovable construction. This situation is described by the boundary value problem (3).

Let us denote $\Omega = (0,\pi) \times (0,2\pi)$ the considered domain, $H = L_2(\Omega)$ the usual Hilbert space with the corresponding L_2-norm

$$\|u(x,t)\| = \left[\int_\Omega |u(x,t)|^2 \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}}$$

and $\mathcal{D}$ the set of all smooth functions satisfying the boundary conditions from equation (3). Now we can generalize the notion of a classical solution by which we mean a continuous function with continuous derivatives up to the fourth order with respect to x and up to the second order with respect to t in the set $[0,\pi] \times [0,2\pi]$, satisfying the boundary value problem (3), and define a so called generalized solution of (3).

Definition 1. A function $u(x,t) \in H$ is called a generalized solution of the boundary value problem (3) if and only if the integral identity

$$\int_\Omega u(v_{tt} + \alpha^2 v_{xxxx} - \beta v_t)\,\mathrm{d}x\mathrm{d}t = \int_\Omega (W + \varepsilon f \ - ku^+)v\,\mathrm{d}x\mathrm{d}t$$

holds for all $v \in \mathcal{D}$.

Remark 2. We can extend the generalized solution $u = u(x,t)$ by 2π-periodicity in t to $(0,\pi)\times\mathbb{R}$. So, any generalized solution can be regarded as a function defined on $(0,\pi)\times\mathbb{R}$.

Let us consider a complex Sobolev space $\tilde{H} = H + iH$. As the set

$$\{\mathrm{e}^{int}\sin mx; n \ \in \mathbb{Z}, m \in \mathbb{N}\}$$

forms a complete orthogonal system in this space, each function $u(x,t)$ can be represented by Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} u_{nm}\mathrm{e}^{int}\sin mx. \tag{6}$$

Moreover, we have

$$\sum_n\sum_m |u_{nm}|^2 < \infty, \quad \text{and} \quad u_{-nm} = \bar{u}_{nm}$$

(see J. Berkovits and V. Mustonen [3]).

Let $p, r \in \mathbb{Z}^+$. If we use this Fourier interpretation, we can define the following spaces

$$H^{p,r} = \{h \in H; \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}(n^{2r} + m^{2p})|h_{nm}|^2 < \infty\} \tag{7}$$

and the corresponding norm

$$\|h\|_{H^{p,r}} = \left(\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}(n^{2r} + m^{2p})|h_{nm}|^2\right)^{\frac{1}{2}}. \tag{8}$$

Then $H^{p,r}$ equipped with the norm $\|\cdot\|_{H^{p,r}}$ is the Sobolev space. In particular, $H^{0,0} = H$.

First of all, we will treat the solvability of the linear equation

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t - \lambda u \ = h. \tag{9}$$

If we define a generalized solution of this equation in an analogous way as in Definition 1, then the following lemma is a consequence of the expansion (6) (cf. [2]).

Lemma 3. *If u_{nm} and h_{nm} are the corresponding Fourier coefficients of the functions u and h, then the equation (9) has a generalized solution if and only if*

$$(-n^2 + \alpha^2 m^4 + \mathrm{i}\beta n \ - \lambda) u_{nm} = h_{nm} \tag{10}$$

holds for all $n \in \mathbb{Z}$, $m \in \mathbb{N}$.

If we denote

$$L(u) = u_{tt} + \alpha^2 u_{xxxx} + \beta u_t$$

the linear operator, and put

$$N_\lambda = \{(m,n) \in \mathbb{N} \times \mathbb{Z};\ \alpha^2 m^4 - n^2 - \lambda = 0\},$$

$$S \ = \{\lambda \in \mathbb{R};\ N_\lambda \neq \emptyset\},$$

$$\sigma = \{\lambda \in \mathbb{R};\ \lambda = \alpha^2 q^4, q \in \mathbb{N}\},$$

then σ is a set of eigenvalues of the operator L, and $\sigma \subset S$ holds. Further, we can rewrite the equation (9) into a new form

$$L(u) - \lambda u \ = h$$

and formulate the following theorem (for the proof see G. Tajčová [20]).

Theorem 4. *Let $\lambda \in \mathbb{R}$. Then for an arbitrary $h \in H$ the equation (9) has a unique generalized solution $u \in H$ if and only if*

$$\lambda \notin \sigma.$$

If $\lambda \notin \sigma$, then there exists a mapping

$$T_\lambda \ : \ H \to H, \quad T_\lambda \ : \ h \mapsto u$$

with the following properties:

(i) *T_λ is linear and $\mathcal{R}(T_\lambda) \subset C(\bar{\Omega})$;*

(ii) *$T_\lambda : H^{p,r} \to H^{p+2,r+1}$ and there exists a constant $c > 0$ such that for any $h \in H^{p,r}$, $p, r \in \mathbb{N} \cup \{0\}$, we have*

$$\|u\|_{H^{p+2,r+1}} \leq c \|h\|_{H^{p,r}},$$

whenever $u = T_\lambda h$;

(iii) *T_λ is compact from H into $C(\bar{\Omega})$ (and thus from H into H) and for its norm we have*

$$\|T_\lambda\| \leq \frac{1}{\max\{dist(\lambda, S),\ \min\{\beta, dist(\lambda,\sigma)\}\}} = \frac{1}{\min\{dist(\lambda,\sigma),\ \max\{\beta, dist(\lambda, S)\}\}}.$$

Now we turn our attention to the equation (3) and deal with its solvability.

As zero is not an eigenvalue of the operator L, we can rewrite this equation — in accordance with the previous paragraph — into an equivalent form

$$u \ = T_0(-ku^+ + W + \varepsilon f). \tag{11}$$

Moreover, we have for the norm of the operator T_0 the following estimate

$$\|T_0\| \leq \max_{m\in\mathbb{N},\ n\in\mathbb{Z}} \frac{1}{\sqrt{\beta^2 n^2 + (\alpha^2 m^4 - n^2)^2}} \leq \frac{1}{\min\{\alpha^2, \beta\}} = K_0.$$

If we want to find out conditions for the existence of a unique solution, it is suitable to use the *Banach contraction principle* which reads as follows:

Let the operator $G : H \to H$ be a contraction, i.e. there exists $c \in (0, 1)$ such that

$$\|G(u) - G(v)\| \leq c\|u - v\| \quad \forall u, v \in H.$$

Then there exists a unique u_0 such that

$$G(u_0) = u_0.$$

In our case $G(u) = T_0(-ku^+ + W + \varepsilon f)$ and

$$\begin{aligned}\|G(u) - G(v)\| &= \|T_0(W + \varepsilon f - ku^+) - T_0(W + \varepsilon f - kv^+)\| = \\ &= \|T_0(kv^+ - ku^+)\| \leq \\ &\leq k\|T_0\|\|v^+ - u^+\| \leq \\ &\leq kK_0\|v - u\|.\end{aligned}$$

If we require the operator G to be a contraction, the condition

$$0 < kK_0 < 1$$

must be satisfied, and thus

$$0 < \frac{k}{\min\{\alpha^2, \beta\}} < 1.$$

Hence, if we put again $k = \frac{\kappa}{m}$, a sufficient condition for the existence of a unique solution of our boundary value problem has a form

$$\boxed{\kappa < m \cdot \min\{\alpha^2, \beta\}.} \tag{12}$$

3.2 The second case — the coupling of a beam and a string

In this part we complete the previous model by a movable main cable, which holds the nonlinear cable stays and which is represented by a vibrating string. Our model is described by a coupled system of partial differential equations (5) (see A. C. Lazer, P. J. McKenna [16])

If we introduce a new vector function

$$\mathbf{w} = \begin{bmatrix} v \\ u \end{bmatrix}, \tag{13}$$

we can rewrite the system (5) into the following matrix form

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}} \mathbf{w}_{tt} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \alpha_2^2 \end{bmatrix}}_{\mathbf{A}_2} \mathbf{w}_{xxxx} + \underbrace{\begin{bmatrix} -\alpha_1^2 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{A}_1} \mathbf{w}_{xx} + \\ + \underbrace{\begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}}_{\mathbf{B}} \mathbf{w}_t + \mathbf{F}(\mathbf{w}) = \underbrace{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}_{\mathbf{h}}, \tag{14}$$

and thus

$$\mathbf{w}_{tt} + \mathbf{A}_2 \mathbf{w}_{xxxx} + \mathbf{A}_1 \mathbf{w}_{xx} + \mathbf{B}\mathbf{w}_t + \mathbf{F}(\mathbf{w}) = \mathbf{h}, \tag{15}$$

where $\mathbf{F}(\mathbf{w})$ is a nonlinear vector function

$$\mathbf{F}(\mathbf{w}) = \begin{bmatrix} -k_1(u \ - v)^+ \\ k_2(u \ - v)^+ \end{bmatrix}.$$

Moreover, we require the unknown function $\mathbf{w}(x,t)$ to be time-periodic and to satisfy the boundary conditions prescribed for a vibrating string in its first component, and the boundary conditions prescribed for a supported beam in its second component.

Let us denote

$$\mathbf{L}(\mathbf{w}) = \mathbf{w}_{tt} + \mathbf{A}_2 \mathbf{w}_{xxxx} + \mathbf{A}_1 \mathbf{w}_{xx} + \mathbf{B}\mathbf{w}_t.$$

Then $\mathbf{L}$ is a linear operator and the equation (15) can be written in the following way

$$\mathbf{L}(\mathbf{w}) = -\mathbf{F}(\mathbf{w}) + \mathbf{h}. \tag{16}$$

The set of all real eigenvalues of the operator $\mathbf{L}$ has a form

$$\sigma = \{\lambda \in \mathbb{R};\ \lambda = \alpha_1^2 m^2 \ \vee \ \lambda = \alpha_2^2 m^4,\ \forall m \in \mathbb{N}\}.$$

Now, we have our system described by an operator equation which has a similar character as the operator equation for the single beam. It allows us to use the same methods and formulate the analogous statements as in previous section.

We can again define the notion of generalized solution and use the Fourier representation of the considered functions. Moreover, we can as well prove the existence of the resolvent $\mathbf{T}_\lambda$ (see G. Tajčová [20]).

Theorem 5. *Let $\lambda \in \mathbb{R}$. Then for an arbitrary $\mathbf{h} \in \mathbf{H}$ the equation $\mathbf{Lw} - \lambda\mathbf{w} = \mathbf{h}$ has a unique solution $\mathbf{w} \in \mathbf{H}$ if and only if*

$$\lambda \notin \sigma.$$

If $\lambda \notin \sigma$ then there exists the mapping

$$\mathbf{T}_\lambda \;:\; \mathbf{H} \to \mathbf{H}, \quad \mathbf{T}_\lambda \;:\; \mathbf{h} \mapsto \mathbf{w}$$

with the following properties:

(i) $\mathbf{T}_\lambda$ *is linear and* $\mathrm{Im}\mathbf{T}_\lambda \subset C(\bar{\Omega}) \times C(\bar{\Omega})$*;*

(ii) $\mathbf{T}_\lambda : H^{p,r} \times H^{p,r} \to H^{p+1,r+1} \times H^{p+2,r+1}$ *and there exists a constant $c > 0$ such that for any $\mathbf{h} \in H^{p,r} \times H^{p,r}$, $p, r \in \mathbb{N} \cup \{0\}$, we have*

$$\|\mathbf{w}\|_{H^{p+1,r+1} \times H^{p+2,r+1}} \le c\|\mathbf{h}\|_{H^{p,r} \times H^{p,r}},$$

whenever $\mathbf{w} = \mathbf{T}_\lambda \mathbf{h}$.

(iii) $\mathbf{T}_\lambda$ *is compact from $\mathbf{H}$ into $C(\bar{\Omega}) \times C(\bar{\Omega})$ (and thus from $\mathbf{H}$ into $\mathbf{H}$), and for its norm we have an estimate*

$$\|\mathbf{T}_\lambda\| \le \max\left\{\max_{m,n} \frac{1}{|A^\lambda_{nm}|};\ \max_{m,n} \frac{1}{|B^\lambda_{nm}|}\right\},$$

where
$$A^\lambda_{nm} = -n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n - \lambda,$$
$$B^\lambda_{nm} = -n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n - \lambda.$$

As zero is not the eigenvalue of the operator $\mathbf{L}$, we can define the operator $\mathbf{T}_0$ and to estimate its norm as follows

$$\|\mathbf{T}_0\| \le \max\left\{\max_{m,n} \frac{1}{|A^0_{nm}|};\ \max_{m,n} \frac{1}{|B^0_{nm}|}\right\}.$$

Further,

$$\begin{aligned}
\max_{m,n} \frac{1}{|A^0_{nm}|} &= \max_{m,n} \frac{1}{|-n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n|} = \max_{m,n} \frac{1}{\sqrt{\beta_1^2 n^2 + (\alpha_1^2 m^2 - n^2)^2}} \le \\
&\le \frac{1}{\min\{\alpha_1^2, \beta_1\}}, \\
\max_{m,n} \frac{1}{|B^0_{nm}|} &= \max_{m,n} \frac{1}{|-n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n|} = \max_{m,n} \frac{1}{\sqrt{\beta_2^2 n^2 + (\alpha_2^2 m^4 - n^2)^2}} \le \\
&\le \frac{1}{\min\{\alpha_2^2, \beta_2\}}.
\end{aligned}$$

Hence we finally obtain

$$\|\mathbf{T}_0\| \leq \max\left\{\frac{1}{\min\{\alpha_1^2,\beta_1\}};\ \frac{1}{\min\{\alpha_2^2,\beta_2\}}\right\} = \\ = \frac{1}{\min\left\{\alpha_1^2,\alpha_2^2,\beta_1,\beta_2\right\}} = \bar{K}_0. \quad (17)$$

If we use this operator $\mathbf{T}_0$, we can rewrite our equation (16) in the equivalent form

$$\mathbf{w} = \mathbf{T}_0(\mathbf{h} - \mathbf{F}(\mathbf{w})). \quad (18)$$

Since we want to prove its unique solvability, it is again suitable to apply the Banach contraction principle.

In our case $\mathbf{G}(\mathbf{w}) = \mathbf{T}_0(\mathbf{h} - \mathbf{F}(\mathbf{w}))$. We have to verify, whether this operator is a contraction:

$$\begin{aligned}\|\mathbf{G}(\mathbf{w}_1) - \mathbf{G}(\mathbf{w}_2)\| &= \|\mathbf{T}_0(\mathbf{h} - \mathbf{F}(\mathbf{w}_1)) - \mathbf{T}_0(\mathbf{h} - \mathbf{F}(\mathbf{w}_2))\| = \\ &= \|\mathbf{T}_0\|\|\mathbf{F}(\mathbf{w}_2) - \mathbf{F}(\mathbf{w}_1)\| \leq \\ &\leq \|\mathbf{T}_0\|(k_1 + k_2)\|(u_2 - v_2)^+ - (u_1 - v_1)^+\| \leq \\ &\leq \|\mathbf{T}_0\|(k_1 + k_2)\|(u_2 - v_2) - (u_1 - v_1)\| = \\ &= \|\mathbf{T}_0\|(k_1 + k_2)\|(u_2 - u_1) - (v_2 - v_1)\| \leq \\ &\leq \|\mathbf{T}_0\|(k_1 + k_2)\left[\|u_2 - u_1\| + \|v_2 - v_1\|\right] \leq \\ &\leq (k_1 + k_2)\bar{K}_0\|\mathbf{w}_2 - \mathbf{w}_1\|.\end{aligned}$$

Hence it follows that the operator $\mathbf{G}$ is a contraction if the condition

$$0 < (k_1 + k_2)\bar{K}_0 < 1$$

holds. Equivalently,

$$k_1 + k_2 < \min\left\{\alpha_1^2, \alpha_2^2, \beta_1, \beta_2\right\}.$$

As we have $k_1 = \frac{\kappa}{m_1}$, $k_2 = \frac{\kappa}{m_2}$, we obtain a condition of the existence of a unique solution of the operator equation (16) in the following form

$$\boxed{\kappa < \frac{m_1 m_2}{m_1 + m_2}\min\left\{\alpha_1^2, \alpha_2^2, \beta_1, \beta_2\right\}.} \quad (19)$$

Remark 6. The question left is whether the condition (19) is stronger or weaker than the condition

$$\kappa = m_2 k \ < m_2 \min\{\alpha_2^2, \beta_2\},$$

obtained by the same way for the bridge modelled only as a supported beam (i.e. by a scalar equation — see (12)), and whether they have any practical sense.

4 General existence and uniqueness result

In the previous chapter, we proved the existence and uniqueness of the solution, but the price we had to pay was a certain restriction on the coefficient κ representing the stiffness of the nonlinear cable stays. On the other hand, the advantage was an arbitrary right hand side. Now we can convert the situation and prove the mentioned existence and uniqueness of the solution in a relative generality of the structure coefficient, but with some special assumptions on the external forcing terms.

We again pay our attention to two chosen mathematical models of suspension bridges. The first one consists of the single beam equation and the second one respects the coupling of the main cable and the roadbed — i.e. the string-beam system.

4.1 The first case — a single beam

We again consider the periodic-boundary value problem (3) for the beam equation which serves as a simple one-dimensional model of a suspension bridge.

Before we state our main result, we formulate some auxiliary assertions which are necessary for its full understanding and which are proved by J. Berkovits, P. Drábek, H. Leinfelder, V. Mustonen and G. Tajčová in [2].

Proposition 7. *Let $u \in H$ and $h \in H$, h is independent of t. Then u is a unique generalized solution of*

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ = h(x) \tag{20}$$

if and only if the function u is independent of the variable t and $\tilde{u}(x) = u(x,t)$ is a classical solution of the boundary value problem

$$\begin{aligned} &\alpha \tilde{u}^{(4)} + k\tilde{u}^+ = h(x) \quad \text{in } (0,\pi), \\ &\tilde{u}(0) = \tilde{u}(\pi) = \tilde{u}''(0) = \tilde{u}''(\pi) = 0. \end{aligned} \tag{21}$$

Under even more special assumption that the right hand side is a constant function, we can prove some other properties of the generalized solution.

Proposition 8. *Assume in (3) that $\varepsilon = 0$ and $W(x) \equiv W_0$ (nonzero constant). Then the corresponding generalized solution u_0 of (3) is unique, positive, time-independent, symmetric with respect to the line $x = \frac{\pi}{2}$ and satisfies*

$$(u_0)_x(0,t) > 0, \quad (u_0)_x(\pi,t) < 0 \tag{22}$$

for every $t \in \mathbb{R}$.

Remark 9. In particular, this means that the equation

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ = 0$$

has due to uniqueness only a trivial generalized solution for any $k \in \mathbb{R}$.

Our main result is the following.

Theorem 10. *Let $\varepsilon \in \mathbb{R}$, $k > 0$, $W(x) \equiv W_0 > 0$, $f \in H^{1,1}$. Then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the problem (3) has a unique generalized solution $u \in H^{3,2}$. Moreover, this generalized solution is strictly positive in $(0, \pi) \times \mathbb{R}$.*

The proof of this main result would be carried out in several steps. We know that there exists at least one generalized solution of the equation (3) for any right hand side (see P. Drábek [5]). Moreover, by Proposition 8, there exists a positive, time-independent solution $u_0(x,t) = \tilde{u}_0(x)$ of the equation

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + k u^+ = W_0,$$

with $\tilde{u}_0'(0) > 0$ and $\tilde{u}_0'(\pi) < 0$.

Step 1. *We prove that there exists a positive generalized solution $u \in H^{3,2}$ of (3) which is "close" to u_0 from Proposition 8 with respect to the norm in $H^{3,2}$.*

Step 2. *There is no other positive generalized solution of (3) than $u = u_0 + u_\varepsilon$.*

Step 3. *There is no other generalized solution of (3) (changing signs) than $\tilde{u}_\varepsilon = u_0 + u_\varepsilon$ if $|\varepsilon| < \varepsilon_0$ and ε_0 is small enough.*

(For the complete proof see G. Tajčová [21] or J. Berkovits, P. Drábek, H. Leinfelder, V. Mustonen and G. Tajčová [2].)

4.2 The second case — the coupling of a beam and a string

Now, we can try to apply the previous ideas on the system of two coupled equations which model the suspension bridge as a simply supported beam and a string connected by nonlinear cable stays.

We work again with a periodic-boundary value problem (5). We would like to formulate a similar assertion as in the previous section, it means to prove under some additional assumptions that if the weight of the bridge W_1 and the weight of the main cable W_2 are constant and the external forces $\varepsilon f_1(x,t)$ and $\varepsilon f_2(x,t)$ are sufficiently small, then our problem (5) has a unique solution which is symmetric and strictly positive in its both components and close to the stationary solution. However, as it can be seen later, we are not able to overcome some problems with regularity of the solution and thus we formulate statements which are more general and — in some sense — weaker.

Similar argument as that used in [2] enables us to prove the following assertion.

Proposition 11. *Let $u, v \in H$ and $h_1,\ h_2 \in H$, h_1, h_2 are independent of t. Then $[v, u]^{\mathrm{T}}$ is a generalize solution of*

$$\begin{aligned} v_{tt} - \alpha_1^2 v_{xx} + \beta_1 v_t - k_1(u-v)^+ =& h_1(x), \\ u_{tt} + \alpha_2^2 u_{xxxx} + \beta_2 u_t + k_2(u-v)^+ =& h_2(x) \end{aligned} \tag{23}$$

if and only if the functions $v,\ u$ are independent of the variable t and $[\tilde{v}(x), \tilde{u}(x)]^{\mathrm{T}} = [v(x,t), u(x,t)]^{\mathrm{T}}$ is a solution of the boundary value problem

$$\begin{aligned} &-\alpha_1 \tilde{v}'' - k_1(\tilde{u}-\tilde{v})^+ = h_1(x), \\ &\alpha_2 \tilde{u}^{(4)} + k_2(\tilde{u}-\tilde{v})^+ = h_2(x) \quad \text{in } (0,\pi), \\ &\tilde{v}(0) = \tilde{v}(\pi) = \tilde{u}(0) = \tilde{u}(\pi) = \tilde{u}''(0) = \tilde{u}''(\pi) = 0. \end{aligned} \tag{24}$$

As for as the uniqueness of the solution, the following statement holds.

Proposition 12. *Let $k_1,\ k_2 > 0$ and $h_1,\ h_2 \in H$, h_1 and h_2 are independent of t. Then (23) has at most one generalized solution $\mathbf{w}_0 = [v_0, u_0]^{\mathrm{T}} \in \mathbf{H}$ which is independent of t.*

Remark 13. As a consequence of Propositions 11 and 12, we can state that for $\varepsilon = 0$ and $W_1 = W_2 = 0$ (it means no loading), the nonlinear system (5)

$$\begin{aligned} v_{tt} - \alpha_1^2 v_{xx} + \beta_1 v_t - k_1(u-v)^+ &= 0, \\ u_{tt} + \alpha_2^2 u_{xxxx} + \beta_2 u_t + k_2(u-v)^+ &= 0, \end{aligned}$$

with standard string-beam boundary conditions, has only a trivial solution.

Now, we have all auxiliary assertions to formulate the following theorem concerning the general existence of a solution of the system (5) for an arbitrary right hand side. The proof is based on the degree theory and is a direct analogy to the proof by P. Drábek in [5].

Theorem 14. *Let $\varepsilon \in \mathbb{R}$, $k_1,\ k_2 > 0$, $W_1(x),\ W_2(x) \in L_2(0,\pi)$, and $f_1(x,t)$, $f_2(x,t) \in H$. Then the system (5) has at least one generalized solution $\mathbf{w} = [v, u]^{\mathrm{T}} \in \mathbf{H}$.*

Now, we can have a look at the case when the right hand sides are constant functions. It means that the corresponding solution is (according to Proposition 11) a stationary solution and should express the equilibrium of the suspension bridge.

By a detailed analysis of a linear system

$$\begin{aligned} &-\gamma_1 v'' - u + v = h_1, \quad x \in (0,\pi), \\ &\gamma_2 u^{(4)} + u - v = h_2, \\ &v(0) = v(\pi) = u(0) = u(\pi) = u''(0) = u''(\pi) = 0 \end{aligned} \tag{25}$$

we can prove the following assertion.

Proposition 15. *Assume in a boundary value problem (5) that $W_1(x) \equiv W_1$ and $W_2(x) \equiv W_2$ are nonzero constants and $\varepsilon = 0$. Moreover, let the weight W_2 is "large enough". Then (5) has a unique generalized solution $\mathbf{w}_0$ which is positive, time-independent, symmetric with respect to the line $x = \frac{\pi}{2}$ in its both components and satisfies*

$$u_0(x,t) > v_0(x,t) \qquad \forall (x,t) \in (0,\pi) \times \mathbb{R}$$

and

$$(u_0 - v_0)_x(0,t) > 0, \qquad (u_0 - v_0)_x(\pi,t) < 0$$

for every $t \in \mathbb{R}$.

Now, we can have a closer look at the solution of the system (5) and its properties. We would like — on the basis of the previous statements — to formulate the analogy of Theorem 10.

However, the only thing we know is that there exists at least one generalized solution of the boundary value problem (5) and, moreover, (see Proposition 15), that under some additional assumptions, there exists a symmetric, strictly positive, time-independent solution $\mathbf{w}_0 = [v_0, u_0]^{\mathrm{T}}$ of the system

$$\begin{aligned} v_{tt} - \alpha_1^2 v_{xx} + \beta_1 v_t - k_1(u-v)^+ &= W_1, \\ u_{tt} + \alpha_2^2 u_{xxxx} + \beta_2 u_t + k_2(u-v)^+ &= W_2, \end{aligned}$$

where W_1 and W_2 are positive constants, and the conditions

$$(u_0 - v_0)_x(0,t) > 0, \qquad (u_0 - v_0)_x(\pi,t) < 0,$$

hold.

But we are not able to prove the existence and uniqueness of the solution of the system (5) which would be "close" to this $\mathbf{w}_0$. The obstacle is the fact that we have not manage to prove a better regularity that $\mathbf{w} \in H^{2,2} \times H^{3,2}$. It means (due to embedding theorems) that $\mathbf{w} \in C^{0,0} \times C^{1,0}$. And this is not enough to guarantee the existence of the positive solution

$$\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_\varepsilon$$

neither for ε sufficiently small.

5 Final remarks and discussion

In this chapter, we would like to clear up our results and compare them with known facts mentioned in Chapter 2.

Our main effort was to determine sufficient conditions for the existence and uniqueness of the solution. Let us have a closer critical look at them.

5.1 The application of Banach contraction principle

First of all, we dealt with the uniqueness using the Banach contraction principle. The price we had to pay was a certain restriction on the magnitude of the stiffness κ of the cable stays.

For the single beam model, it was in the roughest form (cf. (12))

$$\kappa < m \ \min\{\alpha^2, \beta\}. \tag{26}$$

The corresponding result for the string-beam model was (cf. (19))

$$\kappa < \frac{m_1 m_2}{m_1 + m_2} \min\{\alpha_1^2, \alpha_2^2, \beta_1, \beta_2\}. \tag{27}$$

In Chapter 2, we mentioned two similar results. The first one was obtained by W. Walter and P. J. McKenna in [19] for a non-damped single beam model under an additional assumption $\alpha = 1$. It says that the solution of such a system is unique in case that

$$0 < k < 3, \tag{28}$$

where $k = \kappa/m$.

The second result was obtained by J. M. Alonso and R. Ortega in [1] for a discrete system of ordinary differential equations derived from a damped single beam model using the spatial discretization by finite differences. It says again that the solution of such a system is unique if

$$k < \beta^2 + 2\alpha\beta. \tag{29}$$

We have again $k = \kappa/m$.

Obviously, all these results have a similar sense — the more flexible the cable stays are, the better the situation is, because the nonlinearity is less pronounced and we have guaranteed the uniqueness of the solution.

We can make short discussion where we compare our result (26) with (29) derived by J. M. Alonso and R. Ortega, and the results for single beam with that ones for a string-beam model.

We ask whether the result (29)

$$k \ < \beta^2 + 2\beta\alpha$$

by J. M. Alonso, R. Ortega is stronger or weaker than our relation (26) which can be formulated as

$$k \ < \min\{\alpha^2, \beta\}.$$

(In both cases, $k = \kappa/m$, where κ is the stiffness of the cable stays and m is the mass of the bridge.)

We can make the following discussion.

(i) If the condition

$$\alpha \in \left(0; \beta + \sqrt{2}\beta\right\rangle \ \cup\ \left\langle \frac{1-\beta}{2}; \infty\right)$$

is satisfied, which means (in an equivalent form)

$$\beta \in \left\langle \sqrt{2}\alpha - \alpha; \infty\right) \ \cup\ \langle 1 - 2\alpha; \infty),$$

(see Fig. 4), then the implication

$$\boxed{k < \min\{\alpha^2, \beta\} \quad \Longrightarrow \quad k < \beta^2 + 2\alpha\beta}$$

holds and the result of J. M. Alonso and R. Ortega is stronger than (26).

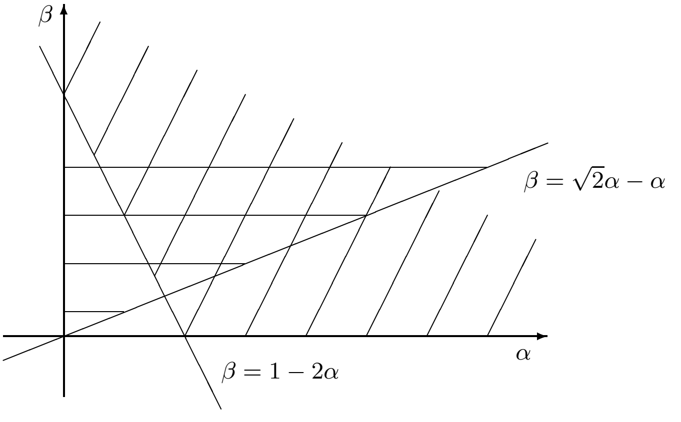

Fig. 4. *The shaded region where the result of J. M. Alonso and R. Ortega is stronger than our condition.*

(ii) But if the condition

$$\alpha \in \left\langle \beta + \sqrt{2}\beta; \frac{1-\beta}{2}\right\rangle$$

is satisfied, which again means

$$\beta \in \left(0; \sqrt{2}\alpha - \alpha\right\rangle \cap (0; 1 - 2\alpha\rangle,$$

(see Fig. 5), then the implication

$$\boxed{k < \beta^2 + 2\alpha\beta \quad \Longrightarrow \quad k < \min\{\alpha^2, \beta\}}$$

holds and our result (26) is stronger than that of J. M. Alonso and R. Ortega.

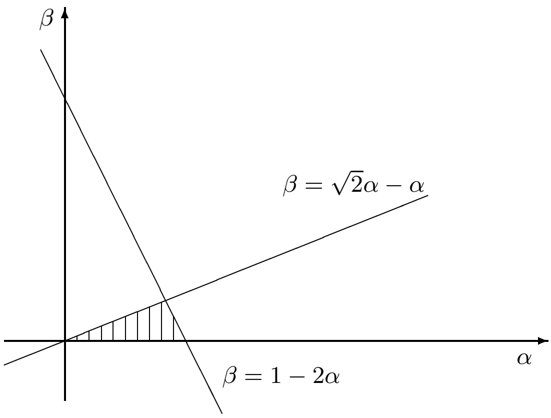

Fig. 5. *The shaded region where our condition is stronger than the result of J. M. Alonso and R. Ortega.*

Remark 16.

1. By physical reason we take into account only positive values of the parameters α and β.
2. In particular, the previous discussion means that for sufficiently small α and β in a certain relation, our result is stronger than the result published in J. M. Alonso, R. Ortega [1].
3. The question left concerns the real values of bridge parameters.

Now, we can have a look at the conditions (26) and (27). It means to compare the relation

$$\kappa < m_2 \min\{\alpha_2^2, \beta_2\},$$

obtained for the single beam model (we use the notation $m = m_2$, $\alpha = \alpha_2$, $\beta = \beta_2$), with the condition

$$\kappa < \frac{m_1 m_2}{m_1 + m_2} \min\{\alpha_1^2, \alpha_2^2, \beta_1, \beta_2\}$$

concerning the string-beam model.

We can expect that the mass of the main cable m_1 will be considerably less than the mass of the roadbed m_2, and thus

$$\frac{m_1 m_2}{m_1 + m_2} \simeq m_1.$$

The damping coefficients β_1 a β_2 can be considered almost the same.

The relation between α_1^2 and α_2^2 is still an open problem for us.

As for the real parameters of particular suspension bridges, we have found in the paper [7] by A. Fonda, Z. Schneider and F. Zanolin the following values.

	Tacoma	Golden Gate	Bronx-Whitestone
m	8.5×10^3 kg m^{-1}	3.1×10^4 kg m^{-1}	1.6×10^4 kg m^{-1}
I	0.2 m^4	5.3 m^4	0.4 m^4
L	855 m	1 280 m	700 m

The acceleration of gravity at earth's surface and the steel's modulus of Young are usually taken to be

$$\begin{aligned} g &= 9.8 \text{ m s}^{-2}, \\ E &= 2 \times 10^{11} \text{ kg m}^{-1}\text{s}^{-2}. \end{aligned}$$

However, we still have not found anything about the real values of the stiffness of the cable stays k, of the inner tension T and the mass m_1 of the main cable.

On the other hand, we succeeded to gain approximate values concerning a similar structure — a concrete suspension footbridge. The corresponding parameters are as follows.

$$\begin{aligned} m_1 &\doteq 256 \text{ kg m}^{-1}, \\ m_2 &\doteq 7\ 300 \text{ kg m}^{-1}, \\ L &\doteq 103 \text{ m}, \\ E &\doteq 30\ 000 \text{ MPa} \\ I &\doteq 1 \text{ m}^4 \\ T &\doteq 2\ 708\ 000 \text{ N} \\ \kappa &\doteq 4.5\ 10^5 \text{ kg m}^{-1} \text{ s}^{-2}. \end{aligned}$$

It means that

$$\begin{aligned} \alpha_1^2 &\doteq 9.8, \\ \alpha_2^2 &\doteq 3.5. \end{aligned}$$

However, we still do not know anything about the damping coefficients β_1, β_2.

Unfortunately, on basis of this information, we can say that our conditions (26), (27) are too restrictive and cannot be satisfied in practice.

Remark 17. Another aspect we have not mentioned so far is the periodicity of the external force and of the solution with respect to the time variable. We assume from the beginning that the period is equal to 2π. Of course, the reality is a little bit different and if we consider a different period, we can obtain new values of the mentioned parameters and the situation can change considerably.

5.2 The general existence and uniqueness

Now, we can make a short discussion about our main result of Chapter 4, which is summed up in Theorem 10. It says that in case of constant weight and small external force (e.g. due to the wind), the bridge stays in a unique position near the equilibrium.

This is surprisingly different from previous results obtained in this direction.

This problem was also studied under the special assumption that the weight of the bridge has a form $W(x) = W_0 \sin x$. Moreover, it was assumed that the external force $f(x,t) = f(t)\sin x$, as well as the displacement of the bridge $u(x,t) = y(t)\sin x$ have a similar character.

This assumptions lead to an ordinary differential equation and results obtained by A. C. Lazer and P. J. McKenna in [15], [16], [17], and by J. Glover, A. C. Lazer and P. J. McKenna in [10] are, roughly speaking, of the following spirit:

> Even if the external force is small enough, the system admits at least two (small amplitude and large amplitude, asymptotically stable) solutions.

These results are illustrated by several interesting numerical experiments (see e.g. A. C. Lazer, P. J. McKenna [16], J. Glover, A. C. Lazer, P. J. McKenna [10], A. Fonda, Z. Schneider, F. Zanolin [7], etc.).

However, from the practical point of view, the assumptions on W and f seem to be somewhat peculiar and it seems to be more natural to assume that the weight of the roadbed is constant along the bridge instead of having distribution as a function $W_0 \sin x$. This, more natural situation is discussed in [19] by P. J. McKenna and W. Walter. However, also in this case the problem is not studied in its full generality and some oversimplifications are made. First of all, the authors neglect the damping term. Second, the data as well as the solution are considered in the space of functions with certain symmetries with respect to both variables x and t.

The main result is summed up in [19] and says that under the assumptions mentioned above, the external force sufficiently small, and $3 < k < 15$, the non-damped problem has at least two solutions.

Also this result supports the idea of multiple solutions of a single beam model under more general assumptions.

However, our Theorem 10 shows that the presence of nonzero damping in the model changes the situation qualitatively and we get uniqueness result.

Moreover, our result describes that the problem is well-posed. If there is no external disturbance (no wind, no cars driving across the bridge, etc.) then the bridge achieves unique steady state position (the equilibrium) determined only by its weight W_0. In the case of "small external disturbances" represented by the term $\varepsilon f(x,t)$ there is always unique solution which is "near" the steady state position when the bridge is not disturbed. This fact illustrates the stability of the solution with respect to small perturbations given by $\varepsilon f(x,t)$.

Of course, there are still many open questions. We can expect that for a certain critical value of parameter $\varepsilon_1 > 0$ we have lack of uniqueness of the solution when $\varepsilon \geq \varepsilon_1$. Another question concerns asymptotical stability of the unique solution.

Unfortunately, for the system of two coupled equations describing the motion of the main cable and of the roadbed, we are not able to obtain a similar result as in the case of one single beam equation. The only thing we can prove is the general existence of at least one solution for any right hand side.

The problem is the lack of regularity in the string equation and this fact is the obstacle for the proof of uniqueness. Moreover, it does not allow us to state that there is at least one solution which is "close" is some sense to the steady state position (the equilibrium) determined only by the weight of the main cable and by the weight of the roadbed.

This unpleasant problem could be solved for example by the following way. We can put an additional small term

$$\varepsilon v_{xxxx}$$

into the string equation and thus modify the model little bit. The presence of such a term can ensure that we obtain higher degree of regularity and, moreover, it expresses a relatively natural fact that the main cable has some stiffness and it is not only a simple string.

However, we have not considered this situation yet and thus we have no idea whether adding this term into the model cannot cause some other troubles.

Another element, which could be added into the problem and which would have a reasonable interpretation, is a certain "pretension". It can be represented e.g. by a function $h(x)$ which would appear in the nonlinear terms. It means, that we could replace the term $(u-v)^+$ with the term $(u+h-v)^+$, or — in case of a single beam model — replace the term u^+ with the term $(u+h)^+$, respectively. Such a modification can cause that in case of no external force (no wind, no cars driving across the bridge), the beam representing the roadbed achieves a negative (or zero) position. This result would be more realistic since the real suspension bridges are never bent in a downward direction if they are in a steady state position.

This pretension was considered e.g. by A. Fonda, Z. Schneider and F. Zanolin in [7]. In this paper, the function $h(x)$ was used in a form

$$h(x) = h \sin \frac{\pi x}{L}$$

which allowed under some additional assumptions on the right hand side to eliminate the space variable x from the boundary value problem.

It would be interesting to include this term into the models considered in our paper as well and find out how it influences our results and whether it can draw them near to the real behaviour of a suspension bridge.

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