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L^p Solutions of Non-linear Integral Equations

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Abstract. We study nonlinear Volterra integral operators of first and second kind on unbounded domains. We get bounded and L^p solutions on all $[0, \infty)$ as domain with Schauder's fixed point theorem over unbounded sets.

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1 Introduction

We wish to find solutions $x(t)$ for the following nonlinear problems of Volterra's kind:

$$g(t) = \int_0^t F(t, s, x(s))\, ds, \qquad t \geq 0, \tag{1}$$

$$x(t) = x_0(t) + \int_0^t k(t, s, x(s))\, ds, \qquad t \geq 0. \tag{2}$$

General existence results can be found in [2], [3] essentially using standard techniques of functional analysis. On first kind equation (1), which is very difficult for its implicit character, very few methods have been implemented on its study.

In this article for both equations we look for bounded and L^p solutions with $1 \leq p < +\infty$, when easily checkable conditions are imposed to functions g, F, x_0 and k. The main technique is based in compactness method which do not ensure uniqueness. The uniqueness problem in nonlinear integral equations is very interesting but also not too touched; in [2] and [3] there are some interesting results. For the Lipschitz situation where the Banach contraction theorem is used, there are important results in [5] and [6].

This is the final form of the paper.

In equation (2) (of second kind) the results are not hard to extend to the Fredholm integral equation of second kind, namely

$$x(t) = x_0(t) + \int_0^\infty k(t,s,x(s))\,ds \tag{2'}$$

For our purpose we need a compactness criterion over not bounded subsets of the whole real axis which are given in the first and second lemmas, and are a well known generalization of the Arzela-Ascoli theorem and Fréchet-Kolmogorov theorem [1].

2 Bounded Solutions

Initially, consider the equation (1). Assuming that g is differentiable and F has partial derivative with respect to the first variable (that will be denoted $F_t = \frac{\partial F}{\partial t}$), then differentiating (1) we obtain:

$$g'(t) = F(t,t,x(t)) + \int_0^t F_t(t,s,x(s))\,ds. \tag{3}$$

Let us denote $\mathcal{C}$ the Banach space of continuous and bounded functions over $[0,\infty)$, normed by the supremum over all $[0,\infty)$. Now let us define the operator T: $\mathcal{C} \to \mathcal{C}$, such that given any x in $\mathcal{C}$

$$Tx(t) = \widetilde{g}(t) + \widehat{F}x(t) + \widehat{K}x(t), \qquad t \geq 0, \tag{4}$$

where

$$\begin{aligned}
\widetilde{g}(t) &= g'(t) - F(t,t,0) - \int_0^t F_t(t,s,0)\,ds,\\
\widehat{F}x(t) &= x(t) + F(t,t,0) - F(t,t,x(t)),\\
\widehat{K}x(t) &= -\int_0^t \left(F_t(t,s,x(s)) - F_t(t,s,0)\right)\,ds.
\end{aligned}$$

With this definition any solution of equation (1) satisfies the problem

$$Tx = x.$$

For this approach we will need the following definition and lemma:

Definition 1. Let $f : [0,\infty) \times [0,\infty) \times \mathbb{C}^n \to \mathbb{C}^n$. We say that $f(t,s,u)$ is t-locally equicontinuous with respect to s and u if

$$\forall \varepsilon > 0\ \exists \delta > 0\ s.t.\ |t_1 - t_2| < \delta \Rightarrow |f(t_1,s,u) - f(t_2,s,u)| < \varepsilon,$$

uniformly on s over compact sets and u over bounded sets.

Lemma 2. *Given $A \subset \mathcal{C}$ bounded, locally equicontinuous and equiconvergent, then A is relatively compact on $\mathcal{C}$.*

Theorem 3. *Let $F = F(t,s,u) : [0,\infty) \times [0,\infty) \times \mathbb{C}^n \to \mathbb{C}^n$ be continuous on each variable, derivable with respect to the first one ; F_t continuous on each variable and t-locally equicontinuous with respect to s and u. Assume*

I. a) *The functional $\widehat{F}$ is equicontinuous over any $M \subset \mathcal{C}$ bounded.*

b) $\sup\limits_{t\in[0,\infty)} |F(t,t,0)| < +\infty.$

c) *There exists $a : [0,\infty) \to \mathbb{R}^+ \cup \{0\}$ bounded, continuous, such that $a(t) \to 0$ as $t \to \infty$ and verifying*

$$|x + F(t,t,0) - F(t,t,x)| \leq a(t)|x| \qquad \forall x \in \mathbb{C}^n,\ \forall t \in [0,\infty).$$

II. *There exists a function $L : [0,\infty) \times [0,\infty) \to \mathbb{R}^+ \cup \{0\}$ such that*

a) $\int\limits_0^t L(t,s)ds$ *is continuous, and it goes to zero as $t \to \infty$,*

b) $|F_t(t,s,u) - F_t(t,s,0)| \leq L(t,s)|u| \ \ \forall t,s \in [0,\infty),\ \forall u \in \mathbb{C}^n.$

III. a) $\sup\limits_{t\in[0,\infty)} \int\limits_0^t |F_t(t,s,0)|\, ds < \infty,$

b) $\sup\limits_{t\in[0,\infty)} [a(t) + \int\limits_0^t L(t,s)ds] < 1.$

IV. *$g' \in \mathcal{C}$ and $g(0) = 0$.*

Then, there exists a solution $\bar{x} \in \mathcal{C}$ of first kind equation (1).

Proof. To prove the theorem we use a fixed point approach, showing first that the operator T, given by (4) is well defined in $\mathcal{C}$; let us take any $x \in \mathcal{C}$, then by I.c) and II.b):

$$\begin{aligned}|Tx(t)| &\leq |\widetilde{g}(t)| + |\widehat{F}x(t)| + |\widehat{K}x(t)| \\ &\leq |\widetilde{g}(t)| + [a(t) + \int\limits_0^t L(t,s)\, ds]\|x\|_\infty.\end{aligned}$$

Then $Tx(\cdot)$ is bounded. Moreover, $Tx(\cdot)$ is continuous for all $x \in \mathcal{C}$, indeed: $\widetilde{g}(\cdot)$ is continuous (g', F, and $\int\limits_0^t |F_t(t,s,0)|ds$ are continuous) ; $\widehat{F}x(\cdot)$ is continuous

(x, $F(\cdot,\cdot,0)$, $F(\cdot,\cdot,\cdot)$ are continuous) ; $\widehat{K}x(\cdot)$ is continuous, because

$$|\widehat{K}x(t)-\widehat{K}x(t')|\le \int_0^t |F_t(t,s,x(s))-F_t(t',s,x(s))|\,ds \le$$
$$\int_0^t |F_t(t,s,0)-F_t(t',s,0)|\,ds+\int_{t'}^t L(t',s)ds\|x\|_\infty$$

and F_t is t-locally equicontinuous and $\int_{t'}^t L(t',s)ds \underset{t\to t'}{\longrightarrow} 0$. Then, $Tx(\cdot)$ is well defined and continuous.

Secondly, using Lemma 2 we prove that T is a compact operator. Let $M\subseteq\mathcal{C}$ bounded, *i.e.*, $\forall x\in M$, $\|x\|_\infty\le R<\infty$, we will show that:

i) $T(M)$ is bounded in $\mathcal{C}$;
ii) $T(M)$ is locally equicontinuous in $\mathcal{C}$;
iii) $T(M)$ is equiconvergent.

i) $T(M)$ is bounded. As the functions $\widetilde{g}(.)$, a(.) and L(.,.) are bounded, given $x\in M$, we use I.b), I.c), II.b) and III.b), and we get that:

$$|Tx(t)|\le|\widetilde{g}(t)|+|\widehat{F}x(t)|+|\widehat{K}x(t)|\le$$
$$|\widetilde{g}(t)|+[a(t)+\int_0^t L(t,s)\,ds]R\le|\widetilde{g}(t)|+R.$$

ii) $T(M)$ is equicontinuous. First $\widetilde{g}$ is continuous over $[0,\infty)$ because g', $F(.,.,0)$ and $F_t(.,s,0)$ are continuous on $[0,\infty)$. Moreover, $\widehat{F}x$ is equicontinuous on bounded subsets of $\mathcal{C}$. With these considerations, given $[a,b]$ compact in $[0,\infty)$ and $t_1\le t_2$ on $[a,b]$, then the equicontinuity of $T(M)$ follows easily from

$$|Tx(t_1)-Tx(t_2)|\le|\widetilde{g}(t_1)-\widetilde{g}(t_2)|+|\hat{F}x(t_1)-\hat{F}x(t_2)|+\int_{t_1}^{t_2}|F_t(t_2,s,0)|\,ds$$
$$+\int_0^{t_1}|F_t(t_1,s,x(s))-F_t(t_2,s,x(s))|\,ds$$
$$+\int_{t_1}^{t_2}|F_t(t_2,s,x(s))-F_t(t_2,s,0)|\,ds$$
$$\le|\widetilde{g}(t_1)-\widetilde{g}(t_2)|+|\hat{F}x(t_1)-\hat{F}x(t_2)|+\int_{t_1}^{t_2}|F_t(t_2,s,0)|\,ds$$

$$+\int_0^{t_1} |F_t(t_1,s,x(s)) - F_t(t_2,s,x(s))|\,ds + R\int_{t_1}^{t_2} L(t_2,s)\,ds.$$

iii) $T(M)$ is equiconvergent, because

$$|Tx(t)-\widetilde{g}(t)| \leq |\hat{F}x(t)| + |\hat{K}x(t)| \leq R\{a(t) + \int_0^t L(t,s)\,ds\} \underset{t\to\infty}{\longrightarrow} 0.$$

So, by Lemma 2, $T(M)$ is relatively compact in $\mathcal{C}$. From Schauder's fixed point theorem $\exists \bar{x} \in \mathcal{C}$ such that $\bar{x} = T\bar{x}$. Integrating this last fixed point equation and as $g(0)=0$, implies that $\bar{x}$ is a solution of equation (1). □

Now, consider the equation of second kind:

$$x(t) = x_0(t) + \int_0^t k(t,s,x(s))\,ds,\; t\geq 0, \tag{2}$$

where $k:[0,\infty)\times[0,\infty)\times\mathbb{C}^n \to \mathbb{C}^n$ is continuous in s and x. We can formulate now the following

Theorem 4. *Assume that the function $k:[0,\infty)\times[0,\infty)\times\mathbb{C}^n\to\mathbb{C}^n$ is t-locally equicontinuous and satisfies*

I. $\sup\limits_{t\in[0,\infty)} \int_0^t |k(t,s,0)|ds < \infty.$

II. *There exists $L:[0,\infty)\times[0,\infty)\to\mathbb{R}^+\cup\{0\}$ such that:*
 a) *$\forall (t,s,u)\in[0,\infty)\times[0,\infty)\times\mathbb{C}^n$ with $t\geq s$,*

$$|k(t,s,u)-k(t,s,0)| \leq L(t,s)|u|.$$

 b) *$\int_0^t L(t,s)ds$ is continuous and converges to 0 as $t\to\infty$.*

 c) $\sup\limits_{t\in[0,\infty)} \int_0^t L(t,s)ds < 1.$

Then for all $x_0\in\mathcal{C}$, equation (2) has a solution $\bar{x}\in\mathcal{C}$.

Proof. Consider the operator $T:\mathcal{C}\to\mathcal{C}$ defined as

$$Tx(t) = x_0(t) + \int_0^t k(t,s,0)ds + \int_0^t \left(k(t,s,x(s)) - k(t,s,0)\right)\,ds.$$

Proceeding as in the previous Theorem, from Schauder's fixed point theorem we obtain that there exists $\bar{x}\in\mathcal{C}$ such that $\bar{x} = T\bar{x}$, and then a solution of equation (2). □

3 L^p Solutions

Now, we will find $L^p[0,\infty)$ solutions to equation (2) with $1 \leq p < \infty$. For the whole section, q will be the Holder conjugate of p, i.e., $\frac{1}{p}+\frac{1}{q}=1$. We will need the next definition and lemma for our result over L^p spaces.

Definition 5. The function $k(t,s,u)$ is t-locally L^p equicontinuous if given $a,b,c,d \in \mathbb{R}^+$, such that $a \leq b$ and $c \leq d$, then

$$\int_a^b \int_c^d |k(t+h,s,x(s)) - k(t,s,x(s))|^q \, dsdt \xrightarrow[h\to 0]{} 0,$$

with x on bounded subsets of $L^p[0,\infty)$.

Lemma 6. *$A \subset L^p[0,\infty)$ bounded will be relatively compact if:*

a) *The restriction $A|_{[a,b]}$ where $[a,b] \subset [0,\infty)$ is a compact interval, satisfy the L^p-equicontinuity of the translations (Fréchet-Kolmogorov criterion).*

b) *Equiconvergence: there exists u in $L^p[0,\infty)$ such that $\int_t^\infty |x(s)-u(s)|^p \, ds \to 0$ as $t \to \infty$ uniformly for $x \in A$.*

Now, our next result is

Theorem 7. *Assuming k is t-locally L^p equicontinuous, and*

I. *There exist a function $L : [0,\infty) \times [0,\infty) \to \mathbb{R}^+ \cup \{0\}$ such that:*

a) $\displaystyle\int_0^\infty \left(\int_0^\infty L^q(t,s)\, ds \right)^{\frac{p}{q}} dt < \infty,$

b) $\displaystyle\sup_{t\in[0,\infty)} \int_0^\infty \left(\int_0^t L^q(t,s) ds \right)^{\frac{p}{q}} dt < 1,$

c) *$\forall (t,s,u) \in [0,\infty) \times [0,\infty) \times \mathbb{C}^n$ with $t \geq s$,*

$$|k(t,s,u) - k(t,s,0)| \leq L(t,s)|u|.$$

II. $\displaystyle\int_0^t |k(t,s,0)| \, ds \in L^p[0,\infty).$

Then, given $x_0 \in L^p[0,\infty)$, equation (2) has a solution on $L^p[0,\infty)$.

Proof. Let $T : L^p[0,\infty) \to L^p[0,\infty)$ defined by

$$Tx(t) = \widetilde{x}_0(t) + \int_0^t (k(t,s,x(s)) - k(t,s,0))\, ds,$$

where

$$\widetilde{x}_0(t) = x_0(t) + \int_0^t k(t,s,0)\, ds.$$

Clearly $\widetilde{x}_0$ is in L^p, and there exists a constant $c \geq 0$, such that

$$|Tx(t)|^p \leq c\Big\{|\widetilde{x}_0(t)|^p + \Big(\int_0^t |k(t,s,x(s)) - k(t,s,0)|ds\Big)^p\Big\}$$

and using hypothesis I.b) and Holder's inequality we get that

$$|Tx(t)|^p \leq c\Big\{|\widetilde{x}_0(t)|^p + \Big(\int_0^t L^q(t,s)\, ds\Big)^{\frac{p}{q}} \|x\|_{L^p}^p\Big\}.$$

Then by hypothesis I.a) and II.), we have that $Tx \in L^p[0,\infty)$. Now we want to see that T is a compact operator from $L^p[0,\infty)$ to $L^p[0,\infty)$. To this end, we use Lemma 6. Let us take $M \subseteq L^p[0,\infty)$ bounded, *i.e.*, $\forall x \in M$, $\|x\|_{L^p} \leq R < \infty$, then we must prove that $T(M)$ is relatively compact.

First, by I.a), I.b), II. and the last inequality $T(M)$ is bounded. Moreover, we have

a) Equicontinuity in the translations. Given $[a,b] \subset [0,\infty)$ compact, there exists a constant $c \geq 0$ such that

$$\begin{aligned}
|Tx(t+h) - Tx(t)|^p &\leq c\Big\{|\widetilde{x}_0(t+h) - \widetilde{x}_0(t)|^p + \\
&\quad \Big[\int_t^{t+h} |k(t+h,s,x(s)) - k(t+h,s,0)|\, ds\Big]^p \\
&\quad + \Big[\int_0^t |k(t+h,s,x(s)) - k(t,s,x(s))|\, ds\Big]^p\Big\} \\
&\leq c\Big\{|\widetilde{x}_0(t+h) - \widetilde{x}_0(t)|^p + \Big(\int_t^{t+h} L^q(t+h,s)\, ds\Big)^{\frac{p}{q}} \|x\|_{L^p}^p \\
&\quad + \Big[\int_0^t |k(t+h,s,x(s)) - k(t,s,x(s))|\, ds\Big]^p\Big\}.
\end{aligned}$$

Then, from Holder's inequality, we have

$$\int_a^b |Tx(t+h)-Tx(t)|^p dt \le c\Big\{\int_a^b |\widetilde{x}_0(t+h)-\widetilde{x}_0(t)|^p\,dt + R^p\int_a^b\Big(\int_t^{t+h} L^q(t+h,s)ds\Big)^{\frac{p}{q}}\,dt + b^p\int_a^b\int_0^b |k(t+h,s,x(s))-k(t,s,x(s))|^q\,dsdt\Big\}.$$

and then, due to $\widetilde{x}_0 \in L^p$, the t-equicontinuity of k, and the integrability of $L(.,.)$, we obtain the equicontinuity in the translations.

b) Finally, the L^p-equiconvergence follows from I.a) and I.b) because:

$$|Tx(t)-\widetilde{x}_0(t)|^p \le \Big(\int_0^t L^q(t,s)ds\Big)^{\frac{p}{q}}\Big(\int_0^t |x(t)|^p\,dt\Big) \le R^p\Big(\int_0^t L^q(t,s)\,ds\Big)^{\frac{p}{q}}$$

and

$$\int_{\widetilde{t}}^{\infty} |Tx(t)-\tilde{x}_0(t)|^p\,dt \le R^p\int_{\widetilde{t}}^{\infty}\Big(\int_0^t L^q(t,s)\,ds\Big)^{\frac{p}{q}}\,dt.$$

Thus, T is a compact operator from $L^p[0,\infty)$ to $L^p[0,\infty)$ and Schauder's fixed point theorem implies there exist $\bar{x}\in L^p[0,\infty)$ satisfying $T\bar{x}=\bar{x}$, and then $\bar{x}$ is an L^p solution of equation (2). □

As an example of the first theorem consider a function $F(.,.,.)$ as follows: $F(t,s,u)=\big(1+(t-s)\big)/4e^{-t}u+f(t,s)$, such that

$$\sup_{t\in[0,\infty)}|f(t,t)|<\infty, \qquad \sup_{t\in[0,\infty)}\int_0^t\Big|\frac{\partial f}{\partial t}(t,s)\Big|ds<\infty.$$

The function $a=0$ satisfies I.c). Moreover, conditions II.a) and II.b) are fulfilled since

$$|F_t(t,s,x)-F_t(t,s,0)|\le|1+(t-s)|/4e^{-t}|x|,$$

and

$$\int_0^t |1+(t-s)|/4e^{-t}\,ds\to 0$$

as $t \to \infty$. Then for any function g that satisfies $g(0)$ and g' in $\mathcal{C}$, theorem 3 implies that the equation

$$g(t) = \int_0^t \big((1+(t-s))/4e^{-t}x(s) + f(t,s)\big)\, ds$$

has a continuous and bounded solution.

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