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The Property (A) for a Certain Class of the Third Order ODE

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Abstract. We study oscillatory and non-oscillatory solutions of the third order ODE

$$[g(t)(u''(t) + p(t)u(t))]' = f(t, u, u', u''), \qquad (*)$$

where $g, p : [T, \infty) \to [0, \infty)$ are bounded functions, $g \geq \delta > 0$. The function f is assumed to be continuous and $f(x_1, x_2, x_3) \cdot x_1 \leq 0$.
Many authors have consider ODE's of the form $(*)$, where the main part, i.e. the term $u'' + pu$ is nonoscillatory. By contrast to these results we consider here the case of the oscillatory kernel function $u'' + pu$.
The main goal is to show that any solution u of $(*)$ is either oscillatory or it is a solution of the second order ODE $u''(t) + p(t)u(t) = \beta(t)$ with vanishing right hand side $\beta \geq 0$, $\beta(t) \to 0$ as $t \to \infty$. In the latter case all the derivatives $u^{(n)}(t)$ up to the second order tend to zero as $t \to \infty$, i.e. eq. $(*)$ has the property (A).
The results are generalizations of these obtained by I. T. Kiguradze [1].

AMS Subject Classification. 34C10, 34C15

Keywords. The Property (A), Oscillatory Solutions ODE

1 Introduction

In this paper we consider a nonlinear third order differential equation in the form

$$(g(t) \cdot [u''(t) + p(t)u(t)])' = f(t, u, u', u''). \tag{1}$$

Let T, g_1, g_2, p_2 be positive constants and let

$$g : [T_0, \infty) \to (0, \infty) \quad \text{belong to the class } C^1[T_0, \infty),$$
$$0 < g_1 \leq g(t) \leq g_2 \quad \text{for all } t \in [T_0, \infty), \tag{2}$$
$$p : [T_0, \infty) \to [0, \infty) \quad \text{belong to the class } C^1[T_0, \infty),$$
$$0 \leq p(t) \leq p_2 \quad \text{for all } t \in [T_0, \infty), \tag{3}$$

$f : [T_0, \infty) \times R^3 \to R$ is continuous function having the following sign property

$$f(t, x_1, x_2, x_3) \cdot \operatorname{sign} x_1 \leq 0 \qquad \text{for } x_1 \neq 0. \tag{4}$$

This is the final form of the paper.

The main goal of this paper is to describe oscillatory and nonoscillatory properties of solution of ordinary differential equation (1). This main result is a dichotomy property saying that any solution u of equation (1) is either oscillatory or u together with its derivatives up to the second order tend to zero.

Several papers focused an aforementioned problem. An Oscillatory Criterion for a Class of Ordinary Differential Equations [1] becomes one of the main ones. The author assumes that a left-sided operator $u^{(n)}(t) + u^{(n-2)}(t)$ is oscillatory at first. This assumption was considered as true, he searched necessary and sufficient conditions to fulfill that equation has a property A (B).

Several authors studied differential equation (1), but they assumed that operator $u''(t) + p(t)u(t)$, which forms an equational kernel is nonoscillatorical. By contrast to these results we consider here *the case of oscillatory kernel function* $u''(t) + p(t)u(t)$.

2 Preliminaries

By a solution *(proper solution)* we mean a function u defined on an interval $[T, \infty) \subset [T_0, \infty)$, having a continuous third derivative and such that

$$\sup\{|u(t)| : t > T\} > 0$$

for any $t \in [T, \infty)$ and u satisfies equation.

By an *oscillatory solution* we mean a solution of (1) having arbitrarily large zeroes. Otherwise, a solution is said to be nonoscillatory.

3 Auxiliary lemmata

We begin with several auxiliary lemmata which are needed in order to prove main results in the main section. Let us consider the equation

$$y''(t) + p(t)y(t) = r(t), \tag{5}$$

where $p : [T, \infty) \to [0, \infty)$ and $r : [T, \infty) \to (0, \infty)$ are continuous functions such that

$$p(t) \le p_2 \quad \text{and} \quad 0 < r_1 \le r(t) \le r_2 \qquad \text{for all } t \in [T, \infty), \tag{6}$$

where r_1, r_2, p_2 are positive constants, with coefficients p, r satisfying (6).

Lemma 1. *Let $y \in C^2[T, \infty)$ be a positive solution of differential equation (1) and let r_1, r_2, p_2 be positive constants which fulfill (6) the conditions .*

Let $p_0 > 0$ be arbitrary large and put $\varepsilon_1 = \frac{r_1}{2p_2}$. Then for any $\delta > 0$, small enough, and $0 < \varepsilon \le min\left(\frac{\varepsilon_1 \cdot r_1}{2(1+2p_0)^2 \cdot r_2 + r_1}, \frac{r_1 \delta^2}{16}\right)$ the solution y has the following property.

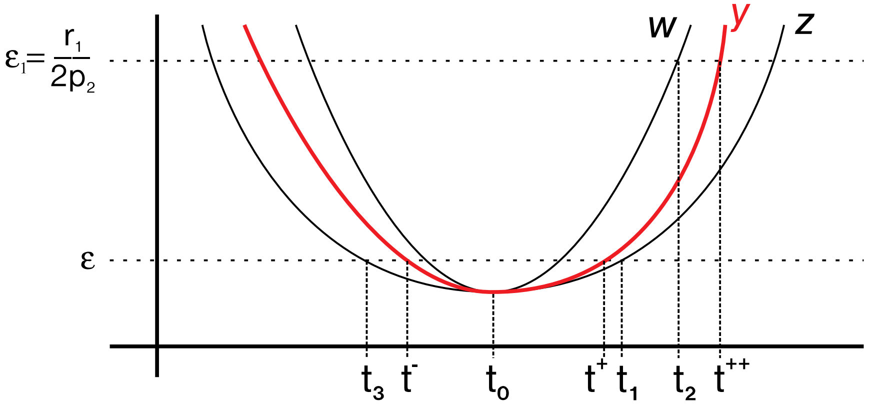
$$\begin{aligned} &\textit{If we have} \quad 0 < y(t) < \varepsilon \quad \textit{on some interval } (t^-, t^+) \quad \textit{and} \\ &\qquad y(t^-) = y(t^+) = \varepsilon, && (7) \\ &\textit{then} \quad y(t) \geq \varepsilon \quad \textit{for any} \quad t \in [t^+, t^+ + p_0(t^+ - t^-)], \\ &\qquad t^+ - t^- \leq \delta. && (8) \end{aligned}$$

Proof. Suppose that $0 < y(t) < \varepsilon$ for any $t \in [t^-, t^+]$ and $y(t^-) = y(t^+) = \varepsilon$. Since y is a solution of (1) we have

$$\begin{aligned} &y''(t) = r(t) - p(t) \cdot y(t) \geq r_1 - p_2\varepsilon_1 = \frac{r_1}{2} \\ \text{and} \quad &y''(t) \leq r_2 \quad \text{for each } t \text{ such that} \quad 0 < y(t) \leq \varepsilon_1 \end{aligned} \tag{9}$$

Put $t_0 = \min\limits_{t \in [t^-, t^+]} y(t)$.

Let us introduce the following auxiliary functions:



$$\begin{aligned} z(t) &= y(t_0) + y'(t_0)(t - t_0) + \frac{r_1}{2} \cdot \frac{(t-t_0)^2}{2} t, \\ w(t) &= y(t_0) + y'(t_0)(t - t_0) + r_2 \cdot \frac{(t-t_0)^2}{2}, && (10) \\ \text{and} \quad y(t) &= y(t_0) + y'(t_0)(t - t_0) + y''(\xi) \cdot \frac{(t-t_0)^2}{2}, \qquad \xi \in [t_0, t]. \end{aligned}$$

According to (9) and (10) we have the estimate

$$z(t) \leq y(t) \leq w(t) \quad \text{provided that} \quad 0 < y(t) \leq \varepsilon_1. \tag{11}$$

Then with regard to (10) there exist $t_1, t_2, t_3 \in [T, \infty)$ such that $t_0 > t_3$ and $t_1, t_2 > t_0$ roots

$$w(t_2) = \varepsilon_1, \qquad z(t_1) = \varepsilon, \qquad z(t_3) = \varepsilon. \tag{12}$$

Furthermore, there exists $t^{++} > t^+$ such that $y(t^{++}) \geq \varepsilon$

We conclude from the definitions of the functions $w(t), z(t)$ and (11) that

$$\begin{aligned} t^{++} - t^+ &\geq t_2 - t_1, \\ t_1 - t_3 &\geq t^+ - t^-. \end{aligned}$$

In what follows, we will prove that

$$t^{++} - t^+ \geq p_0 \cdot (t^+ - t^-).$$

As a consequence of this inequality we will obtain the statement (8).

Assume that $0 < \varepsilon \leq \frac{\varepsilon_1 r_1}{r_1 + 2r_2(1+2p_0)^2}$. Then

$$\varepsilon_1 \geq \varepsilon \left((1+2p_0)^2 \cdot \frac{2r_2}{r_1} + 1 \right) > 0. \tag{13}$$

As $0 < y(t_0) = \min_{t \in [t^-, t^+]} y(t) < \varepsilon$ we obtain

$$\frac{\varepsilon_1 - y(t_0)}{r_2} \geq \frac{\varepsilon_1 - \varepsilon}{r_2} \geq (1+2p_0)^2 \cdot \frac{2\varepsilon}{r_1} \geq 2 \cdot (1+2p_0)^2 \cdot \frac{(\varepsilon - y(t_0))}{r_1}. \tag{14}$$

It easily follows from (10) and (12) that

$$(t_2 - t_1)^2 = (\varepsilon_1 - y(t_0)) \cdot \frac{2}{r_2}, \tag{15}$$

$$(t_1 - t_0)^2 = (\varepsilon - y(t_0)) \cdot \frac{4}{r_1}. \tag{16}$$

Therefore

$$\frac{\varepsilon_1 - y(t_0)}{r_2} = \frac{w(t_2) - y(t_0)}{r_2} = \frac{\frac{r_2}{2} \cdot (t_2 - t_0)^2}{r_2}.$$

With regard to (13), (14), (15), (16) we have

$$\frac{\varepsilon_1 - y(t_0)}{r_2} \geq 2(1+2p_0)^2 \cdot \frac{z(t_1) - y(t_0)}{r_1} = 2(1+2p_0)^2 \cdot \frac{r_1}{4r_1} \cdot (t_1 - t_0)^2.$$

Straightforward computations yield

$$\begin{aligned} \frac{1}{2}(1+2p_0)^2 \cdot (t_1 - t_0)^2 &\leq \frac{1}{2}(t_2 - t_0)^2, \\ (t_1 - t_0) \cdot (1+2p_0) &\leq t_2 - t_0, \\ 2p_0 \cdot (t_1 - t_0) &\leq t_2 - t_1, \\ p_0(t^+ - t^-) \leq (t_1 - t_3) \cdot p_0 &\leq t_2 - t_1 \leq t^{++} - t^+. \end{aligned}$$

Hence

$$t^{++} - t^{+} \geq p_0(t^{+} - t^{-}).$$

Thus $y(t) \geq \varepsilon$ on $[t^{+}, t^{+} + p_0(t^{+} - t^{-})]$.

It remains to show the estimate

$$t^{+} - t^{-} \leq \delta.$$

Due to (10), (12)

$$\varepsilon = z(t_1) = y(t_0) + \frac{r_1}{2}(t_1 - t_0)^2 \geq \frac{r_1}{2}(t_1 - t_0)^2.$$

This is why $t_1 - t_0 \leq \sqrt{\frac{2\varepsilon}{r_1}}$. Since

$$t^{+} - t^{-} \leq t_1 - t_3 = 2(t_1 - t_0) \leq 2\sqrt{2}\sqrt{\frac{\varepsilon}{r_1}} \leq 4 \cdot \sqrt{\frac{\varepsilon}{r_1}}$$

and $\varepsilon < \frac{r_1 \delta^2}{16}$, we finally obtain

$$t^{+} - t^{-} \leq \delta.$$

Lemma 2. *Assume that*

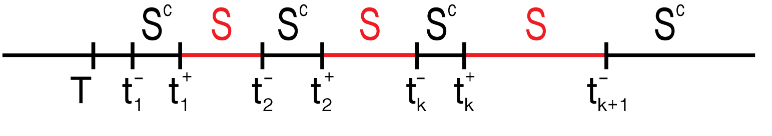
(i) $h \in C[T, \infty), \quad h(t) > 0, \quad \int\limits_{T}^{\infty} h(t)dt = +\infty.$

(ii) *There exists* $\delta > 0$ *and the sequence*

$$T \leq t_1^{-} < t_1^{+} < t_2^{-} < t_2^{+} < \cdots < t_k^{-} < t_k^{+} \to \infty$$

with the property

$$t_k^{+} - t_k^{-} \leq \delta,$$
$$t_{k+1}^{-} - t_k^{+} \geq t_k^{+} - t_k^{-}.$$



(iii) (a) *Either there is* $h_0 > 0$ *such that* $h(t) \geq h_0 > 0$ *on* $[T, \infty)$

(b) *or* h *is a nonincreasing function on* $[T, \infty)$,
such that $h(t) \to 0$ *as* $t \to \infty$
and there is $k_0 > 1$ *such that* $h(t) \leq k_0 \cdot h(t + \delta)$ *for all* $t \geq T$.

Denote

$$S = \bigcup_{k=1}^{\infty} [t_k^+, t_{k+1}^-], \qquad S^c = \bigcup_{k=1}^{\infty} (t_k^-, t_k^+).$$

Then

$$\int\limits_S h(t)dt = +\infty.$$

Proof.

$$S \cup S^c = (t_1^-, \infty) \subset [T, \infty).$$

With respect to (iii) the proof splits into two parts.

The case (a) If $\mu(S^c) < \infty$ (μ is the Lebesgue measure) then $\mu(S) = \infty$.
If $\mu(S^c) = \infty$ then according to (ii) we again obtain $\mu(S) = \infty$.
Since $h(t) \geq h_0 > 0$ we may conclude $\int\limits_S h(t)dt = \infty$.

The case (b) If $\int\limits_{S^c} h(t)dt < \infty$ then clearly $\int\limits_S h(t)dt = \infty$
because $S \cup S^c = (t_1^-, \infty)$, $h(t) \in C[T, \infty)$ and $\int\limits_T^{\infty} h(t)dt = \infty$.
On the other hand suppose that $\int\limits_{S^c} h(t)dt = \infty$.

Choose $\delta > 0$ sufficiently small and $p_0 > 0$ sufficiently large. Let $\varepsilon > 0$ satisfy the assumptions of Lemma 1. Obviously, there is a sequence

$$T \leq t_1^- < t_1^+ < t_2^- < t_2^+ < \cdots < t_k^- < t_k^+ \to \infty$$

such that

$$\begin{aligned} y(t) &< \varepsilon \qquad \text{for } t \in (t_k^-, t_k^+), \\ y(t) &\geq \varepsilon \qquad \text{for } t \in [t_k^+, t_{k+1}^-], \quad k = 1, 2, \dots. \end{aligned}$$

With regard to Lemma 1 we may conclude $t_k^+ - t_k^- \leq \delta$. Then for any $t \in [t_k^-, t_k^+]$ we have $h(t) \leq k_0 \cdot h(t + \delta) \leq k_0 \cdot h(t + (t_k^+ - t_k^-))$ and thus

$$\int\limits_{t_k^-}^{t_k^+} h(t)dt \leq \int\limits_{t_k^-}^{t_k^+} h(t + (t_k^+ - t_k^-))dt = k_0 \cdot \int\limits_{t_k^+}^{t_k^+ + (t_k^+ - t_k^-)} h(u)du \leq k_0 \cdot \int\limits_{t_k^+}^{t_{k+1}^-} h(t)dt.$$

Hence

$$+\infty = \int\limits_{S^c} h(t)dt \leq \int\limits_S h(t)dt.$$

It completes the proof of Lemma 2.

Lemma 3. *Let*

$$(g(t) \cdot [u''(t) + p(t)u(t)])' = f(t, u, u', u''),$$

where $p(.), g(.)$ *and* $f(.)$ *fulfill the following conditions:*

(i) $p(.), g(.)$ *fulfill conditions* (2) *a* (3).

(ii)

$$f(t, x_1, x_2, x_3) \cdot \operatorname{sign} x_1 \leq 0, \qquad x_1 \neq 0, \tag{17}$$
$$f(t, x_1, x_2, x_3) \cdot \operatorname{sign} x_1 \leq -h(t) \cdot w(|x_1|), \tag{18}$$

where $h(.)$ *fulfill on* $[T, \infty)$ *assumption* (i), (ii), (iii) *from Lemma* 2.

(iii) *Let* $w : [0, \infty) \to [0, \infty)$ *be a nonincreasing function such that*

$$w(0) = 0, \qquad w(s) > 0 \quad \textit{for all } s > 0. \tag{19}$$

Then any proper solution of equation (1) *on* $[T, \infty)$ *is either oscillatory or there exists* $\beta(.) \geq 0$ *such that* $\lim_{t\to\infty} \beta(t) = 0$ *and* $u(.)$ *is a solution of equation*

$$u''(t) + p(t)u(t) = \beta(t) \cdot \operatorname{sign}\ u(t). \tag{20}$$

Proof. Let u be a nonoscillatory solution. We will show the existence of a function β as stated in Lemma 3. According to (4), u solves (1) iff $-u$ does. Therefore, without loss of generality we may assume that

$$u(t) > 0 \qquad \text{for all } t \in [T, \infty) \tag{21}$$

Denote

$$\alpha(t) := g(t) \cdot (u''(t) + p(t)u(t)). \tag{22}$$

Then from (17) and (21) we see $\alpha'(t) = f(t, u(t), u'(t), u''(t)) \leq 0$ for all $t \geq T$, is nonincreasing function on $[T, \infty)$. We will consider three distinct cases:

(i) $\lim_{t\to\infty} \alpha(t) < 0$,

(ii) $\lim_{t\to\infty} \alpha(t) > 0$,

(iii) $\lim_{t\to\infty} \alpha(t) = 0$.

In the case (i) we have:
Let there exist $T_1 > T$ such that $\alpha(t) \leq -\varepsilon < 0$ for all $t \in [T_1, \infty)$. Then

$$u''(t) + p(t)u(t) \overset{(22)}{=} \frac{\alpha(t)}{g(t)} \overset{(2)}{\leq} \frac{-\varepsilon}{g_2} < 0.$$

According to (3), (21) we have $p(.), u(.) > 0$ on $[T_1, \infty)$. Hence $u''(t) \leq \frac{-\varepsilon}{g_2} < 0$ for all $t \in [T_1, \infty)$ and so there is $T_2, T_2 \geq T_1$ such that $u(t) < 0$ for $t \geq T_2$. A contradiction.

In the case (ii)

$u(.)$ is the solution of equation (1). Let us mark:

$$u''(t) + p(t)u(t) = \frac{\alpha(t)}{g(t)} =: r(t) \qquad \text{for all } t \in [T, \infty).$$

We know that $\alpha(.)$ is a nonincreasing function on $[T, \infty)$.

Take $\alpha_1 := \lim\limits_{t\to\infty} \alpha(t) > 0$ and $\alpha_2 := \alpha(T)$. Then according to the definition of function α we have $\alpha_2 \geq \alpha(t) \geq \alpha_1 > 0$ for any $t \geq T$ and therefore

$$0 < \frac{\alpha_1}{g_2} \leq \frac{\alpha(t)}{g(t)} \leq \frac{\alpha_2}{g_1} \qquad \Rightarrow \qquad 0 < r_1 \leq r(t) \leq r_2,$$

which means that the function $r(.)$ satisfies assumptions (5), (6) of Lemma 1 with constants $r_1 = \frac{\alpha_1}{g_2}$ and $r_2 = \frac{\alpha_2}{g_1}$.

According to Lemma 1, for any $\delta > 0$ sufficiently small and any $p_0 > 0$ sufficiently large there is $\varepsilon > 0$ and a sequence $t_1^- < t_1^+ < t_2^- < t_2^+ \ldots \to \infty$ such that

$$\begin{aligned} & t_k^+ - t_k^- \leq \delta, \\ & \quad y(t) < \varepsilon \qquad \text{for all } t \in (t_k^-, t_k^+), \\ & \quad y(t) \geq \varepsilon \qquad \text{for all } t \in [t_k^+, t_{k+1}^-], \qquad k = 1, 2, \ldots . \end{aligned}$$

Thus $u(t) \geq \varepsilon \Rightarrow \alpha'(t) \leq -h(t)w(\varepsilon)$ on $[t_k^+, t_{k+1}^-]$.

And $\alpha(t) \leq \alpha(t_k^+) - \int_{t_k^+}^{t} h(t)w(\varepsilon)dt$ for any $t \in [t_k^+, t_{k+1}^-]$. This yields the following estimates.

$$\alpha(t) \leq \alpha(t_1^+) - w(\varepsilon) \int_{t_1^+}^{t} h(t)dt,$$

$$\alpha(t_2^-) \leq \alpha(t_1^+) - w(\varepsilon) \int_{t_1^+}^{t_2^-} h(t)dt,$$

$$\begin{aligned} \alpha(t_3^-) &\leq \alpha(t_2^+) - w(\varepsilon) \int_{t_2^+}^{t_3^-} h(t)dt \leq \alpha(t_2^-) - w(\varepsilon) \int_{t_2^+}^{t_3^-} h(t)dt \\ &\leq \alpha(t_1^+) - w(\varepsilon) \Big[\int_{t_1^+}^{t_2^-} h(t)dt + \int_{t_2^+}^{t_3^-} h(t)dt \Big]. \end{aligned}$$

And in general:

$$\alpha(t_{k+1}^-) \le \alpha(t_1^+) + w(\varepsilon)\Big[\int_{t_1^+}^{t_2^-} h(t)dt + \int_{t_2^+}^{t_3^-} h(t)dt + \cdots + \int_{t_k^+}^{t_{k+1}^-} h(t)dt\Big].$$

Hence

$$\lim_{k\to\infty} \alpha(t_{k+1}^-) \le \alpha(t_1^+) - w(\varepsilon)\Big[\int_S h(t)dt\Big] \to -\infty$$

because $\int_S h(t)dt = \infty$ (see the Lemma 2), a contradiction.

This way we have excluded the cases (i) and (ii). Thus the case (iii) must occur, i.e.

$$\lim_{t\to\infty} \alpha(t) = 0\,.$$

Finally, if we put $\beta(t) = \frac{\alpha(t)}{g(t)}$ for all $t \ge T$, then we have $\beta(t) \ge 0$ (α is nonincreasing function) and $\lim_{t\to\infty} \beta(t) = 0$ and the proof of Lemma 3 follows.

4 Main Theorems

Theorem 4. *Let u be a solution of equation* (1)

$$(g(t)[u''(t) + p(t)u(t)])' = f(t, u, u', u''),$$

where $p(.)$, $g(.)$ and $f(.)$ fulfill conditions (2), (3), (17), (18) *and* (19).

Let further,

$$u''(t) + p(t)u(t) = \beta(t) \qquad \text{on interval } [T, \infty), \tag{23}$$

where

$$\begin{aligned} &u \in C^2[T,\infty), && u(t) > 0 && \text{for all } t \ge T, \\ &\beta \in C^2[T,\infty), && \beta(t) \ge 0 && \text{for all } t \ge T \end{aligned} \tag{24}$$

and

$$\lim_{t\to\infty} \beta(t) = 0. \tag{25}$$

If u is a nonoscillatory solution of (1)*, then*

$$\lim_{t\to\infty} u(t) = 0.$$

Proof. It is sufficient to prove

$$\liminf_{t\to\infty} u(t) = 0 = \limsup_{t\to\infty} u(t).$$

At first we show that

$$\liminf_{t\to\infty} u(t) = 0 \tag{26}$$

We proceed by contradiction.

If (26) is not valid, then according to (24) and (25) suppose that there is $\alpha > 0$ and $T_1 \geq T$ such that $u(t) \geq \alpha$ for each $t \geq T_1$. Thus

$$u''(t) = \beta(t) - p(t)u(t) \leq \beta(t) - p_1\alpha \quad \text{for all } t \in [T_1, \infty).$$

As $\lim\limits_{t\to\infty} \beta(t) = 0$, there exists $T_2 \geq T_1$,

$$u''(t) \leq -\frac{p_1}{2} \cdot \alpha < 0 \quad \text{for } t \geq T_2$$

and therefore $u(t) < 0$ on $[T_3, \infty)$, where $T_3 \geq T_2$. This is a contradiction because u is positive in $[T, \infty)$.

Now we show that

$$\limsup_{t\to\infty} u(t) = 0. \tag{27}$$

Again we will proceed by contradiction. Suppose that (27) is not true. Then two cases can occur:

(i) $\limsup\limits_{t\to\infty} u(t) = \infty$,

(ii) There is ε such that $0 < \varepsilon < \limsup\limits_{t\to\infty} u(t) < 2\varepsilon$.

First we exclude the case (i).

If $\limsup\limits_{t\to\infty} u(t) = +\infty$, then there is a sequence $\{t_k\}_{k=1}^{\infty}$ such that $u(t_k) \to \infty$ for $t_k \to \infty$ and simultaneously with regard to the previous part of the proof $\liminf\limits_{t\to\infty} u(t) = 0$.

Thus there exist $t^*, t^{**} \in [T_1, \infty)$, $T_1 \geq T$ such that

$$\begin{aligned} u'(t^*) &= 0, \qquad u''(t^*) > 0, \\ u'(t^{**}) &= 0, \qquad u''(t^{**}) < 0. \end{aligned}$$

Then according to (23) we have

$$u''(t)u'(t) + p(t)u(t)u'(t) = \beta(t)u'(t) \quad \text{in } [T_1, \infty)$$

and after the integration

$$\int_{t^*}^{t^{**}} u''(t)u'(t)dt + \int_{t^*}^{t^{**}} p(t)u(t)u'(t)dt = \int_{t^*}^{t^{**}} \beta(t)u'(t)dt.$$

Because

$$\int_{t^*}^{t^{**}} p_1 u(t)u'(t)dt \le \int_{t^*}^{t^{**}} p(t)u(t)u'(t)dt \le \int_{t^*}^{t^{**}} p_2 u(t)u'(t)dt,$$

we get

$$\frac{1}{2}[\underbrace{u'(t^{**})^2}_{=0} - \underbrace{u'(t^*)^2}_{=0}] + \frac{p_1}{2} \cdot (u(t^{**}) - u(t^*)) \le \varepsilon[u(t^{**}) - u(t^*)]. \tag{28}$$

Due to the assumption $\lim_{t\to\infty} \beta(t) = 0$, therefore there exists T_1; $T_1 \ge T$ such that $\beta(t) < \varepsilon$ for all $t \ge T_1$.

According to inequality (28) we have

$$\frac{p_1}{2}[u(t^{**})^2 - u(t^*)^2] \le \varepsilon[u(t^{**}) - u(t^*)],$$

$$u(t^{**}) \le u(t^{**}) + u(t^*) \le \frac{2\varepsilon}{p_1}$$

and this is a contradiction to (i)

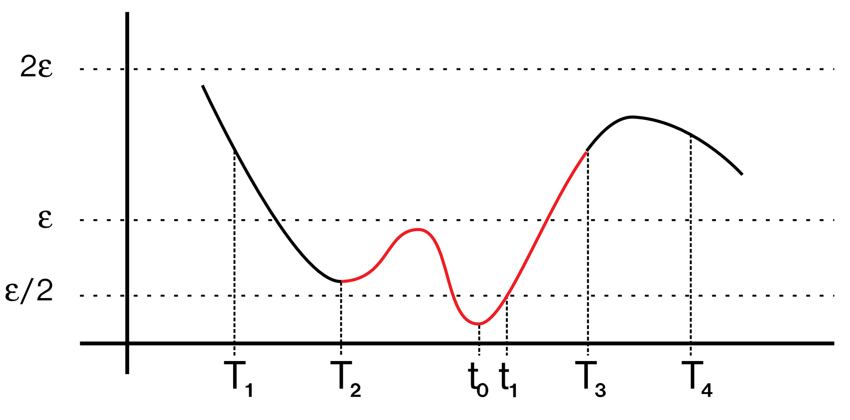
Now consider the case (ii).

Suppose that there exists $\varepsilon > 0$ such that

$$0 < \varepsilon < \limsup_{t\to\infty} u(t) < 2\varepsilon. \tag{29}$$

Let us choose β_0 such that

$$0 < \beta_0 < \frac{p_1 \varepsilon}{4}. \tag{30}$$



According to assumptions (24), (29) there exists $T_1 \geq T$ such that for all $t \geq T_1$

$$\beta(t) \leq \beta_0, \qquad u(t) \leq 2\varepsilon. \tag{31}$$

We assume $u(T_1) > 0$ thus according to (26) there exists $T_2 \geq T_1$ such that

$$u(T_2) < \min(u(T_1), \varepsilon) \tag{32}$$

and with aspect to (29) we have $T_4 \geq T_2$ with the property $u(T_4) > \varepsilon$. Thus $u(T_2) \overset{(32)}{<} \varepsilon < u(T_4)$ and we can find T_3 such that

$$T_2 \leq T_3 \leq T_4, \qquad u(T_3) > \varepsilon, \qquad u'(T_3) > 0. \tag{33}$$

Let

$$t_0 = \inf\{t \geq T_1, u'(\tau) \geq 0 \text{ for all } \tau \in [t, T_3]\}. \tag{34}$$

Since $u(.)$ is continuous (33) implies the inequality $t_0 < T_3$.

If $t_0 < T_1$, then according to (34), $u(.)$ is nondecreasing in $[T_1, T_3]$, what is a contradiction to $u(T_2) \leq u(T_1)$, which follows from (32).

We have $T_1 \leq t_0 < T_3$. According to definition of t_0, $u'(t_0) = 0$. Then $u''(t_0) = \lim\limits_{\delta \to 0^+} \frac{u'(t_0+\delta) - u'(t_0)}{\delta} \geq 0$. Using (25), (31) and (21) we obtain

$$\begin{aligned} 0 \leq u''(t_0) = \beta(t_0) - p(t_0)u(t_0) &\leq \beta_0 - p_1 u(t_0), \\ u(t_0) \leq \frac{\beta_0}{p_1} &\leq \frac{\varepsilon}{4}. \end{aligned} \tag{35}$$

According to definition (34) we have

$$u'(t) \geq 0 \qquad \text{for all } t \in [t_0, T_3]. \tag{36}$$

And by (35) we have $u'(t_0) < 0$, $u(T_3) > \varepsilon$. Then there exists t_1, $t_1 \in (t_0, T_3)$ such that $u(t_1) = \frac{\varepsilon}{2}$. Hence we can obtain for $u''(t)$ the inequality on interval $[t_0, t_1]$.

$$u''(t) \overset{(24)}{=} \beta(t) - p(t)u(t) \overset{(25),\text{(ii)}}{\leq} \beta_0 \qquad \text{on } [t_0, t_1].$$

By (36) it follows that $u''(t) \cdot u(t) \leq \beta_0 \; u'(t)$. And integrating we get

$$\begin{aligned} \int_{t_0}^{t_1} u''(t)u'(t)dt &\leq \int_{t_0}^{t_1} \beta_0 \; u'(t)dt, \\ \frac{1}{2}[u'(t_1)]^2 - \frac{1}{2}[u'(t_0)]^2 &\leq \beta_0(u(t_1) - u(t_0)), \\ \text{clearly} \qquad \frac{1}{2}[u'(t_1)]^2 &\leq \beta_0 \frac{\varepsilon}{2}. \end{aligned} \tag{37}$$

If $t \in [t_1, T_3]$ then we can obtain from (24), (25), (30)

$$u''(t) = \beta(t) - p(t)u(t) \leq \beta_0 - p_1 u(t_1) \leq -\frac{p_1 \cdot \varepsilon}{4}.$$

Since $u'(t) \geq 0$ on $[t_1, T_3]$ we have

$$\int_{t_1}^{T_3} u''(t)u'(t)dt \leq \int_{t_1}^{T_3} -\frac{\varepsilon p_1}{4} u'(t)dt.$$

Thus

$$\frac{1}{2}[u'(t_1)]^2 \geq \frac{\varepsilon \cdot p_1}{4}[u(T_3) - u(t_1)] \geq \frac{\varepsilon \cdot p_1}{4}\frac{\varepsilon}{2}.$$

And according to (37)

$$\frac{\varepsilon \cdot \beta_0}{2} \geq \frac{1}{2}(u'(t_1))^2 \geq \frac{\varepsilon^2 p_1}{8},$$

which implies $\beta_0 \geq \frac{\varepsilon p_1}{4}$.

The last inequality gives a contradiction to (ii). So

$$\liminf_{t\to\infty} u(t) = \limsup_{t\to\infty} u(t) = \lim_{t\to\infty} u(t) = 0.$$

Theorem 5. *Let u be a solution of equation (1)*

$$(g(t)[u''(t) + p(t)u(t)])' = f(t, u, u', u''),$$

where $p(.)$, $g(.)$ and $f(.)$ fulfill conditions (2), (3), (17), (18) and (19).

Then equation (1) has the property A, so every proper solution of (1) is either oscillatory or it converges with its derivatives to zero as $t \to \infty$.

Proof. We proceed by contradiction.

Let $u(.)$ be a nonoscillatory solution. According to Lemma 3 and Theorem 4 we have $\lim_{t\to\infty} u(t) = 0$. Statement (iii) in Lemma 3 gives us that $\lim_{t\to\infty} u''(t) = 0$. So we need only to prove that $u'(t) \to 0$ for $t \to \infty$.

Let $\lim_{t\to\infty} u'(t) \not\to 0$, then $\limsup_{t\to\infty} |u'(t)| \geq A > 0$. Thus in any neighbourhood of ∞ we can find t_0 such that $|u'(t_0)| \geq \frac{A}{2} > 0$.

We have $\lim_{t\to\infty} u(t) = 0$, $\lim_{t\to\infty} u''(t) = 0$. Then we take $0 < \varepsilon < \frac{A}{6}$ such that $|u''(t)| \leq \varepsilon$, $|u(t)| \leq \varepsilon$ in $[t_0 - 1, t_0 + 1]$.

Then on interval $[t_0 - 1, t_0 + 1]$ we get the following inequalities (t_0 is sufficiently great):

$$u'(t) - u'(t_0) = u''(t_0 + \xi(t - t_0))(t - t_0),$$
$$|u'(t)| = \underbrace{|u'(t_0)|}_{\geq \frac{A}{2}} - \underbrace{|u''(t_0 + \xi(t - t_0))|}_{\leq \varepsilon} \cdot \underbrace{|t - t_0|}_{\leq 1},$$

and therefore

$$|u'(t)| \geq \frac{A}{2} - \varepsilon \qquad \text{for all } t \in [t_0 - 1, t_0 + 1].$$

Farther $|u(t_0)| \leq \varepsilon$ and hence

$$\begin{aligned} u(t) &= u(t_0) + u'(t_0 + \xi(t - t_0))(t - t_0), \\ |u(t)| &\geq |u'(t_0 + \xi(t - t_0)| \cdot |t - t_0| - |u(t_0)|, \\ |u(t)| &\geq (\frac{A}{2} - \varepsilon)|t - t_0| - \varepsilon \qquad \text{for all } t \in [t_0 - 1, t_0 + 1]. \end{aligned}$$

We put $t = t_0 + 1$. Then $|u(t_0 + 1)| \geq (\frac{A}{2} - \varepsilon) - \varepsilon = \frac{A}{2} - 2\varepsilon$. Since we took $0 < \varepsilon < \frac{A}{6}$, we get $|u(t_0 + 1)| > \varepsilon$, what is a contradiction to assumption $|u(t_0 + 1)| \leq \varepsilon$.

Thus $\lim_{t \to \infty} u'(t) = 0$ and so equation (1) has the property A.

References

[1] Kiguradze, I. T. , *An oscillation criterion for a class of ordinary differential equations* , J. Diff. Equations, **28**(1992), 207–219