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Global Qualitative Investigation, Limit Cycle Bifurcations and Applications of Polynomial Dynamical Systems

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Abstract. Two-dimensional polynomial dynamical systems are mainly considered. By means of Erugin's two-isocline method we carry out the global qualitative investigation of such systems, construct canonical systems with field-rotation parameters and study limit cycle bifurcations.
It is known, for example, that generic quadratic systems with limit cycles have three field-rotation parameters and bifurcation surfaces of multiplicity-two and three limit cycles are familiar saddle-node and cusp bifurcation surfaces respectively. We use the canonical systems, cyclicity results and apply Perko's termination principle, stating that the boundary of a global limit cycle bifurcation surface typically consists of Hopf bifurcation surfaces and homoclinic (or heteroclinic) loop bifurcation surfaces, to prove the non-existence of swallow-tail bifurcation surface of multiplicity-four limit cycles for quadratic systems.
We discuss also possibilities of application of obtained results to the study of higher-dimensional dynamical systems and systems with more complicated dynamics.

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1 Introduction

This paper is connected with the development of a global bifurcation theory of dynamical systems and discussing possibilities of its application to more complicated systems. First of all, it is directed to the solution of Hilbert's 16th Problem on the maximum number and relative position of limit cycles of two-dimensional dynamical systems given by the equations

$$\frac{dx}{dt} = P_n(x,y), \qquad \frac{dy}{dt} = Q_n(x,y), \tag{1}$$

This is the final form of the paper.

where $P_n(x, y)$ and $Q_n(x, y)$ are polynomials of real variables x, y with real coefficients and not greater than n degree.

This is the most difficult problem in the qualitative theory of systems (1). There are a lot of methods and results for the study of limit cycles [1], [2]. However, the Problem has not been solved completely even for the case of simplest (quadratic) systems. It is known only that a quadratic system has at least four limit cycles in (3:1) distribution [3]. Besides, we can finally state that a general polynomial system has at most a finite number of limit cycles [4]–[6].

A new impulse to the study of limit cycles was given by the introduction of ideas coming from Bifurcation Theory [7]–[10]. We know three principal bifurcations of limit cycles: 1) Andronov-Hopf bifurcation (from a singular point of centre or focus type); 2) separatrix cycle bifurcation (from a homoclinic or heteroclinic orbit); 3) multiple limit cycle bifurcation. The first bifurcation was studied completely only for quadratic systems: the number of limit cycles bifurcating from a singular point (its cyclicity) is equal to three [11]. For cubic systems the cyclicity of a singular point is not less than eleven [12]. The second bifurcation was studying in [13]–[15]. Now we have the classification of separatrix cycles and know the cyclicity of the most of them (of elementary graphics). The last bifurcation is the most complicated. Multiple limit cycles were considering, for example, in [16] and [17]–[19]. All mentioned bifurcations can be generalized for higher-dimensional dynamical systems [20]–[22] and can be used for the study of systems with more complicated dynamics [23]–[25].

But how to find a way to the solution of Hilbert's 16th Problem? Unfortunately, all known limit cycle bifurcations are local. We consider only a neighborhood of either the point or the separatrix cycle, or the multiple limit cycle. We consider also local unfoldings in the parameter space. It needs a qualitative investigation on the whole (both on the whole phase plane and on the whole parameter space), i.e., it needs a global bifurcation theory. This is the first idea introduced for the first time by N. P. Erugin in [26]. Then we should connect all limit cycle bifurcations. This idea came from the theory of higher-dimensional dynamical systems. It was contained in Wintner's principle of natural termination [27] and was used by L. M. Perko for the study of multiple limit cycles in two-dimensional case [17]–[19]. At last, we must understand how to control the limit cycle bifurcations. The best way to do it is to use field-rotation parameters considered for the first time by G. F. Duff in [28]. If we are able to realize these ideas we will answer many questions: 1) How to prove that the maximum number of limit cycles in a quadratic system is equal to four and their only possible distribution is (3:1)? 2) How to construct a cubic system with more than eleven limit cycles? 3) What is a strategy of the qualitative investigation on the whole for cubic and general polynomial systems? 4) How to generalize obtained results for the study of higher-dimensional dynamical systems and to use them for more complicated systems?

2 Methods and technical difficulties

Methods and approaches used for the study of limit cycles are very different: analytic, algebraic, geometric, etc. There are various combinations. In [4], for example, classical analytic methods are applied to the analysis of normal forms in special cases of polycycles. The techniques of [5] and [6] are much more sophisticated. However, by means of these methods we can prove only the finiteness of the number of limit cycles. In the particular case of quadratic (cubic) systems we do not need such powerful methods, since the number of possible situations is rather limited. It is enough to show that limit cycles cannot accumulate on any separatrix cycle. Other techniques are used in [13]–[15] where families of planar quadratic vector fields are considered and the cyclicity of unfoldings for various limit periodic sets is estimated. This is a new combination: of analytic and bifurcation methods. But it does not work for non-elementary (non-monodromic) limit periodic sets. It needs a global blow-up of some unfoldings. Even after such a desingularization we will have only an upper bound of the number of limit cycles. We must find the least upper bound of the number and estimate the relative position of limit cycles!

Purely algebraic methods, for instance, are not able to solve even simpler problems: to distinguish centre and focus or to give the number of small amplitude limit cycles bifurcating from a singular point at least for cubic systems. These problems are algorithmically solvable. Nevertheless, it is still complicated to calculate all the Liapunov focus quantities and to estimate their independence. There are some types of integrable cubic systems: reversible, Hamiltonian, Darboux integrable. For the study of limit cycles we perturbate such systems, consider linear and higher order Abelian integrals (monodromy variations). But only eleven small amplitude limit cycles can be obtained in this way at present [12].

We will develop a geometric aspect of Bifurcation Theory. It will give a global approach to the qualitative investigation and will help to combine all other approaches, their methods and results. We will use the two-isocline method, which was developed by N. P. Erugin for two-dimensional systems [26] and then was generalized by his pupil V. A. Pliss for three-dimensional case [29]. An isocline portrait is the most natural construction in the corresponding polynomial equation. It is enough to have only two isoclines (of zero and infinity) to obtain a principal information on the original system, because these two isoclines are right-hand sides of the system. We know geometric properties of isoclines (conics, cubics, etc.) and can easily get all isoclines portraits. By means of them we can obtain all topologically different qualitative pictures of integral curves to within a number of limit cycles and distinguishing centre and focus. So, we will be able to carry out a rough topological classification of the phase portraits for the polynomial systems. It is the first application of Erugin's two-isocline method.

Studying contact and rotation properties of isoclines we can also construct the simplest (canonical) systems containing limit cycles. Two groups of parameters can be distinguished in such systems: static and dynamic. Static parameters determine a behavior of the phase trajectories in principle, since they control the number, position and type of singular points in finite part of the plane (finite singularities). Parameters from the first group determine also a possible behavior of separatrices and singular points at infinity (infinite singularities) under the variation of parameters from the second group. Dynamic parameters are rotation parameters. They do not change the number, position and index of finite singularities and involve a directional rotation in the vector field. The rotation parameters allow to control infinite singularities, a behavior of limit cycles and separatrices. The cyclicity of singular points and separatrix cycles, the behavior of semistable and other multiple limit cycles are controled by these parameters as well. Thus, with the help of rotation parameters, we can control all limit cycle bifurcations, i.e., we can solve the most fine qualitative problems and carry out the global qualitative investigation of the polynomial systems.

Of course, some technical difficulties may arise in such investigation. We have a good tool: rotation parameters. But we have no enough experience to use them. To control all limit cycle bifurcations (especially, of multiple limit cycles), we should know the properties and combine the effects of all rotation parameters. These difficulties can be overcome by means of the development of new methods based on Perko's planar termination principle stating that the maximal one-parameter family of multiple limit cycles terminates either at a singular point, which is typically of the same multiplicity, or on a separatrix cycle, which is also typically of the same multiplicity [19]. This principle is a consequence of Wintner's principle of natural termination, which was stated for higher-dimensional dynamical systems in [27] where A. Wintner studied one-parameter families of periodic orbits of the restricted three-body problem and used Puiseux series to show that in the analytic case any one-parameter family of periodic orbits can be uniquely continued through any bifurcation except a period-doubling bifurcation. Such a bifurcation can happen, for example, in a three-dimensional Lorenz system. Besides, the periods in a one-parameter family of a higher-dimensional system can become unbounded in strange ways: for example, the periodic orbits may belong to a strange invariant set (strange attractor) generated at a bifurcation value for which there is a homoclinic tangency of the stable and unstable manifolds of the Poincaré map [17]. This cannot happen for planar systems. It would be interesting (in the case of success) to generalize results on multiple limit cycle bifurcations to more complicated systems.

3 Aims and preliminary results

Global bifurcation theory of quadratic systems. It is known that the generic quadratic system with limit cycles has three rotation parameters and bifurcation surfaces of multiplicity-two and three limit cycles are familiar saddle-node and cusp bifurcation surfaces respectively. We will apply Perko's termination principle to prove the non-existence of swallow-tail bifurcation surface of multiplicity-four limit cycles, i.e., using the data on the cyclicity of singular points and separatrix cycles we will prove by contradiction that a quadratic system cannot have more than four limit cycles, that the distributions (4:0), (2:2) are impossible and the multiplicity of a limit cycle is not higher than three. Thus we intend to prove that for quadratic systems the maximum number of limit cycles is equal to four and the only possible their distribution is (3:1).

Cubic and general polynomial systems. First of all, a general strategy of the qualitative investigation on the whole will be developed. The main results by analogy with quadratic systems will be obtained and systematized. We will apply Erugin's two-isocline method to get all isocline portraits of cubic systems, to carry out the rough topological classification of their phase portraits and to construct the canonical systems with field-rotation parameters, which will be used for various aims: study of limit cycle bifurcations, classification of separatrix cycles, obtaining bifurcation diagrams. We will use Poincaré topographical systems and small parameter method, Abelian integrals and variational methods to construct a cubic system with more than eleven limit cycles. All these results will be generalized to develop a global bifurcation theory of planar polynomial systems.

Higher-dimensional dynamical systems and applications. We will apply the theory of planar dynamical systems to the qualitative investigation of higher-dimensional systems. Various bifurcations in reversible, Hamiltonian and conservative systems will be considered: Hopf bifurcation, bifurcations of homoclinic and heteroclinic orbits (including degenerate cases). Multiple limit cycle bifurcations with the application of Wintner's principle of natural termination will be studied as well. Since theory of dynamical systems and bifurcation methods can be used for the mathematical modelling natural systems with complicated dynamics, we will consider possibilities of the application of global bifurcation theory, for instance, to the study of Josephson junctions in forsed non-linear dynamical networks, non-linear evolution systems in Belousov-Zhabotinskii reaction, etc.

Results. A particular preliminary work in this direction has already been carried out in [30]–[41]. By means of Erugin's two-isocline method we carried out the global qualitative investigation of quadratic systems, constructed the canonical systems with field-rotation parameters and applied them for the study of limit cycle bifurcations: Andronov-Hopf bifurcation, bifurcations of homoclinic and heteroclinic orbits (separatrix cycles), multiple limit cycle bifurcations. We studied the bifurcations of various codimensions and introduced so-called a function of limit cycles: a cross-section of Andronov-Hopf surface formed by limit cycles and the corresponding values of rotation parameters. Using numerical and analytic methods, we constructed concrete examples of systems with different number and relative position of limit cycles. In particular, the following theorems have been proved:

Theorem 1. *A quadratic system has at least four limit cycles in (3:1) distribution.*

Theorem 2. *. Any quadratic system with limit (separatrix) cycles can be reduced to one of the systems:*

$$\frac{dx}{dt} = -(x+1)y + \alpha Q(x,y), \qquad \frac{dy}{dt} = Q(x,y) \tag{2}$$

or

$$\frac{dx}{dt} = -y + \alpha y^2, \qquad \frac{dy}{dt} = Q(x,y), \tag{3}$$

where

$$Q(x,y) = x + \lambda y + ax^2 + b(x+1)y + cy^2.$$

We developed a new approach to the classification of separatrix cycles. It is based on the application of canonical systems (2) and (3). The classification was carried out according to the number and type of finite singularities of the original reversible systems and with the help of the successive variation of rotation parameters. That approach allowed not only to define all possible types of separatrix cycles, but also to determine their cyclicity and relative position, to obtain both the corresponding phase portraits and the division of parameter space. By means of the field-rotation parameters and function of limit cycles we showed how to control semistable limit cycles: we were changing the rotation parameters so that to push the semistable cycles either to a singular point of focus (centre) type or to some separatrix cycle and to obtain the contradiction with their cyclicity. On the basis of reversible systems we constructed Poincaré topographical systems and with the help of small parameter method studied various periodic orbits: limit cycles, centre curves. We developed also a control theory of quadratic systems and considered possibilities of the application of our results to general polynomial systems.

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