EQUADIFF 9 CD ROM, Brno 1997 PROCEEDINGS
Masaryk University pp. 49–61

Transformations and Oscillatory Properties of Linear Hamiltonian Systems — Continuous versus Discret

Ondřej Došlý

Department of Mathematics, Masaryk University, Janáčkovo nám. 2a,
662 95 Brno, Czech Republic
Email: dosly@math.muni.cz

Abstract. Transformations and oscillatory properties of discrete and continuous linear Hamiltonian systems are investigated. A particular attention is devoted to the so-called reciprocity principle and trigonometric transformation for these systems.

AMS Subject Classification. 34C10, 39A10

Keywords. Linear Hamiltonian system, trigonometric tranformation, reciprocity principle, symplectic system, principal solution, time scale

1 Introduction

The aim of this contribution is to present a survey of the recent results on transformations and oscillatory behaviour of solutions of linear Hamiltonian systems — both differential and difference – and to suggest some directions for the further investigation.

We consider the differential Hamiltonian system

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u, \tag{1.1}$$

and its difference (= discrete) counterpart

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k. \tag{1.2}$$

We suppose that $t \in I \subseteq \mathbb{R}$, $k \in [0, N] \cap \mathbb{N}$, $N \in \mathbb{N}$, both in continuous and discrete case A, B, C are $n \times n$ matrices, B, C are symmetric, i. e. $B = B^T$, $C = C^T$. Moreover, in the continuous case we suppose that the matrix B is nonnegative definite and in the discrete case that the matrix $(I - A_k)$ is nonsingular, its inverse we denote by $\tilde{A}_k$.

Linear Hamiltonian systems cover a large variety of linear equations. For example, the even order, self-adjoint, differential equation

$$\sum_{\nu=0}^{n} (-1)^\nu \left(r_\nu(t) y^{(\nu)} \right)^{(\nu)} = 0 \tag{1.3}$$

can be written as LHS (1.1) using the substitution

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} (-1)^{n-1}(r_n y^{(n)})^{(n-1)} + \cdots + r_1 y' \\ \vdots \\ -(r_n y^{(n)})' + r_{n-1} y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix}.$$

Then (x, u) solves (1.1) with A, B, C given by

$$B(t) = \operatorname{diag}\{0, \dots, 0, r_n^{-1}(t)\}, \quad C(t) = \operatorname{diag}\{r_0(t), \dots, r_{n-1}(t)\},$$
$$A = A_{ij} = \begin{cases} 1, & \text{if } j = i+1,\ i = 1, \dots, n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

Another example is the second order system

$$[R(t)x' + Q(t)x]' - \left[Q^T(t)x' + P(t)x\right] = 0 \tag{1.4}$$

with $n \times n$ matrices P, Q, R, whereby P, R are symmetric and R nonsingular. Putting $u = R(t)x' + Q(t)x$, the pair of n-vectors (x, u) solves (1.1) with $B = R^{-1}$, $A = -R^{-1}Q$, $C = P - Q^T R^{-1} Q$. Discrete analogies of (1.3) and (1.4) can be written in the form (1.2) using essentially the same substitutions as in the continuous case.

Investigation of oscillatory properties of continuous system (1.1) has a relatively long history and was initiated by the paper of Morse [17] from 1930. Since that time oscillation theory of (1.1) attracted a considerable attention and the results of this investigation up to seventies of this century can be found in the monograph of Reid [18]. In 1995 Kratz [16] published another comprehensive monograph which in addition to the classical results contains also the results achieved in the period 1980–95.

In contrast to the continuous case, oscillation theory of discrete systems (1.2) is much less developed and the fundamental result of this theory, a discrete version of the so-called Roundabout Theorem, was established only very recently by Bohner [8]. This paper accomplished the effort of several mathematicians in the last decade to prove the discrete Roundabout Theorem in its full generality, see [4].

Here we concentrate our attention to the investigation of oscillatory properties and transformations of Hamiltonian systems (1.1) and (1.2). The paper is organized as follows. In the next section we present basic facts of oscillation and transformation theory of continuous systems (1.1), in particular, we formulate trigonometric transformation and reciprocity principle for these systems. Section 3 is devoted to some aspects of transformation theory of Hamiltonian difference systems (1.2) and we give here essentially discrete versions of statements of Section 2. In the last section we discuss some aspects of unified approach to continuous and discrete systems via theory of differential equations on the so-called time scales.

2 Continuous Hamiltonian Systems

We start with basic concepts of oscillation theory of (1.1).

Definition 1. Two points t_1, t_2 are said to be *conjugate* relative to (1.1) if there exists a solution (x, u) such that $x(t_1) = 0 = x(t_2)$ and $x(t) \not\equiv 0$ in $[t_1, t_2]$. System (1.1) is said to be *conjugate* in an interval $[a, b]$ if there exist $t_1, t_2 \in [a, b]$ which are conjugate relative to (1.1), in the opposite case (1.1) is said to be *disconjugate*. System (1.1) is said to be *oscillatory* if for every $c \in \mathbb{R}$ this system is conjugate in $[c, \infty)$, in the opposite case (1.1) is said to be *nonoscillatory*.

As mentioned in the previous section, principal statement concerning oscillatory properties of (1.1) is the so-called Reid Roundabout Theorem [18]. Before formulating it, we recall some very elementary properties of solutions of Hamiltonian systems (1.1).

Simultaneously with (1.1) we consider its matrix analogy

$$X' = A(t)X + B(t)U, \quad U' = C(t)X - A^T(t)U, \tag{2.1}$$

where X, U are $n \times n$ matrices. If $(X, U), (\tilde{X}, \tilde{U})$ are two solutions of (2.1) then the "Wronskian-type" identity $X^T\tilde{U} - U^T\tilde{X} \equiv K$ holds, where K is a constant $n \times n$ matrix. A solution (X, U) of (2.1) is said to be *conjoined* if X^TU is symmetric and it is said to be *conjoined basis* if, moreover, rank $(X^T, U^T) = n$. Recall also that (1.1) is said to be *controllable* in an interval I whenever the trivial solution $(x, u) \equiv (0, 0)$ is the only solution of (1.1) for which $x(t) \equiv 0$ on some nondegenerate subinterval $I_0 \subseteq I$.

Proposition 1 (Reid [18]). *Suppose that the matrix $B(t)$ is nonnegative in the interval $[a, b]$ and that (1.1) is controllable in this interval. Then the following statements are equivalent:*

(i) *System (1.1) is disconjugate in the interval $[a, b]$.*

(ii) *The quadratic functional*

$$\mathcal{F}(x, u) = \int_a^b [u^T(t)B(t)u(t) + x^T(t)C(t)x(t)]dt$$

is positive for every nontrivial (x, u) satisfying $x' = A(t)x + B(t)u$ and $x(a) = 0 = x(b)$.

(iii) *The solution (X, U) of (2.1) given by the initial condition $X(a) = 0$, $U(a) = I$ satisfies $\det X(t) \neq 0$, $t \in (a, b]$.*

(iv) *There exists a conjoined basis (X, U) of (2.1) such that $X(t)$ is nonsingular for $t \in [a, b]$.*

(v) *There exists a symmetric matrix Q which for $t \in [a, b]$ solves the Riccati matrix differential equation*

$$Q' - C(t) + A^T(t)Q + QA(t) + QB(t)Q = 0 \tag{2.2}$$

related to (2.1) by the substitution $Q = UX^{-1}$.

For a better understanding of this statement we suggest the reader to see (1.1) as a rewritten second order equation and to compare this statement with the well known results of oscillation theory of second order equations, see e.g. Swanson [20].

Now we state some results concerning transformations of LHS. A $2n \times 2n$ matrix $\mathcal{R}$ is said to be *symplectic* if $\mathcal{R}^T \mathcal{J} \mathcal{R} = \mathcal{J}$, where $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I being the $n \times n$ identity matrix. If $\mathcal{R} = \begin{pmatrix} H & K \\ M & N \end{pmatrix}$, where H, K, M, N are $n \times n$ matrices then $\mathcal{R}$ is symplectic if and only if

$$H^T K = K^T H, \;\; M^T N = N^T M, \;\; H^T N - K^T M = I. \tag{2.3}$$

Consider the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \mathcal{R}(t) \begin{pmatrix} y \\ z \end{pmatrix}, \;\; \text{where } \mathcal{R}(t) = \begin{pmatrix} H(t) & M(t) \\ K(t) & N(t) \end{pmatrix} \tag{2.4}$$

is symplectic and continuously differentiable. Then the new variables y, z satisfy the LHS

$$y' = \bar{A}(t)y + \bar{B}(t)z, \quad z' = \bar{C}(t)y - \bar{A}^T(t)z, \tag{2.5}$$

where

$$\begin{aligned}
\bar{A} &= N^T[-H' + AH + BK] - M^T[-K' + CH - A^T K], \\
\bar{B} &= N^T[-M' + AM + BN] - M^T[-N' + CM - A^T N], \\
\bar{C} &= -K^T[-H' + AH + BK] + H^T[-N' + CM - A^T N].
\end{aligned}$$

Observe that in the case $M(t) \equiv 0$ transformation (2.4) preserves oscillatory properties of transformed systems since then $H(t)$ is nonsingular (compare (2.3)), hence t_1, t_2 are conjugate relative to (1.1) if and only if they are conjugate relative to (2.5). Consequently, transformation (2.4) with $M(t) \equiv 0$ is the powerful tool for the investigation of oscillatory properties of (1.1). This system is transformed into an “easier” system and from oscillatory properties of this “easy” system we deduce oscillatory properties of (1.1). One of such “easy systems” is the so-called trigonometric system.

Theorem 1 (Došlý [10]). *There exist continuously differentiable $n \times n$ matrices H, K such that H is nonsingular, $H^T K \equiv K^T H$ and the transformation*

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} H(t) & 0 \\ K(t) & (H^T(t))^{-1} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{2.6}$$

transforms (1.1) *into the trigonometric system*

$$y' = Q(t)z, \quad z' = -Q(t)y, \tag{2.7}$$

where Q is a nonnegative definite symmetric $n \times n$ matrix.

The trigonometric system was introduced in [7] in connection with the Prüfer-type transformation for (1.1) and the terminology *trigonometric system* is justified by the fact that in the scalar case, i.e. when y, z, q are scalar quantities, then

$$(y_1, z_1) = (\sin \int^t q(s)\,ds, \cos \int^t q(s)\,ds),$$
$$(y_2, z_2) = (\cos \int^t q(s)\,ds, -\sin \int^t q(s)\,ds)$$

form the basis of the solution space of (2.7). In the n-dimensional case system (2.7) cannot be in general solved explicitly, but it may be proved that its solutions have many properties which in the scalar case reduce to the well-known trigonometric identities and these properties we may use to study properties of (1.1). For example, (2.7) is oscillatory if and only if

$$\int^\infty \operatorname{Tr} Q(t)\,dt = \infty,$$

where Tr stands for the trace, i.e. the sum of the diagonal entries of the matrix indicated.

Now turn our attention to the reciprocity principle for LHS. We start with the following elementary example. Consider the second order equation

$$(r(t)y')' + p(t)y = 0 \tag{2.8}$$

with *positive* coefficients r, p. If we denote $z = r(t)y'$ then this function verifies the so-called *reciprocal equation*

$$\left(\frac{1}{p(t)}z'\right)' + \frac{1}{r(t)}z = 0. \tag{2.9}$$

Using an elementary argument it is easy to see that a solution y of (2.8) oscillates if and only if its derivatives y' oscillates, i.e. (2.8) is oscillatory if and only if (2.9) is oscillatory. These equations may be written in the form of Hamiltonian system (1.1)

$$y' = \frac{1}{r(t)}z, \quad z' = -p(t)y \tag{2.10}$$

and

$$\tilde{y}' = p(t)\tilde{z}, \quad \tilde{z}' = -\frac{1}{r(t)}\tilde{y} \tag{2.11}$$

and these systems are related by the transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix}. \tag{2.12}$$

The statement concerning relation between oscillatory behaviour of these systems may be now formulated as follows: If the functions r, p are positive then transformation (2.12) preserves oscillation properties of transformed 2×2 systems, i.e. (2.10) is oscillatory if and only (2.11) is oscillatory.

Reciprocity principle concerns extension of this statement to Hamiltonian system (1.1).

Theorem 2 (Ahlbrandt [2]). *Suppose that $B(t) \geq 0$, $C(t) \leq 0$ (this means that B is nonnegative definite and C nonpositive definite) for large t and both system* (1.1) *and its reciprocal system*

$$y' = -A^T(t)y - C(t)z, \quad z' = -B(t)y + A(t)z \tag{2.13}$$

are eventually controllable (i.e., the trivial solution $(x, u) = (0, 0)$ is the only solution of (1.1) *for which one of the components x, u is eventually vanishing). Then* (1.1) *is oscillatory if and only if* (2.13) *is oscillatory.*

Obviously, this statement is a generalization of the relationship between (2.10) and (2.11) and claims, roughly speaking, that (1.1) is oscillatory with respect to the first component x if and only if it is oscillatory with respect to the second component u (compare Definition 1). Indeed, (2.13) results from (1.1) upon the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix} \tag{2.14}$$

which essentially only reverses the order of equations in (1.1). In another words, under definiteness assumption on the matrices B, C, transformation (2.14) preserves oscillatory properties of transformed systems.

The above mentioned reciprocity principle may be easily shown to be a particular case of the following general statement concerning transformations of (1.1) preserving oscillatory behaviour of transformed systems.

Theorem 3 (Došlý [11]). *Consider Hamiltonian systems* (1.1) *and* (2.5) *related by transformation* (2.4) *and suppose that matrices $B(t), \bar{B}(t)$ in these systems are nonnegative definite for large t. Then* (1.1) *is oscillatory if and only if* (2.5) *is oscillatory.*

This statement is proved using the trigonometric transformation given in Theorem 1. Systems (1.1) and (2.5) are transformed into trigonometric systems (using transformation of the form (2.6) which preserves oscillatory properties) with matrices Q and $\bar{Q}$ and then it is shown that $\int^\infty \operatorname{Tr} Q(t)\, dt = \infty$ if and only if $\int^\infty \operatorname{Tr} \bar{Q}(t)\, dt = \infty$. This means that these trigonometric systems and hence also systems (1.1), (2.5) are simultaneously oscillatory or nonoscillatory.

3 Discrete Hamiltonian Systems

Similar to the continuous case, we start with the definition of basic concepts.

Definition 2. We say that an interval $(k, k+1]$, $k \in \mathbb{N}$, contains a *generalized zero* of a solution (x, u) of (1.2) if $x_k \neq 0$ and there exists $c \in \mathbb{R}^n$ such that

$$x_{k+1} = \tilde{A}_k B_k c, \quad \text{and} \quad x_k^T B_k^\dagger (I - A_k) x_{k+1} \leq 0.$$

System (1.2) is said to be *disconjugate* in an interval $[n, m]$ if any solution of (1.2) has at most one generalized zero in $[n, m+1]$ and, moreover, any solution satisfying $x_n = 0$ has no generalized zero in $(n, m+1]$, in the opposite case (1.2) is said to be *conjugate* in $[n, m]$. System (1.2) is said to be *nonoscillatory* if there exists $n \in \mathbb{N}$ such that (1.2) is disconjugate on $[n, m]$ for every $m > n$, in the opposite case (1.2) is said to be *oscillatory*.

In the above definition † denotes the Moore-Penrose generalized inverse matrix, for an $n \times n$ matrix V its generalized inverse $V^\dagger$ is the (unique) $n \times n$ matrix such that $VV^\dagger$, $V^\dagger V$ are symmetric and $V^\dagger V V^\dagger = V^\dagger$, $VV^\dagger V = V$.

Basic oscillatory properties of discrete Hamiltonian systems are summarized in the discrete version of Roundabout Theorem.

Proposition 2 (Bohner [8]). *The following statements are equivalent:*

(i) *System (1.2) is disconjugate in the interval* $[0, N]$, $N \in \mathbb{N}$.

(ii) *The discrete quadratic functional*

$$\mathcal{F}(x, u) = \sum_{k=0}^{N} \left\{ u_k^T B_k u_k + x_{k+1}^T C_k x_{k+1} \right\}$$

is positive for every (x, u) *satisfying* $\Delta x_k = A_k x_{k+1} + B_k u_k$ *with* $x_0 = 0 = x_{N+1}$ *and* $x \not\equiv 0$.

(iii) *The matrix solution* (X, U) *of (1.2) given by the initial condition* $X_0 = 0$, $U_0 = I$ *satisfies*

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \qquad \text{and} \qquad X_k X_{k+1}^\dagger \tilde{A}_k B_k \geq 0, \;\; k = 1, \ldots, N.$$

(iv) *There exists a conjoined basis* (X, U) *of (1.2) (this is defined in the same way as for (1.1)) such that* X_k *are nonsingular and* $X_k X_{k+1}^{-1} \tilde{A}_k B_k \geq 0$, $k = 0, \ldots, N$.

(v) *There exist symmetric matrices* Q_k *such that* $(I + B_k Q_k)$ *are nonsingular,* $(I + B_k Q_k)^{-1} B_k \geq 0$, *and verify the discrete Riccati matrix difference equation*

$$Q_{k+1} = C_k + (I - A_k^T) Q_k (I + B_k Q_k)^{-1} (I - A_k), \tag{3.1}$$

$k = 0, \ldots, N$.

Concerning transformations of discrete LHS (1.2), the situation is not so easy as in the continuous case. In the discrete case it is supposed that the matrices $(I - A_k)$ are nonsingular and this assumption must satisfy also the system resulting after a transformation. To ensure this, we need an extra assumption as shows the next theorem.

Theorem 4 (Došlý [12]). *Let $\mathcal{R}_k$ be a $2n \times 2n$ symplectic matrix consisting of $n \times n$ matrices $\mathcal{R}_k = \begin{pmatrix} H_k & M_k \\ K_k & N_k \end{pmatrix}$ such that the matrix*

$$\begin{pmatrix} H_k + B_k K_k & (I - A_k) M_{k+1} \\ (I - A_k^T) K_k & N_{k+1} - C_k M_{k+1} \end{pmatrix} \tag{3.2}$$

is nonsingular and denote $\begin{pmatrix} D_k & F_k \\ E_k & G_k \end{pmatrix}$ its inverse. The transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \mathcal{R}_k \begin{pmatrix} y \\ z \end{pmatrix} \tag{3.3}$$

transforms (1.2) into the system

$$\Delta y_k = \bar{A}_k y_{k+1} + \bar{B}_k z_k, \quad \Delta z_k = \bar{C}_k y_{k+1} - \bar{A}_k^T z_k, \tag{3.4}$$

where

$$\begin{aligned} \bar{A}_k &= D_k(-\Delta H_k + A_k H_{k+1} + B_k K_k) + F_k(-\Delta K_k + C_k H_{k+1} - A_k^T K_k), \\ \bar{B}_k &= D_k(-\Delta M_k + A_k M_{k+1} + B_k N_k) + F_k(-\Delta N_k + C_k M_{k+1} - A_k^T N_k), \\ \bar{C}_k &= E_k(-\Delta H_k + A_k H_{k+1} + B_k K_k) + G_k(-\Delta K_k + C_k H_{k+1} - A_k^T K_k), \end{aligned}$$

in particular, the matrices $\bar{B}_k, \bar{C}_k$ are symmetric and $(I - \bar{A}_k)$ are nonsingular, i.e. (3.4) is again a difference LHS.

Having now in disposal the above given statements, we may try to extend the reciprocity principle and trigonometric transformation to discrete systems. Let us start with the reciprocity principle. If we apply transformation (3.3) with $\mathcal{R} = \mathcal{J}$ to (1.2) (this transformation relates (1.1) and (2.13) in the continuous case), it is easy to see that the assumption of Theorem 4 concerning nonsingularity of the matrix in (3.2) is not generally satisfied, i.e. the resulting *(reciprocal)* system

$$\Delta y_k = -A_k^T y_k - C_k z_{k+1}, \quad \Delta z_k = -B_k y_k + A_k z_{k+1} \tag{3.5}$$

is the system of a different kind than (1.2). In fact, the variable x which defines oscillatory properties of (1.2) appears in the right-hand-sides of this system with indices $k+1$, whereas the variable y which should define oscillations of (3.5) appears there with indices k. For this reason, Definition 2 and Proposition 2 do not apply to (3.5). However, as suggests the equivalence between oscillatory properties of the pair of second order equations $\Delta(r_k \Delta x_k) + p_k x_{k+1} = 0$ and $\Delta(p_k^{-1} \Delta z_k) + r_{k+1}^{-1} z_{k+1} = 0$ with positive r_k, p_k, which follows using the same argument as in the continuous case, one can expect some kind of similarity between oscillatory properties of (1.2) and (3.5).

In studying the relationship between (1.2) and (3.5), the principal role play the so-called *symplectic systems*, i.e. systems of the form

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = S_k \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad S_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \tag{3.6}$$

where S_k are symplectic $2n \times 2n$ matrices. Expanding forward differences in (1.2) and (3.5), it is not difficult to see that these systems are symplectic systems. Oscillation theory of symplectic systems was established in [9] and fundamental definition is the following:

Definition 3. We say that the interval $(k, k+1]$ contains the *generalized zero* of a solution (x, u) of (3.6) if $x_k \neq 0$, there exists $c \in \mathbb{R}^n$ such that

$$x_{k+1} = \mathcal{B}_k c \quad \text{and} \quad x_{k+1}^T \mathcal{B}_k^{\dagger} x_k \leq 0.$$

Oscillation and nonoscillation of symplectic systems are defined via generalized zeros in the same way as for Hamiltonian systems. Applying these definitions to (1.2) and (3.5) we get the following discrete version of the reciprocity principle.

Theorem 5 (Došlý-Bohner [9]). *Suppose that both systems (1.2) and (3.5) are eventually controllable. If $C_k \leq 0$ for large k and (1.2) is nonoscillatory, then reciprocal system (3.5) is also nonoscillatory. Conversely, if $B_k \geq 0$ for large k and (3.5) is nonoscillatory then (1.2) is also nonoscillatory.*

Essentially the same difficulty as in the the case of the reciprocity principle we meet when trying to extend the trigonometric transformation to difference Hamiltonian systems (1.2). Trigonometric system (2.7) may be characterized as a Hamiltonian system which complies with its reciprocal system. Since the reciprocity transformation does not preserve the Hamiltonian structure of transformed difference systems, also in this case we have to pass to symplectic systems. By a direct computation one may verify that the transformation $\binom{x}{u} = \mathcal{J}\binom{\tilde{x}}{\tilde{u}}$ transforms (3.6) into itself if and only if $\mathcal{D} = \mathcal{A}$, $\mathcal{C} = -\mathcal{B}$. A symplectic system (3.6) having this property we will call *self-reciprocal* and such system may be regarded as a discrete analogue of the trigonometric differential system (2.7). However, it is an open problem whether any symplectic system may be transformed (by a transformation preserving oscillatory properties, i.e. by (3.3) with $M \equiv 0$) into a self-reciprocal system. Moreover, in contrast to trigonometric systems, till now no necessary and sufficient condition for oscillation of self-reciprocal symplectic systems is known.

We finish this section with discrete version of Theorem 5. To introduce this statement, consider the transformation of symplectic system (3.6)

$$\begin{pmatrix} x \\ u \end{pmatrix} = \mathcal{R}_k \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}, \quad R_k = \begin{pmatrix} H_k & M_k \\ K_k & N_k \end{pmatrix} \tag{3.7}$$

with a symplectic $2n \times 2n$ matrix $\mathcal{R}$. Directly one can verify that this transformation transforms (2.7) into another symplectic system

$$\begin{pmatrix} \tilde{x}_{k+1} \\ \tilde{u}_{k+1} \end{pmatrix} = \tilde{S}_k \begin{pmatrix} \tilde{x}_k \\ \tilde{u}_k \end{pmatrix}, \quad \tilde{S}_k = \begin{pmatrix} \tilde{\mathcal{A}}_k & \tilde{\mathcal{B}}_k \\ \tilde{\mathcal{C}}_k & \tilde{\mathcal{D}}_k \end{pmatrix}. \tag{3.8}$$

The $n \times n$ matrices $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}}$ can be expressed via matrices H, K, M, N in a similar way as in Theorem 4 but we will not need these formulas.

Theorem 6 (Došlý - Hilscher [13]). *Suppose that systems (3.6) and (3.8) are related by transformation (3.7) with a symplectic matrix $\mathcal{R}$ and consider the following hypotheses:*

(i) *Both systems (3.6) and (3.8) are eventually controllable;*

(ii) *The matrices M and $\mathcal{A}M + \mathcal{B}N$ are eventually nonsingular;*

(iii) *Eventually, $R(NM^{-1}) \geq 0$, where $R(\cdot)$ is given by*

$$R(Q)_k \equiv Q_{k+1} - (\mathcal{C}_k + \mathcal{D}_k Q_k)(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}.$$

(iv) *Eventually, $\tilde{R}(-M^{-1}H) \geq 0$, where*

$$\tilde{R}(\tilde{Q}) := -\tilde{Q}_k + (-\tilde{Q}_{k+1}\tilde{\mathcal{B}}_k + \tilde{\mathcal{D}}_k)^{-1}(\tilde{Q}_{k+1}\tilde{\mathcal{A}}_k - \tilde{\mathcal{C}}_k).$$

If the assumptions (i), (ii), (iii) *hold and (3.6) is eventually disconjugate then (3.8) is also eventually disconjugate. Conversely, if* (i), (ii), (iv) *hold and (3.8) is eventually disconjugate then (3.6) is eventually disconjugate.*

Obviously, if $\mathcal{R} = \mathcal{J}$, i.e. $H = 0 = N$, $-K = I = M$ and (3.6) corresponds to (1.2), i.e.

$$S = \begin{pmatrix} \tilde{A} & \tilde{A}B \\ C\tilde{A} & C\tilde{A}B + I + A^T \end{pmatrix},$$

then this statement reduces to reciprocity principle given in Theorem 6.

4 Hamiltonian Systems on Time Scales

In this section we discuss briefly possibilities of a unified approach to the investigation of discrete and continuous Hamiltonian systems. One of such possibilities consists in transforming both (1.1) and (1.2) into an integral equation with Riemann-Stiltjes integrals. This approach has been offered by Reid in [19], where the Roundabout Theorem for these generalized systems is presented. However, as pointed out in [5], this method when applied to difference systems (1.2) requires the matrix B to be nonnegative definite and as shows the Roundabout Theorem for difference systems (Proposition 2) this assumption is not needed there.

Another unified approach to continuous and discrete systems is based on the theory of equations on the so-called time scales. A *time scale* $\mathbb{T}$ is defined to be any closed subset of real numbers $\mathbb{R}$ (an alternative terminology for time scale is *measure chain* [14]). On this set there are defined operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$

$$\sigma(t) := \inf\{s \in \mathbb{T},\ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T},\ s < t\}.$$

A point $t \in \mathbb{T}$ is said to be *left-dense* (l-d) if $\rho(t) = t$, *right-dense* (r-d) if $\sigma(t) = t$, *left-scattered* (l-s) if $\rho(t) < t$, *right-scattered* (r-s) if $\sigma(t) > t$ and it is said to be *dense* if it is r-d or l-d. The *graininess* μ of a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) it is defined the *generalized derivative*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where } s \in \mathbb{T}\backslash\{\sigma(t)\}.$$

As a basic reference concerning the differential and integral calculus on time scales we suggest the monograph [6] and the paper [14]. In particular cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ the generalized derivative $f^{\Delta}(t)$ reduces to the usual derivative $f'(t)$ and to the usual forward difference $\Delta f(t) = f(t+1) - f(t)$, respectively.

Linear Hamiltonian system on a time scale $\mathbb{T}$ is the system

$$x^{\Delta}(t) = A(t)x(\sigma(t)) + B(t)u(t), \quad u^{\Delta}(t) = C(t)x(\sigma(t)) - A^T(t)u(t),$$

where it is supposed that $A, B, C : \mathbb{T} \to \mathbb{R}^{n \times n}$, B, C are symmetric and $\tilde{A} = (I - \mu A)^{-1}$ exists. The corresponding quadratic functional and the Riccati matrix equation are of the form

$$\mathcal{F}(x,u) = \int_a^b \left\{u^T(t)B(t)u(t) + x^T(\sigma(t))C(t)x(\sigma(t))\right\} \Delta t \tag{4.1}$$

and

$$Q^{\Delta}(t) - C(t) + A^T(t)Q + (Q(\sigma(t)) - \mu(t)C(t))\tilde{A}(t)(A(t) + B(t)Q(t)) = 0. \tag{4.2}$$

respectively. Concerning the definition of the integral over a subset of a time scale appearing in (4.1), we will not specify explicitly this definition and we note only that this integral reduces to the usual Riemann integral in case $\mathbb{T} = \mathbb{R}$ and to the usual sum if $\mathbb{T} = \mathbb{Z}$, the exact definition of this integral is given e.g. in [6]. Substituting $\mu \equiv 0$ (continuous case) in (4.2) we get equation (2.2) and substituting $\mu \equiv 1$ (discrete case) we have (3.1). As a basic reference concerning qualitative theory of Hamiltonian systems on time scales may be regarded the recent papers of Agarwal and Bohner [1] and of Hilscher [15]. Here the main result is the "Partly Roundabout Theorem", relating positivity of the functional (4.1), existence of a symmetric solution of (4.2) and the existence of a self-conjoined basis of the matrix system

$$X^{\Delta}(t) = A(t)X(\sigma(t)) + B(t)U(t), \quad U^{\Delta}(t) = C(t)X(\sigma(t)) - A^T(t)U(t) \tag{4.3}$$

without focal points (the conjoined basis of (4.3) is defined in the same way as for (1.1) and (1.2)). The word "partly" in the name of this statement is motivated by the fact that the proofs of some implications with respect to the "classical" Roundabout Theorem are still missing and these proofs are subject of the present investigation.

The advantages of the time scale approach to Hamiltonian systems well illustrates the explanation why in the discrete oscillation theory no assumption concerning definiteness of the matrix B and controllability of (1.2) is needed, whereas in the continuous oscillation theory controllability of (1.1) it is necessary (at least for the formulation of the Roundabout Theorem in the form given here) and the assumption $B(t) \geq 0$ plays there a crucial role here — in the calculus of variations it is known as the Legendre necessary condition for positivity of the functional $\mathcal{F}$ given in Proposition 1.

Following [15], a conjoined basis (X, U) of (4.3) has no focal point in an interval $\mathcal{I} := (a, b] \cap \mathbb{T}$ provided $X(t)$ is invertible in all dense points of $\mathcal{I}$,

$$\operatorname{Ker} X(\sigma(t)) \subseteq \operatorname{Ker} X(t) \quad \text{and} \quad D(t) := X(t)(X(\sigma(t))^{\dagger}\tilde{A}(t)B(t) \geq 0 \tag{4.4}$$

in this interval. Consequently, $\mathcal{I}$ contains a focal point whenever one of the following conditions holds:

(i) There exists $s \in \mathcal{I}$ such that $\operatorname{Ker} X(\sigma(t)) \not\subseteq \operatorname{Ker} X(t)$, or
(ii) $\operatorname{Ker} X(\sigma(t)) \subseteq \operatorname{Ker} X(t)$ on $\mathcal{I}$ and X is singular at some dense point $s \in \mathcal{I}$, or
(iii) For every $t \in \mathcal{I}$ we have $\operatorname{Ker} X(\sigma(t)) \subseteq \operatorname{Ker} X(t)$, X is nonsingular in all dense points of $\mathcal{I}$, but $D(s) := X(s)X^{\dagger}(\sigma(s))\tilde{A}(s)B(s) \not\geq 0$ at some $s \in \mathcal{I}$.

Nonexistence of a focal point of the matrix solution (X, U) of (4.3) given by the initial condition $X(a) = 0, U(a) = I$ is sufficient for positivity of the functional (4.1) in the class of n-dimensional pairs (x, u) satisfying $x^{\Delta}(t) = A(t)x(\sigma(t)) + B(t)u(t)$ and $x(a) = 0 = x(b)$, see [15]. The proof of this statements is based on the generalized Picone identity where the quantity $D(t)$ defined in (4.4) plays a crucial role.

In the continuous case $\mathbb{T} = \mathbb{R}$, controllability of (1.1) implies that singularities of X are isolated, in particular, that X is nonsingular in some right neighbourhood of $t = a$. Since $\sigma(t) = t$, $\mu(t) \equiv 0$, we have $D(t) = B(t)$ and focal points of X are singularities of X or points where B fails to be nonnegative definite. However, the last possibility is eliminated by the *apriori* assumption $B \geq 0$ and focal points of X are just singularities of this matrix as it is usual in oscillation theory of differential systems. In the discrete case $\mathbb{T} = \mathbb{Z}$ all points are automatically isolated and this explains why controllability assumption is not needed in this case.

Finally, one may also easily see why the assumption of invertibility of the matrix $(I - A_k)$ (supposed in the discrete case) has no continuous analogue. This is a particular case of the general assumption of invertibility of $(I - \mu(t)A(t))$ which is in the continuous case $\mu(t) \equiv 0$ trivially satisfied.

Supported by the Grant No. 201/98/0677 of the Czech Grant Agency (Prague).

References

1. R. P. Agarwal, M. Bohner, *Quadratic functionals for second order matrix equations on time scales*, submitted
2. C. D. Ahlbrandt, *Equivalent boundary value problems for self-adjoint differential system*, J. Diff. Equations, **9** (1971), 420–435.
3. C. D. Ahlbrandt, *Recessive solutions of symmetric three term recurrence relations*, Canad. Math. Soc. Proceedings, **8** (1987), 3–42.
4. C. D. Ahlbrandt, A. C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer, Boston 1996.
5. C. D. Ahlbrandt, S. L. Clark, J. W. Hooker, W. T. Patula, *A Discrete interpretation of Reid's Roundabout Theorem for generalized differential systems*, Comput. Appl. Math. **28** (1994), 11–21.
6. B. Aulbach, S. Hilger, *Linear dynamic processes with inhomogeneous time scale*, In Nonlinear Dynamics and Quantum Dynamical Systems, Akademie Verlag, Berlin, 1990.
7. J. H. Barrett, *A Prüfer transformation for matrix differential system*, Proc. Amer. Math. Soc., **8** (1957), 510–518.
8. M. Bohner, *Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions*, J. Math. Anal. Appl., **199** (1996), 804–826.
9. M. Bohner, O. Došlý, *Disconjugacy and transformations for symplectic systems*, Rocky Mountain J. Math., **27** (1997), 707–743.
10. O. Došlý, *On transformations of self-adjoint linear differential systems and their reciprocals*, Annal. Pol. Math., **50** (1990), 223–224.
11. O. Došlý, *Transformations of linear Hamiltonian systems preserving oscillatory behaviour*, Arch. Math. (Brno), **27b** (1991), 211–219.
12. O. Došlý, *Transformations of linear Hamiltonian difference systems and some of their applications*, J. Math. Anal. Appl., **191** (1995), 250–265.
13. O. Došlý, R. Hilscher, *Linear Hamiltonian difference systems: transformations, recessive solutions, generalized reciprocity,* submitted, 1997.
14. S. Hilger, *Analysis on measure chains — a unified approach to continuous and discrete calculus*, Res. Math. **18** (1990), 18–56.
15. R. Hilscher, *A unified approach to continuous and discrete linear Hamiltonian systems via the calculus on time scales*, submitted 1998.
16. W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademie Verlag, Berlin, 1995.
17. M. Morse, *A generalization of Sturm separation and comparison theorems in n-space*, *Math. Ann.*, **103** (1930), 52–69.
18. W. T. Reid, *Sturmian Theory for Ordinary Differential Equations*, Springer Verlag, New York-Heidelberg-Berlin 1980.
19. W. T. Reid, *Generalized linear differential systems and related Riccati matrix integral equation*, Illinois J. Math. **10** (1966), 701–722.
20. C. A. Swanson, *Comparison and Oscillation Theorems for Linear Differential Equations*, Acad. Press, London 1968.