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Compactness Condition for Boundary Value Problems

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Dedicated to Professor Lloyd K. Jackson

Abstract. In existence and uniqueness theory of boundary value problems for ordinary differential equations *Compactness Condition* plays an important role. It has been a long standing problem whether other conditions imposed on the differential equations imply this compactness condition. In this lecture we shall survey known results on this problem, including its complete unpublished proof essentially due to L. Jackson and K. Schrader. We shall also discuss some related problems.

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1 Introduction

In this lecture we shall consider the following n (≥ 2)th order nonlinear differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(q)}), \quad 0 \leq q \leq n-1, \quad \text{but fixed.} \tag{1.1}$$

With respect to (1.1) we shall assume that

(A) $f(x, u_0, u_1, \ldots, u_q) : (a, b) \times \mathbb{R}^{q+1} \to \mathbb{R}$ is continuous.

(B) Solutions of initial value problems for (1.1) are unique.

(C) Solutions of (1.1) extend to (a, b).

(D_n) For any $a < a_1 < a_2 < \cdots < a_n < b$ and any solutions $y(x)$ and $z(x)$ of (1.1), it follows that $y(a_i) = z(a_i)$, $1 \leq i \leq n$ implies $y(x) \equiv z(x)$, i.e., the differential equation (1.1) is n-point *disconjugate* on (a, b).

In the study of boundary value problems for the differential equation (1.1), one of the Propositions which has attracted several Mathematicians and has led to substantially new mathematics is *whether conditions* (A) – (D_n) *imply the following* ***compactness condition:***

(E) If $[c, d]$ is a compact subinterval of (a, b) and $\{y_m(x)\}$ is a sequence of solutions of (1.1) which is uniformly bounded, i.e., $|y_m(x)| \le M$ on $[c, d]$ for some $M > 0$ and all $m = 1, 2, \ldots$, then there is a subsequence $\{y_{m(j)}(x)\}$ such that $\{y^{(i)}_{m(j)}(x)\}$ converges uniformly on $[c, d]$ for each $0 \le i \le n - 1$.

In this lecture we shall survey most of the known results on this Proposition, and touch on some related topics.

2 Preliminary Results

We shall need the following version of Kamke's convergence theorem.

Theorem 2.1. **([5, p. 14])** *Assume that for the differential equation* (1.1) *the conditions* (A) *and* (C) *are satisfied. Then, if* $\{y_m(x)\}$ *is a sequence of solutions of* (1.1) *such that there exists a sequence* $\{x_m\} \subset (a, b)$ *with* $\lim_{m\to\infty} x_m = x_0 \in (a, b)$, $\lim_{m\to\infty} y^{(i)}_m(x_m) = y_i$, $0 \le i \le n - 1$. *Then, there is a solution* $y(x)$ *of the differential equation* (1.1) *satisfying the initial conditions* $y^{(i)}(x_0) = y_i$, $0 \le i \le n-1$, *and a subsequence* $\{y_{m(j)}(x)\}$ *of* $\{y_m(x)\}$ *such that* $\lim_{j\to\infty} y^{(i)}_{m(j)}(x) = y^{(i)}(x)$, $0 \le i \le n - 1$, *uniformly on each compact subinterval of* (a, b).

Lemma 2.2. *Let* $y(x) \in C^{(n)}[a_1, a_r]$, *satisfying*

$$\begin{gathered} y(a_i) = y'(a_i) = \cdots = y^{(k_i)}(a_i) = 0, \quad 1 \le i \le r \ (\ge 2) \\ a < a_1 < a_2 < \cdots < a_r < b, \quad k_i \ge 0, \quad \sum_{i=1}^{r} k_i + r = n. \end{gathered} \tag{2.1}$$

Then, there exist constants $C_{n,k}$, $0 \le k \le n - 1$, *such that*

$$|y^{(k)}(x)| \le C_{n,k}(a_r - a_1)^{n-k} \max_{a_1 \le x \le a_r} |y^{(n)}(x)|. \tag{2.2}$$

The problem of finding the best possible constants $C_{n,k}$ in (2.2) is one of the most outstanding problems in polynomial interpolation theory [1,2].

Inequalities (2.2) will be used now to prove local existence of solutions of the differential equation (1.1) satisfying the *r-point conjugate boundary conditions*

$$y(a_i) = A_{1,i},\ y'(a_i) = A_{2,i}, \ldots, y^{(k_i)}(a_i) = A_{k_i+1,i}, \quad 1 \le i \le r. \tag{2.3}$$

Theorem 2.3 ([1]). *Assume that for the differential equation* (1.1) *the condition* (A) *is satisfied. Further, assume that*

(i) $K_i > 0,\ 0 \le i \le q$ *are given real numbers and let* Q *be the maximum of* $|f(x, u_0, u_1, \ldots, u_q)|$ *on the compact set* $[a_1, a_r] \times D_0$, *where*

$$D_0 = \{(u_0, u_1, \ldots, u_q) \ :\ |u_i| \le 2K_i,\ 0 \le i \le q\},$$

(ii) $\max_{a_1 \le x \le a_r} |p^{(i)}(x)| \le K_i,\ 0 \le i \le q$, *where* $p(x)$ *is the Hermite interpolating polynomial*

$$p(x) = \sum_{i=1}^{r} \sum_{j=0}^{k_i} \sum_{\ell=0}^{k_i-j} \frac{1}{j!\ell!} \left[\frac{(x-a_i)^{k_i+1}}{\Omega(x)}\right]^{(\ell)}_{x=a_i} \frac{\Omega(x)}{(x-a_i)^{k_i+1-j-\ell}} A_{j+1,i}$$

and

$$\Omega(x) = \prod_{i=1}^{r} (x-a_i)^{k_i+1},$$

(iii) $(a_r - a_1) \le \left(\dfrac{K_i}{QC_{n,i}}\right)^{1/(n-i)},\ 0 \le i \le q.$

Then, the boundary value problem (1.1), (2.3) *has a solution in* D_0.

Proof. The set

$$B[a_1, a_r] = \left\{ y(x) \in C^{(q)}[a_1, a_r] \ :\ \|y^{(i)}\| \le 2K_i,\ 0 \le i \le q \right\},$$

where

$$\|y^{(i)}\| = \max_{a_1 \le x \le a_r} |y^{(i)}(x)|$$

is a closed convex subset of the Banach space $C^{(q)}[a_1, a_r]$. Consider an operator $T : C^{(q)}[a_1, a_r] \to C^{(n)}[a_1, a_r]$ as follows

$$(Ty)(x) = p(x) + \int_{a_1}^{a_r} g(x,t) f(t, y(t), y'(t), \ldots, y^{(q)}(t)) dt, \tag{2.4}$$

where $g(x,t)$ is the Green's function of the boundary value problem $y^{(n)} = 0$, (2.1). Obviously, any fixed point of (2.4) is a solution of (1.1), (2.3).

We note that $(Ty)(x) - p(x)$ satisfies the conditions of Lemma 2.2, and

$$(Ty)^{(n)}(x) - p^{(n)}(x) = (Ty)^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(q)}(x)).$$

Thus, for all $y(x) \in B[a_1, a_r]$, $\|(Ty)^{(n)}\| \le Q$, and

$$\|(Ty)^{(i)} - p^{(i)}\| \le C_{n,i} Q (a_r - a_1)^{n-i},\quad 0 \le i \le q$$

which also implies that

$$\|(Ty)^{(i)}\| \le \|p^{(i)}\| + C_{n,i} Q (a_r - a_1)^{n-i} \le K_i + K_i = 2K_i,\quad 0 \le i \le q. \tag{2.5}$$

Thus, the operator T maps $B[a_1, a_r]$ into itself. Further, the inequalities (2.5) imply that the sets $\{(Ty)^{(i)}(x) \ : \ y(x) \in B[a_1, a_r]\}$, $0 \le i \le q$ are uniformly bounded and equicontinuous on $[a_1, a_r]$. Hence, $\overline{TB}[a_1, a_r]$ is compact follows from the Ascoli-Arzela theorem. The Schauder fixed point theorem is applicable and a fixed point of (2.4) in D_0 exits.

Corollary 2.4. *Assume that for the differential equation (1.1) the condition (A) is satisfied. Further, assume that there exist constants $N_i \ge 0$, $0 \le i \le q$ such that $\max_{a_1 \le x \le a_r} |p^{(i)}(x)| \le N_i$, $0 \le i \le q$. Then, there exists a $\delta = \delta(N_0, N_1, \ldots, N_q) > 0$ such that if $a_r - a_1 \le \delta$, the boundary value problem (1.1), (2.3) has a solution $y(x)$. Furthermore,*

$$|y^{(i)}(x)| \le N_i + 1, \ 0 \le i \le q \qquad on \quad [a_1, a_r].$$

Theorem 2.5. *Assume that for the differential equation (1.1) the condition (A) is satisfied. Further, assume that the conditions (i) and (iii) of Theorem 2.3 are satisfied. Then, for any $g(x) \in C^{(n-1)}[a_1, a_r]$ the differential equation (1.1) together with*

$$y^{(j)}(a_i) = g^{(j)}(a_i), \qquad 0 \le j \le k_i, \quad 1 \le i \le r \tag{2.6}$$

has a solution, if

$$\sum_{j=i}^{n-1} M_j (a_r - a_1)^{j-i} \le K_i, \qquad 0 \le i \le q$$

where

$$M_j = \max_{a_1 \le x \le a_r} |g^{(j)}(x)|, \qquad 0 \le j \le n-1.$$

Proof. We need to verify that the condition (ii) of Theorem 2.3 is satisfied. For this, in $p(x)$ we take $A_{j+1,i} = g^{(j)}(a_i)$, $0 \le j \le k_i$, $1 \le i \le r$. Then, the function $\phi(x) = g(x) - p(x)$ has n zeros in $[a_1, a_r]$. Thus, from the generalized Rolle's theorem $\phi^{(k)}(x)$, $1 \le k \le n-1$ vanishes at least $n-k$ times in (a_1, a_r). Let $x_k \in (a_1, a_r)$ be any zero of $\phi^{(k)}(x)$, then

$$\left|p^{(n-1)}(x)\right| = \left|p^{(n-1)}(x_{n-1})\right| = \left|g^{(n-1)}(x_{n-1})\right| \le \max_{a_1 \le x \le a_r} \left|g^{(n-1)}(x)\right| = M_{n-1}$$

and

$$\begin{aligned} \left|p^{(n-2)}(x)\right| &\le \left|p^{(n-2)}(x_{n-2})\right| + \left|\int_{x_{n-2}}^{x} \left|p^{(n-1)}(t)\right| dt\right| \\ &= \left|g^{(n-2)}(x_{n-2})\right| + \left|\int_{x_{n-2}}^{x} \left|g^{(n-1)}(x_{n-1})\right| dt\right| \\ &\le M_{n-2} + M_{n-1}(a_r - a_1). \end{aligned}$$

Using the same argument repeatedly, we obtain

$$\left|p^{(i)}(x)\right| \leq \sum_{j=i}^{n-1} M_j(a_r - a_1)^{j-i}.$$

Corollary 2.6. *Assume that for the differential equation (1.1) the condition (A) is satisfied. Then, for any $g(x) \in C^{(n-1)}[a_1, a_r]$ there exist constants $\delta > 0$, $N_i \geq 0,\ 0 \leq i \leq q$, all depending on $g(x)$ such that the boundary value problem (1.1), (2.6) has a solution $y(x)$, provided $a_r - a_1 \leq \delta$. Furthermore, $|y^{(i)}(x)| \leq N_i + 1,\ 0 \leq i \leq q$ on $[a_1, a_r]$.*

Corollary 2.7. *Assume that for the differential equation (1.1) the condition (A) is satisfied. Further, assume that there exist constants $N_i \geq 0,\ 0 \leq i \leq q$ such that $\max_{a_1 \leq x \leq a_r} |p^{(i)}(x)| \leq N_i,\ 0 \leq i \leq q$. Then, there exist a $\delta = \delta(N_0, N_1, \ldots, N_q) > 0$, and an $\varepsilon = \varepsilon(a_1, \ldots, a_r)$ such that for $a_r - a_1 \leq \delta$, the boundary value problem (1.1),*

$$y(a_i) = A_{1,i} + \varepsilon_{1,i},\ y'(a_i) = A_{2,i} + \varepsilon_{2,i}, \ldots, y^{(k_i)}(a_i) = A_{k_i+1,i} + \varepsilon_{k_i+1,i},\ \ 1 \leq i \leq r$$

has a solution $y_\varepsilon(x)$, provided $|\varepsilon_{j,i}| \leq \varepsilon,\ 0 \leq j \leq k_i,\ 1 \leq i \leq r$. Furthermore, $|y_\varepsilon^{(i)}(x)| \leq N_i + 1,\ 0 \leq i \leq q$ on $[a_1, a_r]$.

3 The Case $n = 2$

For the second order differential equation (1.1) only conditions (A) and (C) imply (E). We shall prove this in the following:

Theorem 3.1. *If the differential equation (1.1) is of second order and satisfies conditions (A) and (C), then (1.1) also satisfies condition (E).*

Proof. If $\{y_m(x)\}$ is a sequence of solutions of (1.1) with $|y_m(x)| \leq M$ on $[c, d] \subset (a, b)$ for some $M > 0$, and each $m \geq 1$, then for each m there is a $x_m \in (c, d)$ such that

$$|y'_m(x_m)| = \frac{|y_m(d) - y_m(c)|}{d - c} \leq \frac{2M}{d - c}.$$

Consequently, $\{x_m\},\ \{y_m(x_m)\}$ and $\{y'_m(x_m)\}$ are bounded sequences. By taking subsequences in succession which converge, we conclude that there exist values $x_0,\ y_0,\ y'_0$ such that $x_{m(1)} \to x_0,\ y_{m(1)}(x_{m(1)}) \to y_0\ y'_{m(1)}(x_{m(1)}) \to y'_0$, where $\{m(1)\}$ is some subsequence of $\{m\}$. Thus, by Theorem 2.1 there is a subsequence $\{y_{m(2)}(x)\}$ of $\{y_{m(1)}(x)\}$ and a solution $y(x)$ of (1.1) satisfying $y(x_0) = y_0,\ y'(x_0) = y'_0$ such that $\lim_{m\to\infty} y^{(i)}_{m(2)}(x) = y^{(i)}(x),\ i = 0, 1,$ uniformly on $[c, d]$.

4 Case $n = 3$

If the differential equation (1.1) is of third order, then conditions (A) and (C) are not enough for (E). In fact, the equation $y''' = -[y']^3$ satisfies (A) and (C) on $\mathbb{R}^4$, but the sequence $\{y_m(x)\}$ of solutions of the initial value problem

$$y''' = -[y']^3, \quad y(0) = y'(0) = 0, \quad y''(0) = m$$

for $m = 1, 2, \ldots$, is uniformly bounded on $\mathbb{R}$ and does not contain a subsequence satisfying (E) on any compact subinterval of $\mathbb{R}$. Here, we shall show that conditions (A), (C) and (D_3) do imply (E). However, for this an immediate appeal to Theorem 2.1 is not possible and we shall need the following lemmas.

Lemma 4.1. *Assume that the differential equation (1.1) is of third order and satisfies the condition (A). Then, given any compact subinterval $[c, d] \subset (a, b)$ and any fixed $M > 0$, there is a $\delta(M) > 0$ such that for any $[a_1, a_2] \subset [c, d]$ with $a_2 - a_1 \le \delta(M)$, and any real α with $|\alpha| \le M$, (1.1) has solutions satisfying each of the boundary conditions*

$$y(a_1) = y(a_2) = \alpha, \ y'(a_1) = 0 \quad \text{and} \quad y(a_1) = y(a_2) = \alpha, \ y'(a_2) = 0.$$

Furthermore, for any such solution $|y'(x)| \le 1$ and $|y''(x)| \le 1$ on $[a_1, a_2]$.

Proof. This is a particular case of Corollary 2.4.

Lemma 4.2 ([11]). *Assume that the differential equation (1.1) is of third order and satisfies the condition (A). Let $\phi(x)$, $\psi(x)$ be of class $C^{(2)}$ on $[a_1 - \tau, a_1 + \tau] \subset (a, b)$ with $\phi(a_1) = \psi(a_1)$, $\phi'(a_1) = \psi'(a_1)$ and $\phi''(a_1) < \psi''(a_1)$. Then, there is a δ, $0 < \delta \le \tau$, such that all solutions $y(x)$ of (1.1) with the initial conditions $y(a_1) = y_0 = \phi(a_1)$, $y'(a_1) = y_1 = \phi'(a_1)$, and $y''(a_1) = y_2 = \frac{1}{2}[\phi''(a_1) + \psi''(a_1)]$ exist on $[a_1 - \delta, a_1 + \delta]$ and satisfy $\phi(x) < y(x) < \psi(x)$ for $0 < |x - a_1| \le \delta$.*

Proof. Let $8\rho = \psi''(a_1) - \phi''(a_1)$ and choose δ_0, $0 < \delta_0 \le \tau$ such that $|\phi''(x) - \phi''(a_1)| \le \rho$ and $|\psi''(x) - \psi''(a_1)| \le \rho$ for $|x - a_1| \le \delta_0$. Let $M > 0$ be a bound for $f(x, y, y', y'')$ on the compact set

$$\{(x, y, y', y'') \ : \ |x - a_1| \ \le \ \delta_0, \ |y - y_0| \le 1, \ |y' - y_1| \le 1, \ |y'' - y_2| \le 1\}.$$

Then, it follows from the relations

$$y(x) = y_0 + \int_{a_1}^{x} y'(t)dt, \qquad y'(x) = y_1 + \int_{a_1}^{x} y''(t)dt$$

$$y''(x) = y_2 + \int_{a_1}^{x} f(t, y(t), y'(t), y''(t))dt$$

that all solutions of the stated initial value problem exist on the closed interval $[a_1 - \delta_1, a_1 + \delta_1]$, where

$$\delta_1 = \min\left\{\delta_0, \frac{1}{M}, \frac{1}{1+|y_2|}, \frac{1}{1+|y_1|}\right\}.$$

Thus, if $\delta = \min\{\delta_1, \rho/M\}$, it follows that for all solutions $y(x)$ of the initial value problem $|y''(x) - y_2| \le \rho$ for $|x - a_1| \le \delta$. Hence, for all solutions $y(x)$ of the initial value problem

$$y''(x) - \phi''(x) \ge 2\rho \quad \text{and} \quad \psi''(x) - y''(x) \ge 2\rho$$

on $[a_1 - \delta, a_1 + \delta]$.

Lemma 4.3 ([11]). *Assume that the differential equation (1.1) is of third order and satisfies the conditions (A), (C) and (D_3). Then, solutions of two point boundary value problems for (1.1) are unique, i.e., the following condition (D_2) holds:*

(D_2) *If $a < a_1 < a_2 < b$, and $y(x)$, $z(x)$ are both solutions of (1.1) satisfying $y(a_1) = z(a_1)$, $y'(a_1) = z'(a_1)$, $y(a_2) = z(a_2)$, or $y(a_1) = z(a_1)$, $y(a_2) = z(a_2)$, $y'(a_2) = z'(a_2)$, then it follows that $y(x) \equiv z(x)$ on $[a_1, a_2]$.*

Proof. We shall consider only the case where $y(a_1) = z(a_1)$, $y'(a_1) = z'(a_1)$, $y(a_2) = z(a_2)$. We first assume that $y''(a_1) \ne z''(a_1)$, and to be specific assume that $y''(a_1) > z''(a_1)$. Then, by Lemma 4.2 there is a $\delta > 0$ with $a < a_1 - \delta < a_1 + \delta < a_2$ such that all solutions $w(x)$ of the initial value problem for (1.1) with the initial conditions

$$w(a_1) = y_0 = z(a_1), \quad w'(a_1) = y_1 = z'(a_1), \quad w''(a_1) = y_2 = \frac{1}{2}\left[y''(a_1) + z''(a_1)\right] \tag{4.1}$$

satisfy $z(x) < w(x) < y(x)$ for $0 < |x - a_1| \le \delta$. Let $\{\varepsilon_m\}$ be a monotone decreasing sequence of positive numbers converging to zero, and let $z_m(x)$ be a solution of (1.1) with the initial conditions

$$z_m(a_1) = y_0, \quad z'_m(a_1) = y_1 + \varepsilon_m, \quad z''_m(a_1) = y_2. \tag{4.2}$$

Then, $\{z_m(x)\}$ contains a subsequence converging uniformly on $[a_1 - \delta, a_1 + \delta]$ to a solution of (1.1), (4.1). Hence, for sufficiently large m, there is a solution $z_m(x)$ of (1.1), (4.2) such that

$$z(a_1 - \delta) < z_m(a_1 - \delta) < y(a_1 - \delta) \quad \text{and} \quad z(a_1 + \delta) < z_m(a_1 + \delta) < y(a_1 + \delta).$$

Since $z_m(a_1) = y(a_1) = z(a_1)$ and $z'_m(a_1) = y_1 + \varepsilon_m > y'(a_1) = z'(a_1)$, it follows that there are x_1, x_2 with $a_1 - \delta < x_1 < a_1 < x_2 < a_1 + \delta$ such that $z_m(x_1) = z(x_1)$, and $z_m(x_2) = y(x_2)$. Since $y(a_1) = z(a_1)$ at $a_2 > a_1 + \delta$, it follows that any extension of $z_m(x)$ intersects either $y(x)$ or $z(x)$ again on

$[a_1+\delta, b)$. Since $z_m(x) \not\equiv z(x)$ on $[x_1, a_1]$ and $z_m(x) \not\equiv y(x)$ on $[a_1, x_2]$, this contradicts condition (D_3).

Thus, if $y(x) \not\equiv z(x)$ on $[a_1, a_2]$, then $y^{(i)}(a_1) = z^{(i)}(a_1)$ for $i = 0, 1, 2$. However, if $y^{(i)}(a_1) = z^{(i)}(a_1)$ for $i = 0, 1, 2$ and $y(x) \not\equiv z(x)$ on $[a_1, a_2]$, then for $a < x_3 < a_1$, $u(x)$ and $v(x)$ defined by

$$u(x) = v(x) = y(x) \quad \text{on} \quad [x_3, a_1], \quad u(x) = y(x) \quad \text{on} \quad (a_1, a_2], \qquad \text{and} \quad v(x) = z(x) \quad \text{on} \quad (a_1, a_2]$$

will be solutions on $[x_3, a_2]$ which again contradicts condition (D_3). Thus, we conclude that $y(x) \equiv z(x)$ on $[a_1, a_2]$.

Lemma 4.4 ([12]). *Let $y(x) \in C^{(2)}[\alpha, \beta]$ and assume that $|y(x)| \le M$ on $[\alpha, \beta]$. There is a $K > 0$ depending on M and $\beta - \alpha$ such that if, $\max\{|y'(x)|, |y''(x)|\} > K$ for all $\alpha \le x \le \beta$, then $y'(x_0) = 0$ for some x_0 with $\alpha < x_0 < \beta$.*

Proof. Assume that the conclusion is false. We shall determine $N > 0$ so that the following inequality holds

$$|y'(x)| + |y''(x)| \ge N + \frac{2M}{\beta - \alpha} + 1 \quad \text{on} \quad [\alpha, \beta]. \tag{4.3}$$

For this, by the Mean value theorem there exists a $x_1 \in (\alpha, \beta)$ such that

$$|y'(x_1)| = \left| \frac{y(\beta) - y(\alpha)}{\beta - \alpha} \right| \le \frac{2M}{\beta - \alpha}.$$

There are now two possible cases, however, since both are similar, we shall consider only the case

$$0 < y'(x_1) \le \frac{2M}{\beta - \alpha} \quad \text{and} \quad \alpha < x_1 \le \frac{\alpha + \beta}{2}.$$

If $y(x_1) = M$, then $y'(x_1) = 0$ and the proof is finished. So, we assume that $y(x_1) \ne M$. (It is clear that we are assuming $y'(x_1) \ne 0$.) We define $\eta = (\beta - \alpha)/8$. Now to complete the proof we need to consider the following two subcases:

Case (i). Assume that $y''(x_1) \le 0$. Then, in order (4.3) holds, it is necessary that $y''(x_1) \le -N$. Thus, $y'(x)$ is decreasing on a right neighborhood of $x = x_1$. In fact, if $0 \le y'(x) \le (2M/(\beta - \alpha))$, then it will follow that $y''(x) \le -N$ on $[x_1, \beta]$. However, then by Taylor's formula, we have

$$\begin{aligned} y(\beta) &= y(x_1) + (x_1 - \beta) y'(x_1) + \frac{(x_1 - \beta)^2}{2} y''(\xi), \qquad \xi \in (x_1, \beta) \\ &< M + \frac{2M(\beta - \alpha)}{\beta - \alpha} - \frac{(\beta - \alpha)^2}{4} N \\ &\le -M \end{aligned}$$

provided

$$N \geq \frac{32M}{(\beta-\alpha)^2} = \frac{M}{2\eta^2}.$$

But, this implies that $|y(\beta)| > M$, which is a contradiction. Thus, there exists a point $x_1 \leq x_0 < \beta$ such that $y'(x_0) = 0$.

Case (ii). Assume that $y''(x_1) > 0$, (we shall work across $[\alpha, \beta]$ on subintervals of length η.) Then, from (4.3) it follows that $y''(x_1) > N$. We assume that $y''(x) \geq N/2$ on $[x_1, x_1 + \eta]$. Again, by Taylor's formula, we have

$$y(x_1+\eta) = y(x_1) + \eta y'(x_1) + \frac{1}{2}\eta^2 y''(\xi), \quad \xi \in (x_1, x_1+\eta).$$

As in Case (i), we find $y(x_1+\eta) > -M + N\eta^2/4 \geq M$, provided $N \geq (8M/\eta^2)$, which is a contradiction.

From this contradiction, we conclude that there exists a $x_1 < x_2 < x_1 + \eta$ such that $y''(x_2) = N/2$. Since, $y''(x)$ is positive up to x_2, we can assume that x_2 is the first point such that $y''(x_2) = N/2$. Thus,

$$y'(x_2) > \frac{1}{2}N + \frac{2M}{\beta-\alpha} \quad \text{on} \quad [x_1, x_2).$$

Now assume that

$$y'(x) \geq \frac{1}{2}N + \frac{2M}{\beta-\alpha} \quad \text{on} \quad [x_2, x_2+\eta).$$

Then, it follows that

$$\begin{aligned} y(x_2+\eta) &= y(x_2) + \eta y'(\xi), \quad \xi \in (x_2, x_2+\eta) \\ &> -M + \frac{1}{2}\eta N + \eta\frac{2M}{\beta-\alpha} \\ &= -M + \frac{1}{2}\eta N + \frac{1}{4}M \\ &\geq M, \end{aligned}$$

provided $N \geq (7M/2\eta)$. But, this implies $y(x_2+\eta) > M$, which is a contradiction.

From this contradiction, there exists a $x_2 < x_3 < x_2 + \eta$ such that

$$y'(x_3) = \frac{1}{2}N + \frac{2M}{\beta-\alpha},$$

and we take x_3 to be the first such point, i.e.,

$$y'(x) > \frac{1}{2}N + \frac{2M}{\beta-\alpha}, \quad \text{on} \quad [x_2, x_3).$$

So, $y'(x)$ is decreasing on $[x_2, x_3)$. Thus, $y''(x_3) < -N/2$. This implies that in a right neighborhood of x_3, $y''(x) < -N/2$ and $y'(x)$ is decreasing. Assume that

$$0 < y'(x) \le \frac{1}{2}N + \frac{2M}{\beta - \alpha} \quad \text{on} \quad [x_3, \beta).$$

Then, by Taylor's formula

$$y(x_3) = y(\beta) + (x_3 - \beta)y'(\beta) + \frac{(x_3 - \beta)^2}{2}y''(\xi), \qquad \xi \in (x_3, \beta).$$

Since in the above relation the second term in the right side is nonpositive, in view of $(\beta - x_3) \ge (\beta - \alpha)/4$, it follows that

$$y(x_3) < M - \frac{1}{4}(\beta - \alpha)^2 N \le -M$$

provided $N \ge M/\eta^2$. But, this leads to the contradiction that $|y(x_3)| > M$.

From this construction we conclude that

$$0 < y'(x) \le \frac{1}{2}N + \frac{2M}{\beta - \alpha} \quad \text{on} \quad [x_3, \beta)$$

is false. Thus, in conclusion $y'(x_0) = 0$, for some $x_3 < x_0 < \beta$.

Theorem 4.5 ([12]). *If the differential equation* (1.1) *is of third order and satisfies conditions* (A), (C) *and* (D_3), *then* (1.1) *also satisfies condition* (E).

Proof. Suppose that the result is not true. Then, since $|y_m(x)| \le M$ on $[c, d]$ for $n \ge 1$, it follows from Theorem 2.1 that $|y'_m(x)| + |y''_m(x)| \to \infty$ uniformly on $[c, d]$. Let $c \le a_1 < a_2 < a_3 < a_4 \le d$ be such that $a_4 - a_1 \le \delta(M)$, where $\delta(M)$ is as defined in Lemma 4.1. By Lemma 4.4 there is a $K > 0$ such that, if $\max\{|y'_m(x)|, |y''_m(x)|\} > K$ for each $x \in [c, d]$, then $y'_m(x)$ has a zero on (a_1, a_2), on (a_2, a_3) and on (a_3, a_4). Furthermore, we can assume that $K > 1$. Now from the fact that $|y'_m(x)| + |y''_m(x)| \to \infty$ uniformly on $[c, d]$ we can conclude that there is a positive integer m_0 such that $\max\{|y'_{m_0}(x)|, |y''_{m_0}(x)|\} > K$ on $[c, d]$. Let $a_1 < x_1 < a_2 < x_2 < a_3 < x_3 < a_4$ be such that $y'_{m_0}(x_i) = 0$ for $i = 1, 2, 3$. Then, $|y''_{m_0}(x_i)| > K > 1$ for $i = 1, 2, 3$. Now we need to consider the following two cases:

If $y_{m_0}(x_i) = y_{m_0}(x_j)$ with $x_i < x_j$, then $y_{m_0}(x)$ is the solution of the differential equation (1.1) together with the two-point boundary conditions $y(x_i) = y(x_j) = y_{m_0}(x_i)$, $y'(x_i) = 0$. However, since $x_j - x_i < \delta(M)$, it follows from Lemma 4.1 that $|y'_{m_0}(x)| \le 1$ and $|y''_{m_0}(x)| \le 1$ on $[x_i, x_j]$, which is a contradiction to $|y''_{m_0}(x_i)| > K > 1$.

If $y_{m_0}(x_i) \ne y_{m_0}(x_j)$ for $x_i \ne x_j$, then it suffices to assume that $y_{m_0}(x_1) < y_{m_0}(x_2) < y_{m_0}(x_3)$. In fact, the same argument applies to the other orderings of the values of $y_{m_0}(x_i)$, $i = 1, 2, 3$. If $y''_{m_0}(x_2) > K$, there is a t_1, $x_1 < t_1 < x_2$, such that $y_{m_0}(t_1) = y_{m_0}(x_2)$. If $y''_{m_0}(x_2) < -K$, there is a t_2, $x_2 < t_2 < x_3$, such that $y_{m_0}(t_2) = y_{m_0}(x_2)$. In either case Lemma 4.1 is again applied to obtain a contradiction.

Hence, the sequence $\{y_m(x)\}$ contains a subsequence converging uniformly on $[c, d]$ along with its first and second order derivative sequences.

5 Weak Compactness Condition

It seems very difficult, if not impossible, to extend the method of Theorem 4.5 to equations of higher orders. Here, we shall show that for equation (1.1) of arbitrary order n, conditions (A), (C) and (D_n) do imply a weaker type of compactness condition for the solutions of (1.1). For this we shall need the following:

Theorem 5.1. (Banach Indicatrix Theorem, [9, p. 271]) *If $h \in C[c,d] \cap BV[c,d]$, then*

$$V_c^d(h) = \int_{-\infty}^{\infty} N_h(\alpha) d\alpha,$$

where

$$N_h(\alpha) = \begin{cases} Card\{x \in [c,d] \ : \ h(x) = \alpha\}, & \text{if this set is finite,} \\ +\infty, & \text{if the above set is infinite,} \end{cases}$$

and where the above integral is in the Lebesgue sense.

Theorem 5.2 ([16]). *Assume that the differential equation (1.1) satisfies the conditions (A), (C), and (D_n). Further, assume that $[c,d]$ is a compact subinterval of (a,b) and $\{y_m(x)\}$ is a sequence of solutions of (1.1) which is uniformly bounded on $[c,d]$. Then, the sequence $\{V_c^d(y_m)\}$ of total variations of the functions $y_m(x)$ on $[c,d]$ is bounded, i.e., there exists an $N > 0$ such that $V_c^d(y_m) \le N$ for all m.*

Proof. Assume the assertion is false. Then there is a sequence of solutions $\{y_m(x)\}$ of (1.1), a compact interval $[c,d] \subset (a,b)$, and an $M > 0$ such that $|y_m(x)| \le M$ on $[c,d]$ for all m, but such that $V_c^d(y_m) \to \infty$, as $m \to \infty$.

We claim that $\sum_{i=0}^{n-1} |y_m^{(i)}(x)| \to \infty$ on $[c,d]$ as $m \to \infty$, i.e., given $R > 0$, there exists a $L > 0$ such that $\sum_{i=0}^{n-1} |y_m^{(i)}(x)| > R$, on $[c,d]$ for all $m \ge L$. If the claim is false, then there exists a $\beta > 0$ and a subsequence $\{y_{m(j)}(x)\}$ such that $\sum_{i=0}^{n-1} |y_{m(j)}^{(i)}(x_j)| \le \beta$, for all $j \ge 1$, and where $\{x_j\} \subset [c,d]$. Now by choosing successive subsequences and relabeling, we obtain points $\{x_p\}$ and solutions $\{y_p(x)\}$ such that $\{x_p\}$ and $\{y_p^{(i)}(x_p)\}$, $0 \le i \le n-1$ all converge. Thus, by Theorem 2.1 there exists a further subsequence $\{y_{p(j)}(x)\}$ such that $\{y_{p(j)}^{(i)}(x)\}$ converges uniformly on $[c,d]$, $0 \le i \le n-1$. This implies that $\{y'_{p(j)}(x)\}$ is a uniformly bounded sequence on $[c,d]$. Now since each $y'_{p(j)}(x)$ is absolutely continuous, it follows that

$$V_c^d(y_{p(j)}) = \int_c^d |y'_{p(j)}(x)| dx.$$

Hence, the sequence $\{V_c^d(y_{p(j)})\}$ is a bounded sequence, which is a contradiction. Thus, $\sum_{i=0}^{n-1} |y_m^{(i)}(x)| \to \infty$ on $[c,d]$ as $m \to \infty$.

We shall now apply Corollary 2.4 with $N_0 = M$ and $N_0 = 0,\ 1 \le i \le n-1$, so that there exists a $\delta(M) > 0$ such that for any α with $|\alpha| \le M$ and any points $c \le a_1 < \cdots < a_n \le d$ with $a_n - a_1 \le \delta$, and with $w(x) \equiv \alpha$ so that $w^{(i)}(x) \equiv 0,\ 1 \le i \le n-1$, the boundary value problem for (1.1) satisfying $y(a_i) = \alpha,\ 1 \le i \le n$ has a solution $y(x)$ with $|y^{(i)}(x)| \le N_i + 1$ on $[a_1, a_n]$ for $0 \le i \le n-1$. In particular, the boundary value problem has a solution $y(x)$ with $|y(x)| \le M+1$, and $|y^{(i)}(x)| \le 1,\ 1 \le i \le n-1$ on $[a_1, a_n]$. Further, for such a solution, we have $\sum_{i=0}^{n-1} |y^{(i)}(x)| \le M + n$ on $[a_1, a_n]$.

Now let L_0 be such that $\sum_{i=0}^{n-1} |y_m^{(i)}(x)| > M + n$ on $[c,d]$ for all $m \ge L_0$. It follows that given $m \ge L_0$ and α, with $|\alpha| \le M$ the solution $y_m(x)$ intersects the line $y = \alpha$ at most $n-1$ times in any closed subinterval of $[c,d]$ of length less than δ. For if, there are points $c \le x_1 < \cdots < x_n \le d,\ x_n - x_1 \le \delta$ such that $y_m(x_i) = \alpha,\ 1 \le i \le n$ where $|\alpha| \le M$ and $m \ge L_0$, there is also the above mentioned solution $y(x)$ by Corollary 2.4 satisfying $y(x_i) = \alpha,\ 1 \le i \le n$. By (D_n), $y_m(x) \equiv y(x)$ on $[x_1, x_n]$. But, then $\sum_{i=0}^{n-1} |y^{(i)}(x)| \le M+n$ on $[a_1, a_n]$ and $\sum_{i=0}^{n-1} |y_m^{(i)}(x)| > M+n$ on $[a_1, a_n]$ is not possible. Thus, for every $m \ge L_0,\ y_m(x)$ intersects each line $y = \alpha,\ |\alpha| \le M$ at most $n-1$ times in any closed subinterval of $[c,d]$ of length less than δ. So, for all $m \ge L_0$,

$$N_{y_m}(\alpha) \le (n-1)\left(\left[\frac{d-c}{\delta}\right] + 1\right), \qquad \text{if} \qquad |\alpha| \le M$$

and $N_{y_m}(\alpha) = 0$, if $|\alpha| > M$. Thus, by the Banach Indicatrix Theorem it follows that for $m \ge L_0$,

$$\begin{aligned} V_c^d(y_m) &= \int_{-M}^{M} N_{y_m}(\alpha) d\alpha \\ &\le \int_{-M}^{M} (n-1)\left(\left[\frac{d-c}{\delta}\right] + 1\right) d\alpha \\ &= 2M(n-1)\left(\left[\frac{d-c}{\delta}\right] + 1\right). \end{aligned}$$

But, this contradicts $V_c^d(y_m) \to \infty$, as $m \to \infty$. Hence, $\{V_c^d(y_m)\}$ is a bounded sequence.

Theorem 5.3. (Helly's Selection (or Choice) Theorem, [22, p. 398]) *If $\{y_m(x)\}$ is a sequence of functions on $[c,d]$ such that for some M, $|y_m(x)| \le M$ on $[c,d]$ for all $m \ge 1$, and such that $|V_c^d(y_m)| \le H$, for all $m \ge 1$, and some $H > 0$, then there exists a subsequence $\{y_{m(j)}(x)\}$ which converges point-wise on $[c,d]$. Moreover, the limit function is of bounded variation on $[c,d]$.*

Corollary 5.4. *Assume that the differential equation (1.1) satisfies the conditions (A), (C) and (D_n). Then, if $[c,d]$ is a compact subinterval of (a,b) and if $\{y_m(x)\}$ is a sequence of solutions of (1.1) which is uniformly bounded on*

$[c,d]$, there is a subsequence $\{y_{m(j)}(x)\}$ which converges point-wise on $[c,d]$ and $z(x) = \lim_{j\to\infty} y_{m(j)}(x)$ is of bounded variation on $[c,d]$ (as a consequence $z(x)$ has finite derivative almost everywhere, and also $\int_c^d z(x)dx$ exists).

Proof. The result follows from Theorems 5.2 and 5.3.

Remark 5.1. In conjunction with Corollary 5.4 we remark that Schrader [19,20] has proven that, if $\{y_m(x)\}$ is a uniformly bounded sequence of functions on a compact interval $[c,d]$, and if the functions $y_m(x)$ satisfy only the uniqueness condition (D_n) on $[c,d]$, then there is a subsequence of $\{y_m(x)\}$ which converges point-wise on $[c,d]$.

6 Generalized Solutions

To prove the Proposition now we shall follow another possible approach. For this, if the differential equation (1.1) satisfies (A) – (D_n), then it is straightforward to show that the compactness condition (E) is equivalent to the following:

(E^*) If $\{y_m(x)\}$ is a sequence of solutions of (1.1) which is monotone and bounded on some compact subinterval $[c,d] \subset (a,b)$, then $\lim_{m\to\infty} y_m(x)$ is a solution of (1.1) on $[c,d]$.

Thus, to prove the Proposition it suffices to show that the conditions (A) – (D_n) imply that the limit of a bounded monotone sequence of solutions of (1.1) is also a solution.

Definition 6.1. A function $\phi(x)$ defined on an interval $J \subset (a,b)$ is said to be a *generalized solution* of (1.1) on J if for each set of points $a_1 < a_2 < \cdots < a_n$ contained in J and any solution $y(x)$ of (1.1), the inequalities $(-1)^{n+i}\,[y(a_i) - \phi(a_i)] < 0$, $1 \le i \le n$ imply that $y(x) < \phi(x)$ on $J \cap [a_n, b)$ and $(-1)^{n+1}[y(x) - \phi(x)] < 0$ on $J \cap (a, a_1]$, and the inequalities $(-1)^{n+i}\,[y(a_i) - \phi(a_i)] > 0$, $1 \le i \le n$ imply $y(x) > \phi(x)$ on $J \cap [a_n, b)$ and $(-1)^{n+1}[y(x) - \phi(x)] > 0$ on $J \cap (a, a_1]$.

Theorem 6.1 ([13,17]). *Assume that the differential equation (1.1) satisfies conditions (A) – (D_n), and that $\lim_{m\to\infty} y_m(x) = \phi(x)$ on $J \subset (a,b)$, where $\{y_m(x)\}$ is a sequence of solutions of (1.1). Then, $\phi(x)$ is a generalized solution of (1.1) on J.*

Proof. Assume that for $a_1 < a_2 < \cdots < a_n$ contained in J there is a solution $y(x)$ of (1.1) such that $(-1)^{n+i}\,[y(a_i) - \phi(a_i)] < 0$ for $1 \le i \le n$, but that also $y(a_0) > \phi(a_0)$ for some $a_0 > a_n$ in J. Then, since $\lim_{m\to\infty} y_m(x) = \phi(x)$, there is a solution $y_m(x)$ of (1.1) such that $(-1)^{n+i}\,[y(a_i) - y_m(a_i)] < 0$ for $1 \le i \le n$ and $y(a_0) > y_m(a_0)$. This contradicts the condition (D_n). The remaining inequalities can be proved in a similar way.

Thus, the limit of a bounded monotone sequence of solutions $\{y_m(x)\}$ of (1.1) satisfying (A) – (D_n) is a generalized solution.

Lemma 6.2. *Assume that the differential equation (1.1) satisfies condition (A) and that $\phi(x) \in C^{(n-1)}[c,d]$, where $[c,d]$ is a compact subinterval of (a,b). Assume that $M > 0$ is such that $|\phi^{(j)}(x)| \le M$ on $[c,d]$ for $0 \le j \le n-1$. Then, there exists a $\delta > 0$ such that, for any $c \le a_1 < a_2 < \cdots < a_n \le d$ with $a_n - a_1 \le \delta$, (1.1) has a solution $y(x)$ with $y(a_i) = \phi(a_i)$, $1 \le i \le n$ and $|y^{(j)}(x)| \le 2M$ on $[a_1, a_n]$ for $0 \le j \le n-1$. Furthermore, δ can be chosen in such a way that, for each fixed set $a_1 < a_2 < \cdots < a_n$ satisfying the above conditions, there is an $\varepsilon > 0$ such that for any y_i, $1 \le i \le n$ with $|y_i - \phi(a_i)| < \varepsilon$, $1 \le i \le n$, (1.1) has a solution $y(x)$ satisfying $y(a_i) = y_i$, $1 \le i \le n$, and $|y^{(j)}(x)| \le 3M$ on $[a_1, a_n]$ for $0 \le j \le n-1$.*

Proof. The proof follows from Corollary 2.7.

Theorem 6.3 ([13,17]). *Assume that the differential equation (1.1) satisfies conditions (A) and (D_n), and that $\lim_{m\to\infty} y_m(x) = \phi(x)$ on $[c,d] \subset (a,b)$, where $\{y_m(x)\}$ is a sequence of solutions of (1.1). Then, if $\phi(x) \in C^{(n-1)}[c,d]$, $\phi(x)$ is a solution of (1.1) on $[c,d]$ and $\lim_{m\to\infty} y_m^{(j)}(x) = \phi^{(j)}(x)$ uniformly on $[c,d]$ for each $0 \le j \le n-1$.*

Proof. Let $M > 0$ be such that $|\phi^{(j)}(x)| \le M$ on $[c,d]$ for $0 \le j \le n-1$. By Lemma 6.2 there is a $\delta > 0$ such that, if $c \le a_1 < a_2 < \cdots < a_n \le d$ is a fixed set of points with $a_n - a_1 \le \delta$, there is an $\varepsilon > 0$ with the property that $|y_i - \phi(a_i)| < \varepsilon$, $1 \le i \le n$ implies that (1.1) has a solution $y(x)$ satisfying $y(a_i) = y_i$, $1 \le i \le n$ and $|y^{(j)}(x)| \le 3M$ on $[a_1, a_n]$ for $0 \le j \le n-1$. It follows that there is an $N > 0$ such that $m \ge N$ implies $|y_m(a_i) - \phi(a_i)| < \varepsilon$, $1 \le i \le n$. Hence, by condition (D_n) and the choice of ε, $|y_m^{(j)}(x)| \le 3M$ on $[a_1, a_n]$ for $0 \le j \le n-1$ and all $m \ge N$. From this the conclusion follows.

Now let $\phi(x)$ be a real valued function defined on (c,d). At a point $x_0 \in (c,d)$ where $\phi(x)$ has a finite right limit $\phi(x_0+0)$, we define

$$D^1\phi(x_0+0) = \lim_{x\to x_0^+} \frac{\phi(x) - \phi(x_0+0)}{x - x_0}$$

provided the limit exists. The left derivative $D^1\phi(x_0-0)$ is similarly defined. Likewise, if $\phi(x_0+0)$ and $D^1\phi(x_0+0)$ exist and are finite, we define

$$D^2\phi(x_0+0) = \lim_{x\to x_0^+} \left\{ \frac{2}{(x-x_0)^2} \left[\phi(x) - \phi(x_0+0) - D^1\phi(x_0+0)(x-x_0)\right] \right\}$$

provided the limit exists. In general, if the limits defining $\phi(x_0+0)$ and $D^j\phi(x_0+0)$, $1 \le j \le k-1$ exist and are finite, we define

$$D^k\phi(x_0+0) = \lim_{x\to x_0^+} \left\{ \frac{k!}{(x-x_0)^k} \left[\phi(x) - \phi(x_0+0) - \sum_{j=1}^{k-1} \frac{D^j\phi(x_0+0)(x-x_0)^j}{j!} \right] \right\}$$

provided the limit exists. The left derivatives $D^j\phi(x_0-0)$ are defined correspondingly.

Theorem 6.4 ([13,17]). *Assume that the differential equation (1.1) satisfies conditions (A) and (D_n), and that $\phi(x)$ is a bounded generalized solution of (1.1) on $(c,d)\subset(a,b)$. Then, $\phi(x)$ has right and left limits at each point of (c,d) and $D^1\phi(x_0-0)$ and $D^1\phi(x_0+0)$ exist in the extended reals for all $x_0\in(c,d)$. Furthermore, if at a point $x_0\in(c,d)$, $D^j\phi(x_0+0)$ exists and is finite for each $1\le j\le k-1\le n-2$, then the limit defining $D^k\phi(x_0+0)$ exists in the extended reals. The same assertion applies to the left derivative $D^k\phi(x_0-0)$.*

Proof. Assume that for some $x_0\in(c,d)$, $\liminf_{x\to x_0^+}\phi(x)<\limsup_{x\to x_0^+}\phi(x)$ and choose a real number r such that $\liminf_{x\to x_0^+}\phi(x)<r<\limsup_{x\to x_0^+}\phi(x)$. Then, there exist sequences $\{t_m\}$ and $\{x_m\}$ in (c,d) such that $\lim t_m=\lim x_m=x_0$, $x_0<t_{m+1}<x_m<t_m$ for each $m\ge 1$, $\lim\phi(t_m)=\limsup_{x\to x_0^+}\phi(x)$, and $\lim\phi(x_m)=\liminf_{x\to x_0^+}\phi(x)$. let $y(x)$ be a solution of (1.1) satisfying the initial conditions $y(x_0)=r$ and $y^{(j)}(x_0)=0$, $1\le j\le n-1$. This solution exists on $[x_0,x_0+\delta]$ for some $\delta>0$, and since $\lim_{x\to x_0}y(x)=r$, there is an $N>0$ such that $m\ge N$ implies that $x_0<t_m<x_0+\delta$ and $\phi(t_m)>y(t_m)$, $\phi(x_m)<y(x_m)$. This contradicts the fact that $\phi(x)$ is a generalized solution on (c,d). The existence of $\phi(x_0-0)$ can be proved similarly.

Now assume that for some $x_0\in(c,d)$ the limit defining $D^1\phi(x_0+0)$ does not exist in the extended reals. Then, choose the real number r such that

$$\liminf_{x\to x_0^+}\frac{\phi(x)-\phi(x_0+0)}{x-x_0}<r<\limsup_{x\to x_0^+}\frac{\phi(x)-\phi(x_0+0)}{x-x_0}.$$

If $y(x)$ is a solution of (1.1) satisfying the initial conditions $y(x_0)=\phi(x_0+0)$, $y'(x_0)=r$, and $y^{(j)}(x_0)=0$, $2\le j\le n-1$, again sequences $\{t_m\}$ and $\{x_m\}$ can be chosen so that $\lim t_m=\lim x_m=x_0$, $x_0<t_{m+1}<x_m<t_m$ for each $m\ge 1$, and $\phi(t_m)>y(t_m)$, $\phi(x_m)<y(x_m)$ for all sufficiently large m. This again contradicts $\phi(x)$ being a generalized solution. Thus, $D^1\phi(x_0+0)$ and $D^1\phi(x_0-0)$ exist in the extended reals for all $x_0\in(c,d)$.

Finally, if we assume that for some $x_0\in(c,d)$, $D^j\phi(x_0+0)$ exists and is finite for each $1\le j\le k-1\le n-2$, then by considering a solution of (1.1) satisfying the initial conditions $y(x_0)=\phi(x_0+0)$, $y^{(j)}(x_0)=D^j\phi(x_0+0)$ for $1\le j\le k-1$, $y^{(k)}(x_0)=r$, and $y^{(j)}(x_0)=0$ for $k+1\le j\le n-1$, we can as above prove that the limit defining $D^k\phi(x_0+0)$ exists in the extended reals.

Corollary 6.5. *Assume that the differential equation (1.1) satisfies conditions (A) and (D_n), and that $\phi(x)$ is a bounded generalized solution of (1.1) on $(c,d)\subset(a,b)$. Then, $\phi(x)$ has a finite derivative $\phi'(x)$ almost everywhere on (c,d).*

Theorem 6.6 ([13,17]). *Assume that the differential equation (1.1) satisfies conditions (A) – (D_n). Let $\{y_m(x)\}$ be a sequence of solutions of (1.1) on $(c,d)\subset(a,b)$ such that $\{y_m(x)\}$ is uniformly bounded on (c,d) and $\lim y_m(x)=\phi(x)$ on*

(c, d). Then, if for some $x_0 \in (c, d)$ the derivatives $D^j\phi(x_0+0)$, $1 \le j \le n-1$ all exist and are finite, or the derivatives $D^j\phi(x_0-0)$, $1 \le j \le n-1$ all exist and are finite, it follows that there is a subsequence $\{y_{m(j)}(x)\}$ such that $\{y^{(i)}_{m(j)}(x)\}$ converges uniformly on each compact subinterval of (a, b) for each $0 \le i \le n-1$.

Proof. Assume that for some $x_0 \in (c, d)$ the derivatives $D^j\phi(x_0+0)$, $1 \le j \le n-1$ exist and are finite. Let $p(x)$ be the polynomial

$$p(x) = \phi(x_0+0) + \sum_{j=1}^{n-1} \frac{D^j\phi(x_0+0)(x-x_0)^j}{j!}$$

then, it follows from the definition of $D^{n-1}\phi(x_0+0)$ that given any $\varepsilon > 0$ there is a $\delta > 0$ such that $x_0 + \delta < d$, and

$$|p(x) - \phi(x)| < \frac{\varepsilon(x-x_0)^{n-1}}{(n-1)!}$$

for $x_0 < x \le x_0 + \delta$. Let d_0 be a fixed number satisfying $x_0 < d_0 < d$. By Lemma 6.2 there is a $\delta_0 > 0$ such that for $x_0 < x_1 < x_2 < \cdots < x_n \le d_0$ with $x_i - x_{i-1} = \eta \le \delta_0$ for each $1 \le i \le n$, (1.1) has a solution $y(x)$ with $y(x_i) = p(x_i)$ for $1 \le i \le n$ and $|y^{(j)}(x)| \le 2M$ on $[x_1, x_n]$ for $0 \le j \le n-1$ where $|p^{(j)}(x)| \le M$ on $[x_0, d_0]$ for $0 \le j \le n-1$. Furthermore, there is an $\varepsilon_0 > 0$ such that, if $|y_i - p(x_i)| < \varepsilon_0$ for $1 \le i \le n$, then (1.1) has a solution $y(x)$ with $y(x_i) = y_i$ for $1 \le i \le n$ and $|y^{(j)}(x)| \le 3M$ on $[x_1, x_n]$ for $0 \le j \le n-1$. It is not difficult to show that with equal spacing η between the $x_i's$ a suitable ε_0 has the form $\varepsilon_0 = Mh_n\eta^{n-1}$, where h_n is a fixed constant depending on n. Now as noted above, if we choose $\varepsilon = Mh_n/(2n^{n-1})$, there is a η, $0 < \eta \le \delta_0$ such that $x_0 < x < x_0 + n\eta$ implies

$$|p(x) - \phi(x)| < \frac{\varepsilon(x-x_0)^{n-1}}{(n-1)!} \le \frac{\varepsilon_0}{2(n-1)!} \le \frac{\varepsilon_0}{2}.$$

For such a choice of $\eta > 0$, we have $|p(x_i) - \phi(x_i)| \le \varepsilon_0/2$ for $1 \le i \le n$ where $x_i - x_{i-1} = \eta$ for $1 \le i \le n$. Consequently, if $N > 0$ is such that $m \ge N$ implies $|y_m(x_i) - \phi(x_i)| < \varepsilon_0/2$ for $1 \le i \le n$, then $|p(x_i) - y_m(x_i)| < \varepsilon_0$ for $m \ge N$ and $1 \le i \le n$. It follows from our construction and condition (D_n) that $|y^{(j)}_m(x)| \le 3M$ on $[x_1, x_n]$ for $0 \le j \le n-1$ and all $m \ge N$. The conclusion of the theorem now follows.

Thus, we see that, in order to prove that conditions (A) – (D_n) imply the compactness condition (E), it is sufficient to prove that, if $\phi(x)$ is the pointwise limit of a bounded sequence of solutions of (1.1) on $(c, d) \subset (a, b)$, then there is at least one $x_0 \in (c, d)$ at which either $D^j\phi(x_0+0)$, $1 \le j \le n-1$ or $D^j\phi(x_0-0)$, $1 \le j \le n-1$ are finite.

7 The Case $q = 0$

As we have remarked in Section 6 for the differential equation (1.1) the compactness condition (E), under the assumptions (A) – (D_n), is equivalent to (E^*). This observation is used in the following result to establish the Proposition for the case $q = 0$.

Theorem 7.1 ([17]). *If the differential equation (1.1) with $q = 0$ satisfies conditions (A) – (D_n), then (1.1) with $q = 0$ also satisfies condition (E^*).*

Proof. Let $\{y_m(x)\}$ be a monotone, bounded sequence of solutions of (1.1) with $q = 0$ which converges point-wise to a function $\phi(x)$ on $[c, d] \subset (a, b)$. Let $c = a_1 < a_2 < \cdots < a_n = d$ and $p_m(x)$ be the unique polynomial of degree $n-1$ such that $p_m(a_i) = y_m(a_i),\ 1 \le i \le n$ and $m = 1, 2, \ldots$. Then, $p_m(x)$ converges uniformly to $p(x)$, where $p(x)$ is the unique polynomial of degree $n-1$ such that $p(a_i) = \lim_{m\to\infty} y_m(a_i),\ 1 \le i \le n$. Now, since $y_m(x)$ are uniformly bounded on $[c, d]$, it is clear that $M = \sup\{|f(x, y_m(x))| : c \le x \le d,\ m \ge 1\}$ exists. Further, from the properties of the Green's function $g(x, t)$, it follows that $\partial g/\partial x$ exists and is continuous on $[c, d] \times [c, d]$, and hence $|\partial g/\partial x| \le K$ for all $x, t \in [c, d]$. Thus, if $x \ne t$ from the integral representation

$$y_m(x) = p_m(x) + \int_c^d g(x, t) f(t, y_m(t)) dt, \tag{7.1}$$

which is the same

$$\omega_m(x) \equiv y_m(x) - p_m(x) = \int_c^d g(x, t) f(t, y_m(t)) dt$$

we find

$$\begin{aligned} |\omega_m(x) - \omega_m(s)| &\le \int_c^d |g(x, t) - g(s, t)||g(t, y_m(t))| dt \\ &\le MK|x - s|(d - c). \end{aligned}$$

Hence, $\{\omega_m(x)\}$ is uniformly bounded and equicontinuous on $[c, d]$. Thus, a subsequence and by monotonicity the whole sequence $\{y_m(x)\}$ converges uniformly to $\phi(x)$ on $[c, d]$. Finally, taking limits through (7.1) yields that $\phi(x)$ is a solution of (1.1) with $q = 0$ on $[c, d]$, and hence the condition (E^*) is satisfied.

8 The Uniform Convergence

In [8] Henderson and Jackson in there closing remarks have mentioned the validity of the Proposition for fourth order differential equations. To prove the Proposition for arbitrary order differential equations we let P_n denote the set of all real-valued polynomials of degree at most n.

Definition 8.1 ([3]). Given $S \subset [c,d]$, x_0 is a *bilateral accumulation point* of S, in case x_0 is an accumulation point of both $S \cap [c, x_0]$ and $S \cap [x_0, d]$.

Definition 8.2. A function $g(x) : I \to \mathbb{R}$, I an interval, is said to be *n-convex (n-concave),* on I in case for any distinct points $x_0, x_1, \ldots, x_n$ in I,

$$\sum_{i=0}^{n} \frac{g(x_i)}{\omega'(x_i)} \geq 0, \ (\leq 0),$$

where

$$\omega(x) = \prod_{i=0}^{n}(x - x_i), \qquad \text{so that} \quad \omega'(x_j) = \prod_{i=0, i\neq j}^{n} (x_j - x_i).$$

The following results for n-convex functions are well known.

Lemma 8.1. *Suppose $g(x) \in C^{(n)}(I)$. Then, $g(x)$ is n-convex on I, if and only if, $g^{(n)}(x) \geq 0$ on I.*

Lemma 8.2. *The function $g(x)$ is n-convex, if and only if, $g(x) \in C^{(n-2)}(I)$ and $g^{(n-2)}(x)$ is convex.*

Remark 8.1. In Lemmas 8.1 and 8.2, 'convex' can be replaced by 'concave'.

Lemma 8.3 ([3]). *Let $g(x) \in C[c,d]$ and assume that, for each $p(x) \in P_n$, the set $\{x : p(x) = g(x)\}$ does not have a bilateral accumulation point in (c,d). Then, there exists a subinterval $I \subseteq [c,d]$ on which $g(x)$ is either $(n+1)$-convex or $(n+1)$-concave.*

Theorem 8.4 ([21]). *Assume that the differential equation (1.1) satisfies the conditions (A) – (D$_n$). Then, (1.1) also satisfies condition (E).*

Proof. Let $\{y_m(x)\}$ be a sequence of solutions of (1.1) which is uniformly bounded on some subinterval $[c,d] \subset (a,b)$. Then, by Corollary 5.4 there exists a subsequence $\{y_{m(j)}(x)\}$ and a function $z(x) \in BV[c,d]$ such that $\lim_{j\to\infty} y_{m(j)}(x) = z(x)$ point-wise on $[c,d]$. Thus, $z'(x)$ exists a.e. on $[c,d]$ and $\int_c^d z(x)dx$ exists. We set

$$Z(x) = \int_c^x z(t)dt.$$

Then, $Z \in C[c,d]$, and by Lemma 8.3, either one of the following holds.

(i) $Z(x)$ is $(n+2)$-convex or $(n+2)$-concave on some $[c_1, d_1] \subseteq [c,d]$, or

(ii) there exists a $p(x) \in P_{n+1}$ such that $\{x : p(x) = Z(x)\}$ has a bilateral accumulation point in (c,d).

Case (i). Let us relabel the sequence $\{y_{m(j)}(x)\}$ as $\{y_m(x)\}$. By Lemma 8.2, $Z(x) \in C^{(n)}[c_1, d_1]$ and $Z^{(n)}(x)$ is convex, (or concave). Thus, $Z'(x) \in$

$C^{(n-1)}[c_1, d_1]$ and $Z'(x) = z(x)$ a.e. on $[c_1, d_1]$. Thus, as a consequence of Corollaries 2.6 and 2.7 there exists a $\delta = \delta(Z', d_1 - c_1) > 0$ such that, for fixed $c_1 \le a_1 < a_2 < \cdots < a_n \le d_1$, with $a_n - a_1 \le \delta$, there exists an $\varepsilon_0 > 0$ such that the boundary value problem for (1.1) satisfying $y(a_j) = Z'(a_j) + \varepsilon_j,\ 1 \le j \le n$, where $|\varepsilon_j| \le \varepsilon_0,\ 1 \le j \le n$, has a solution $y(x)$. Furthermore, the first $n-1$ derivatives of this solution are bounded, with bounds depending on Z' and $d_1 - c_1$. We call these bounds as $N_0 + 1, \ldots, N_{n-1} + 1$.

Let us now choose points $c_1 \le a_1 < a_2 < \cdots < a_n \le d_1$, with $a_n - a_1 \le \delta$ and such that $Z'(a_j) = z(a_j),\ 1 \le j \le n$. Then, there exists an M such that $|y_m(a_j) - z(a_j)| \le \varepsilon_0,\ 1 \le j \le n$ for all $m \ge M$. Now for $m \ge M$ and $1 \le j \le n$, let $\varepsilon_{m(j)} = y_m(a_j) - z(a_j)$. Then, for $m \ge M$,

$$y_m(a_j) = z(a_j) + \varepsilon_{m(j)} = Z'(a_j) + \varepsilon_{m(j)}, \quad 1 \le j \le n$$

and it follows from condition (D_n) that $y_m(x)$ is the solution referred to above resulting from Corollaries 2.6 and 2.7. As a consequence, we have for $m \ge M$, $|y_m^{(i)}(x)| \le N_i + 1$ on $[a_1, a_n]$ for each $0 \le i \le n-1$. Now we can apply Theorem 2.1 to obtain a further subsequence $\{y_{m(\ell)}(x)\}$ such that $\{y_{m(\ell)}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b) for each $0 \le i \le n-1$.

Case (ii). Assume that $\{x_m\} \downarrow x_0$ is such that $p(x_m) = Z(x_m)$, for all $m \ge 1$. Thus, if on some subinterval $[x_{j+1}, x_j]$, $Z'(x) = z(x) \ge p'(x)$ a.e., then we have

$$\begin{aligned} Z(x_j) - Z(x_{j+1}) &= \int_{x_{j+1}}^{x_j} z(t)dt \\ &\ge \int_{x_{j+1}}^{x_j} p'(t)dt \\ &= p(x_j) - p(x_{j+1}) \ = \ Z(x_j) - Z(x_{j+1}), \end{aligned}$$

so that the inequality is in fact an equality. However, from $z(x) - p'(x) \ge 0$, a.e. on $[x_{j+1}, x_j]$, we have $\int_{x_{j+1}}^{x_j} (z(t) - p'(t))dt = 0$, we conclude that $z(x) - p'(x) = 0$ a.e. on $[x_{j+1}, x_j]$. In particular,

$$Z'(x) = z(x) = p'(x) \quad \text{a.e.} \quad \text{on } [x_{j+1}, x_j].$$

Similarly, if we assume that on some subinterval $[x_{j+1}, x_j]$, $Z'(x) = z(x) \le p'(x)$ a.e., then we would arrive at $Z'(x) = z(x) = p'(x)$ a.e. on $[x_{j+1}, x_j]$.

Now that $p'(x) \in C^{(n-1)}[c, d]$, by Corollaries 2.6 and 2.7 there exists a $\delta = \delta(p', d - c) > 0$ such that for fixed $c \le a_1 < a_2 < \cdots < a_n \le d,\ a_n - a_1 \le \delta$, there exists an $\varepsilon_0 > 0$ such that the boundary value problem for (1.1) satisfying $y(a_j) = p'(a_j) + \varepsilon_j,\ 1 \le j \le n$, where $|\varepsilon_j| \le \varepsilon_0,\ 1 \le j \le n$, has a solution $y(x)$. As in Case (i) this solution has bounds on its first $n-1$ derivatives depending only on p' and $d - c$; again we call these bounds as $N_0 + 1, \ldots, N_{n-1} + 1$.

Let us choose points $x_{\ell+n} < x_{\ell+n-1} < \cdots < x_\ell$ from $\{x_m\}$ such that $x_\ell - x_{\ell+n} \le \delta$. We need to consider two subcases:

Case (a). For some $1 \le r \le n$, on $[x_{\ell+r}, x_{\ell+r-1}]$ we have $z(x) \ge p'(x)$ a.e., or $z(x) \le p'(x)$ a.e. From the above arguments we, however, have $z(x) = p'(x)$ a.e. on $[x_{\ell+r}, x_{\ell+r-1}]$. We repeat the arguments of Case (i). Choose points $x_{\ell+r} \le a_1 < \cdots < a_n \le x_{\ell+r-1}$ such that $z(a_j) = p'(a_j),\ 1 \le j \le n$. Then, there exists an M such that $|y_m(a_j) - z(a_j)| \le \varepsilon_0,\ 1 \le j \le n,\ m \ge M$. For $m \ge M$ and $1 \le j \le n$ let $\varepsilon_{m(j)} = y_m(a_j) - z(a_j)$. Then, for $m \ge M$,

$$y_m(a_j) = z(a_j) + \varepsilon_{m(j)} = p'(a_j) + \varepsilon_{m(j)}, \qquad 1 \le j \le n$$

and it follows that, from condition (D_n), $y_m(x)$ is the solution referred to the above problem arising from the Corollaries 2.6 and 2.7. Thus, for all $m \ge M$,

$$|y_m^{(i)}(x)| \le N_i + 1 \quad \text{on} \quad [a_1, a_n]$$

for each $0 \le i \le n-1$. Then, by Theorem 2.1 there exists a further subsequence $\{y_{m(s)}(x)\}$ such that $\{y_{m(s)}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), for each $0 \le i \le n-1$.

Case (b). For each $1 \le r \le n$, there exist sets $A_r,\ B_r \subset [x_{\ell+r}, x_{\ell+r-1}]$, each having positive Lebesgue measure, and $z(x) > p'(x)$ on A_r, and $z(x) < p'(x)$ on B_r. However, since $\lim_{m\to\infty} y_m(x) = z(x)$, and so there exists a M such that for $m \ge M$, $y_m(x) > p'(x)$, for some A_r, and $y_m(x) < p'(x)$, for some $x \in B_r$. By continuity, for all $1 \le r \le n$, there exists $a_r \in (x_{\ell+r}, x_{\ell+r-1})$ such that $y_m(a_r) = p'(a_r)$.

In particular, there are points $x_{\ell+n} \le \tilde{a}_1 < \cdots < \tilde{a}_n \le x_\ell$ $(\tilde{a}_n - \tilde{a}_1 \le \delta)$, so that, for some $M' \ge M$,

$$y_m(\tilde{a}_j) = p'(\tilde{a}_j) + \varepsilon_{m(j)}, \qquad 1 \le j \le n$$

where $|\varepsilon_{m(j)}| \le \varepsilon_0,\ 1 \le j \le n$ and all $m \ge M'$.

It follows from condition (D_n) that $y_m(x)$ is the solution referred to before Case (a) arising from Corollaries 2.6 and 2.7. Thus, for all $k \ge M'$,

$$|y_m^{(i)}(x)| \le N_i + 1 \quad \text{on} \ \ [\tilde{a}_1, \tilde{a}_n],$$

for each $0 \le i \le n-1$. Now an application of Theorem 2.1 leads to a subsequence $\{y_{m(j)}(x)\}$ such that $\{y_{m(j)}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), for each $0 \le i \le n-1$.

9 Problems and Comments

The establishment of Theorem 8.4 implies that in the known results on conjugate boundary value problems the condition (E) is, in fact, superfluous. As an example we state an important result, which was independently proved by Hartman [6] and Klassen [17] with the additional condition (E).

Theorem 9.1. *Assume that for the differential equation* (1.1) *conditions* (A) – (D_n) *are satisfied. Then, each r point boundary value problem, i.e., for any $a < a_1 < a_2 \cdots < a_r < b$ and any $A_{j+1,i}$, $0 \le j \le k_i$, $1 \le i \le r$ the problem* (1.1), (2.3) *has a unique solution.*

Problem 1. For the third order differential equations in Theorem 4.5 we have proved that conditions (A), (C) and (D_3) imply condition (E). It will be interesting to extend this result to equations of arbitrary order, i.e., whether it is possible to prove Theorem 8.4 without the assumption (B).

In [16] Jackson has indicated that for n-point boundary value problems, Klassen has used a result he established in [18] to prove the existence of solutions under the assumptions (A), (C), (D_n) and (E). Thus, if the answer to Problem 1 is affirmative, then for $r = n$, Theorem 9.1 holds without the assumption (B).

Let $2 \le r \le n$ and let m_i, $1 \le i \le r$, be positive integers such that $\sum_{i=1}^{r} m_i = n$. Let $s_0 = 0$ and for $1 \le k \le r$, let $s_k = \sum_{i=1}^{k} m_i$. A boundary value problem for (1.1) with the boundary conditions

$$y^{(i)}(a_k) = y_{i,k}, \quad s_{k-1} \le i \le s_k - 1, \quad 1 \le k \le r \tag{9.1}$$

where $a < a_1 < a_2 < \cdots < a_r < b$ is called a *right* $(m_1, \ldots, m_r)$*-focal point boundary value problem for* (1.1) *on* (a, b).

With respect to the boundary conditions (9.1) we replace the condition (D_n) by the following:

(D_n^{rf}) For any $a < a_1 < a_2 < \cdots < a_n < b$ and any solutions $y(x)$ and $z(x)$ of (1.1), it follows that $y^{(i-1)}(a_i) = z^{(i-1)}(a_i)$, $1 \le i \le n$ implies $y(x) \equiv z(x)$, i.e., the differential equation (1.1) is right $(1, 1, \ldots, 1)$ disfocal on (a, b).

As an application of Rolle's theorem it follows that condition (D_n^{rf}) implies the condition (D_n). Thus, in Theorem 8.4 condition (D_n) can be replaced by (D_n^{rf}). We state this observation in the following result.

Theorem 9.2. *Assume that the differential equation* (1.1) *satisfies conditions* (A) – (C) *and* (D_n^{rf}). *Then,* (1.1) *also satisfies condition* (E).

Of course, in Theorem 8.4, we can always replace condition (D_n) by a stronger condition. The point is now whether it is possible to replace condition (D_n) by some other condition which does not imply (D_n). The first 'round about' result in this direction is the following:

Theorem 9.3. *If the differential equation* (1.1) *is of third order and satisfies conditions* (A), (C) *and* (D_2), *then* (1.1) *also satisfies condition* (E).

Proof. The proof is similar to that of Theorems 4.5.

Problem 2. A result similar to that of Theorem 9.3 for arbitrary order differential equations (1.1) remains undecided, i.e., does conditions (A), (C) and (D_r) imply condition (E).

For arbitrary order differential equations (1.1) Jackson [14] has established that conditions (A), (B) and (D_n) imply (D_r). A converse of this result for third order differential equations (1.1) is that the conditions (A), (C) and (D_2) imply (D_3). Jackson's proof [15] of this converse result uses Theorem 9.3, i.e., under the assumptions, condition (E) is implied, and then this fact is used to prove (D_3). Thus, if we accept Jackson's converse result without looking at its proof, then we can argue that conditions (A), (C) and (D_2) imply (D_3), and therefore Theorem 4.5 gives condition (E).

Problem 3. The question for arbitrary order differential equations (1.1) which remains open is whether conditions (A), (C) and (D_r) imply condition (D_n).

Finally, we state one more result which is similar in nature to that of Theorem 9.3.

Theorem 9.4 ([7]). *If the differential equation* (1.1) *is of third order and satisfies conditions* (A), (C) *and*

(D_2^{rf}) each right $(2, 1)$-focal point boundary value problem for (1.1) on (a, b) has at most one solution,

then (1.1) also satisfies condition (E).

Problem 4. A result similar to that of Theorem 9.4 for arbitrary order differential equations (1.1) is not known.

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