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Michal Greguš

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ON LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ODD ORDER

M. Greguš, Bratislava

In my paper I will consider a differential equation of the *n-th* order, where n is odd of the following form:

$$y^{(n)} + 2A(x)\,y' + [A'(x) + b(x)]\,y = 0. \tag{a}$$

Suppose that $A'(x)$ and $b(x)$ are continuous functions of $x \in (-\infty, \infty)$.
The adjoint differential equation to the equation (a) is of the form

$$z^{(n)} + 2A(x)\,z' + [A'(x) - b(x)]\,z = 0. \tag{b}$$

Between the solutions of the differential equations (a) and (b) hold some relations, for instance:

If $y_1, y_2, \ldots, y_{n-1}$ are linearly independent solutions of the equation (a) then

$$z(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} \\ y_1' & y_2' & \cdots & y_{n-1}' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} \end{vmatrix}$$

is the solution of the equation (b).

If $y(x)$ is the solution of the differential equation (a) with the property

$$y(a) = y'(a) = \ldots = y^{(n-2)}(a) = 0, \qquad y^{(n-1)}(a) \neq 0,$$
$$a \in (-\infty, \infty) \quad \text{and} \quad y(\bar{x}) = 0, \quad \bar{x} \neq a,$$

then the solution $z(x)$ of the differential equation (b) with the property

$$z(\bar{x}) = z'(\bar{x}) = \ldots = z^{(n-2)}(\bar{x}) = 0, \qquad z^{(n-1)}(\bar{x}) \neq 0$$

has also the property $z(a) = 0$.

We can deduce more of such relations.

The solutions of the differential equation (a), respectively (b) fulfil the following integral identities:

$$y^{(n-1)} + 2Ay + \int_a^x (b - A')\, y\, dt = \text{const.} \tag{1}$$

$$yy^{(n-1)} + Ay^2 + \int_a^x [by^2 - y'y^{(n-1)}]\, dt = \text{const.} \tag{2}$$

respectively

$$z^{(n-1)} + 2Az - \int_a^x (A' + b)\, z\, dt = \text{const.} \tag{1'}$$

$$zz^{(n-1)} + Az^2 - \int_a^x [bz^2 + z'z^{(n-1)}]\, dt = \text{const.} \tag{2'}$$

In the following I will introduce the results concerning the solutions without zeros of the differential equations (a) and (b) and the criterion of the divergence of the solutions without zeros of the differential equation (b). In the special case for $n = 3$ I will quote further results concerning this problems.

At the end I will deal with the existence of solutions of certain boundary problems chiefly of the third order.

I. First I will introduce two lemmas.

Lemma 1. *Let $A(x) \leqq 0$, $b(x) \geqq 0$ for $x \in (-\infty, \infty)$. Let $y(x)$ be the solution of the differential equation (a) with properties $y^{(i)}(a) = 0$, $i = 0, 1, \ldots, k - 1, k + 1, \ldots, n - 1$, $y^{(k)}(a) \neq 0$, $1 \leq k \leq n - 1$. Then neither $y(x)$ nor its derivatives $y^{(i)}(x)$, $i = 1, 2, \ldots, n - 1$ have no zeros to the left side of a.*

Lemma 2. *Let the assumptions of Lemma 1 be satisfied and let $z(x)$ be the solution of the differential equation (b) with the properties $\overset{(i)}{z}(a) = 0$, $i = 0, 1, \ldots, k - 1, k + 1, \ldots, n - 1$, $z^{(k)}(a) \neq 0$. Then $z^{(i)}(x)$, $i = 0, 1, \ldots, n - 1$ have no zeros to the right of a and at the same time $z^{(i)}(x) \to \pm\infty$ for $x \to \infty$, $i = 0, 1, \ldots, n - 3$. Here $z^{(i)}(x) \to +\infty$ if $z^{(k)}(a) > 0$ and $z^{(i)}(x) \to -\infty$ if $\overset{(k)}{z}(a) < 0$.*

The proof of Lemma 1 follows from the identity (2) and that of Lemma 2 from the identity (2').

Remark 1. Similarly as Lemma 2 it can be proved that every non-trivial solution $z(x)$ of the differential equation (b) with properties

$$z(a) = 0, \qquad z^{(i)}(a) \geqq 0, \qquad i = 1, 2, \ldots, n - 1, \ -\infty < a < +\infty,$$

has no zero point to the right, and no point of zero of the derivatives up to the order $n - 1$.

Theorem 1. *Let $A(x) \leqq 0$, $b(x) \geqq 0$ for $x \in (-\infty, \infty)$. Then the differential equation (a) [(b)] has at least one solution $u(x)$ [$v(x)$] without zero in the interval $(-\infty, \infty)$ for which holds*

$$\operatorname{sgn} u(x) = \operatorname{sgn} u''(x) = \ldots = \operatorname{sgn} u^{(n-1)}(x) \neq \operatorname{sgn} u'(x) = \operatorname{sgn} u'''(x) = \\ = \ldots = \operatorname{sgn} u^{(n-2)}(x)$$

$$[\operatorname{sgn} v(x) = \operatorname{sgn} v'(x) = \operatorname{sgn} v''(x) = \ldots = \operatorname{sgn} v^{(n-1)}(x)]$$

for all $x \in (-\infty, \infty)$ *at the same time* $u'(x) \to 0$, $u''(x) \to 0$, $\ldots$, $u^{(n-2)}(x) \to 0$ *for* $x \to \infty$ $[v(x) \to \pm\infty$, $\ldots$, $v^{(n-3)}(x) \to \pm\infty$ *for* $x \to \infty]$.

The solution $u(x)$ without zero-points of the properties mentioned in Theor. 1 can be obtained as the limit of the sequence of the solutions $\{u_k(x)\}_{k=1}^{\infty}$ with $u_k^{(i)}(x_k) = 0$, $u_k^{(n-1)}(x_k) > 0$, $i = 0, 1, \ldots, n-2$, where $\{x_k\}_{k=1}^{\infty}$ is a suitable sequence of numbers which diverges to infinity. The integral identity (2) for the solution u_k is of the form

$$u_k u_k^{(n-1)} + A u_k^2 = \int_x^{x_k} [b u_k^2 - u_k' u_k^{(n-1)}] \, \mathrm{d}t.$$

It can be shown that the solution $u(x)$ fulfils the analogical identity:

$$u u^{n-1} + A u^2 = \int_x^{\infty} [b u^2 - u' u^{(n-1)}] \, \mathrm{d}t. \tag{3}$$

Theorem 2. *Let* $A(x) \leqq 0$, $b(x) \geqq 0$ *for* $x \in (-\infty, \infty)$ *and let* $\int_a^{\infty} b \, \mathrm{d}t$ *diverge,* $-\infty < a < \infty$. *Let* $z(x)$ *be the solution of the differential equation* (b) *with the properties* $z^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-2$, $z^{(n-1)}(a) > 0$. *Then the following holds*: $z^{(n-1)}(x) + 2A(x) z(x) \to \infty$ *for* $x \to \infty$.

The statement follows from Lemma 2 and from the identity (1').

Lemma 3. *Let the assumtions of Lemma 2 hold and in additicn let* $A' + b \geqq 0$ *and* $\int_a^{\infty} t^{n-1}[A'(t) + b(t)] \, \mathrm{d}t$ *diverge,* $-\infty < a < \infty$.

Let $z(x)$ *be the solution of the differential equation* (b) *with the properties* $z^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-2$, $z^{(n-1)}(a) > 0$. *Then it holds*: $z^{(n-1)}(x) + 2A(x) z(x) \to \infty$ *for* $x \to \infty$.

The statement follows from Lemma 2 and from the identity (1').

Theorem 3. *Let the assumptions of Lemma* 3 *be satisfied. Then there exists at least one solution of the differential equation* (a) $y(x) \neq 0$ *for* $x \in (-\infty, \infty)$ *which has the following properties*: $y, y', \ldots, y^{(n-1)}$ *are monotonous function of* $x \in (-\infty, \infty)$, $\operatorname{sgn} y = \operatorname{sgn} y'' = \ldots = \operatorname{sgn} y^{(n-1)} \neq \operatorname{sgn} y' = \operatorname{sgn} y''' = \ldots = \operatorname{sgn} y^{(n-2)}$ *for* $x \in (-\infty, \infty)$ *and* $y \to 0$, $y' \to 0$, $\ldots$, $y^{(n-1)} \to 0$ *for* $x \to \infty$.

Let $n = 3$. Then the statement of Theorem 2 and Theorem 3 can be sharpened in the following way:

Theorem 4. *Let the assumptions of Theorem* 2 *and Remark* 2 *respectively Theorem* 3 *for* $n = 3$ *be fulfilled and let* $b(x) \equiv 0$ *do not hold in any interval.*

Then there exists just one solution of the differential equation (a) *with the following properties*: y, y', y'' *are monotonous functions of* $x \in (-\infty, \infty)$, $\operatorname{sgn} y = \operatorname{sgn} y'' \neq \neq \operatorname{sgn} y'$ *for* $x \in (-\infty, \infty)$ *and* $y \to 0$, $y' \to 0$, $y'' \to 0$ *for* $x \to \infty$.

Theorem 5. *Let* $n = 3$. *Let* $A(x) \leqq 0$, $b(x) \geqq 0$, $A'(x) + b(x) \geqq 0$ *for* $x \in (-\infty, \infty)$ *and let* $b(x) \equiv 0$ *do not hold in any interval.*

If the differential equation (a) *has one oscillatory solution in the interval* (a, ∞), $-\infty < a < +\infty$ *(i.e. the solution has an infinite number of zero-points there) then all solutions of the differential equation* (a) *are oscillatory in the interval* (a, ∞) *with one exception of the solution* y *(up to the linear dependence) which has the following properties*: $y(x) \neq 0$, $\operatorname{sgn} y(x) = \operatorname{sgn} y''(x) \neq \operatorname{sgn} y'(x)$ *for* $x \in (-\infty, \infty)$, y, y', y'' *are monotonous functions of* $x \in (-\infty, \infty)$ and $y' \to 0$, $y'' \to 0$ *for* $x \to \infty$. [1].

The question is about the solution without zeros in the case $A(x) \geqq 0$. For $n = 3$ hold the following results [2]:

Theorem 6. *Let* $n = 3$. *Let* $b(x) \geqq 0$ *for* $x \in (-\infty, \infty)$ *and* $b(x) \equiv 0$ *do not hold in any interval. Then the solution of the differential equation* (a) *has at least one solution without zero points in* $(-\infty, \infty)$.

Theorem 7. *Let the assumptions of Theorem* 6 *be fulfilled and let* $\int_{x_0}^{\infty} b \, \mathrm{d}t$ *diverge. Then the differential equation* (a) *has at least one solution without zero-points for which holds* $\liminf_{\langle x_0, \infty)} y(x) = 0$.

If $b(x) \geqq m > 0$ *for* $x \in (x_0, \infty)$ *then* $y(x)$ *belongs to the class* L^2.

M. Zlámal [3] proved the following theorem:

Let $n = 3$. Let $A(x) \geqq m > 0$, $A'(x) + b(x) \geq m$ and $b(x) - A'(x) \geqq 0$ for $x \in (x_0, \infty)$.

Then every solution of the differential equation (a) is either oscillatory in (x_0, ∞) or non oscillatory and then $\lim_{x \to \infty} y = \lim_{x \to \infty} y' = 0$ and $y(x)$ is of the class L^2.

In the paper [2] is shown that under given assumptions all solutions of the differential equation (a) are oscillatory in (x_0, ∞) with the exception of one (up to the linear dependence) which has the mentioned properties.

Now we shall devote our attention to the differential equation (b). According to Theorem 1, the equation (b) has at least one solution without zero points and every solution of the differential equation (b) of the properties given in Lemma 2 and in the Remark 1 has not on the right side of a zero and there are no zeros of the derivatives up to the order $n - 1$. In the following we give the criteria for the rate of divergence of these solutions to the infinity.

Theorem 8. *Let* $A(x) \leqq 0$, $b(x) \geqq 0$ *for* $x \in (-\infty, \infty)$ *and let* $b(x) \equiv 0$ *do not* *hold in any interval. Let $f(x)$ be a non-negative function with the continuous n-th derivative of properties$f^{(n)} + 2Af' + [A' - b]f \leqq 0$ for $x \in (x_0, \infty)$, $-\infty < x_0 < \infty$.*

Then there exists for every non-trivial solution $z(x)$ of the differential equation (b) of the properties $z(a) = 0$, $z^{(i)}(a) \geqq 0$, $i = 1, \ldots, n-1$, $a \geqq x_0$, such a $\alpha \geqq a$ and such a constant $k > 0$ that for $x > \alpha$ holds $z(x) - kf(x) > 0$.

Corollary 1. Let $f(x) = e^x$. Then the non-trivial solution of the differential equation (b) of the properties $z(a) = 0$, $z^{(i)}(a) \geqq 0$ diverges to $+\infty$ faster then e^x if $A(x) \leqq 0$, $b(x) \geqq 0$, $1 + 2A + A' - b \leqq 0$ for $x \in (x_0, \infty)$, $x_0 \leqq a$ and at the same time $b(x) \equiv 0$ does not hold in any interval.

For $n = 3$ and the case $A(x) \geqq 0$ hold the following criteria:

M. Zlámal [3] proved:

Let $A(x) \geqq 0$, $A'(x)$ and $b(x) \geqq 0$ be continuous functions of x for $0 < x_0 \leqq x$. Futher let on (x_0, ∞) $M = \limsup \dfrac{A(x)}{\sqrt{x}} < \infty$, $m = \limsup \sqrt{x}\,[A'(x) - b(x)] < 0$. Then every non-trivial solution $y(x)$ fo the differential equation (b) is either oscillatory or diverges into $\pm\infty$ faster then a certain positive power x_0. The solution $y(x)$ is oscillatory then, and only then when for every $x \in (x_0, \infty)$ holds $yy'' - \dfrac{1}{2}y'^2 + Ay^2 < 0$.

If $b(x) \geqq d > 0$, then every scillatory solution of the differential equation (b) belongs to the class L^2.

Theorem 9. *Let $n = 3$ and let $A(x) \geqq 0$, $b(x) \geqq 0$, $A'(x) - b(x) \leqq 0$ at the same time $b(x) \equiv 0$ do not hold, in any interval and let $\int_{x_0}^{\infty} b\,\mathrm{d}t$ diverge. Further let $f(x)$ be a non-negative function with continuous third derivative on (x_0, ∞) for which holds on (x_0, ∞) the inequality $f''' + 2Af' + (A' - b)f \leqq 0$. Then for every solution $z(x)$ of the differential equation (b) without zeropoints on the interval (α, ∞), $\alpha \geqq x_0$ holds: $|z| - kf > 0$ for $x \in (\alpha, \infty)$, k is a suitable constant.*

The proof is given in the paper [2].

II. We devide this section in two parts. In the first section we shall deal with certain non—homogenous boundary value problems chiefly of the 3^d order, and in the second section we shall show some results of the so called Sturm boundary problems of the 3^d order.

Let the boundary problem

(4) $L(y) = 0$.

(5) $Ui(y) = 0$, $i = 1, 2, \ldots, n$, be given where $L(y)$ is a linear differential operator of the n^{th} order, $n \geqq 2$, with continuous coefficients $p_0 \neq 0$ (coefficient of the highest derivative) $p_1, \ldots, p_n$ on the interval $\langle a_1, a_m \rangle$, Ui, $i = 1, 2, \ldots, n$ are linearly independent forms of $y(a_1), \ldots, y^{(n-1)}(a_1), \ldots,$

$y(a_m), \ldots, y^{(n-1)}(a_m)$, $a_1 < a_2 < \ldots < a_m$, $m \geqq 2$. Suppose that the problem (4), (5) is unsolvable, i.e. its only solution is trivial. Then the following theorem [4] holds.

Theorem 10. *For an arbitrary point $\xi \in (a_k, a_{k+1})$ the function $y = G_k(x, \xi)$ may be constructed (the particular Green's function) which has the following properties:*

1. $G_k(x, \xi)$, $\dfrac{\partial}{\partial x} G_k(x, \xi)$, $\ldots$, $\dfrac{\partial^{n-2}}{\partial x^{n-2}} G_k(x, \xi)$ *are continuous functions of* $x \in \in \langle a_1, a_m \rangle$.

2. *The function* $\dfrac{\partial^{n-1}}{\partial x^{n-1}} G_k(x, \xi)$ *is on* $\langle a_1, a_m \rangle$ *everywhere continuous with the exception of the point* ξ, *where it has a discontinuity of the first order with a jump of the discontinuity* $\dfrac{1}{p_0(\xi)}$, *i.e.*

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G_k(\xi + 0, \xi) - \frac{\partial^{n-1}}{\partial x^{n-1}} G_k(\xi - 0, \xi) = \frac{1}{p_0(\xi)}.$$

3. *The function* $G_k(x, \xi)$ *is the solution of the equation* (4) *on the intervals* $\langle a_1, \xi)$, $(\xi, a_m\rangle$ *and satisfies the boundary conditions* (5).

4. *The function* $G_k(x, \xi)$ *is by the properties* 1., 2., 3., *uniquely defined.*

Theorem 11. *Let* $G_k(x, \xi)$, $k = 1, 2, \ldots, m - 1$, *be the particular Green's functions belonging to the problem* (4), (5). *Then the solution of the problem*

(4′) $L(y) = r(x)$.

(5) $Ui(y) = 0, \qquad i = 1, 2, \ldots, n$

where $r(x)$ *is the continuous function on* $\langle a_1, a_m \rangle$ *is given by the formula*

$$(6) \qquad y(x) = \sum_{k=1}^{m-1} \int_{a_k}^{a_{k+1}} G_k(x, \xi)\, r(\xi)\, \mathrm{d}\xi.$$

Lemma 4. *Let* $n = 3$. *Let* $A(x) \leqq 0$, $A'(x)$, $b(x) \geqq 0$ *be such continuous functions of* $x \in (-\infty, \infty)$ *that* $b(x) - A'(x) \leqq 0$ *and* $b(x) \equiv 0$ *does not hold in any interval. Then every solution of the differential equation* (*a*) *has at most two zero points or one double zero-point* [2].

Lemma 5. *Let* $n = 3$. *Let* $A(x) \leqq 0$, $A'(x)$, $b(x) \geqq 0$ *be such continuous functions of* $x \in (-\infty, \infty)$ *that* $A'(x) + b(x) \leqq 0$ *and* $b(x) \equiv 0$ *does not hold in any interval. Then every solution of the differential equation* (*a*) *has only two zero points or one double zero point* [2].

Theorem 12. *Let* $n = 3$. *Let the coefficients of the differential equation* (*a*) *fulfil the assumptions of Lemma* 4, *resp.* 5. *Then the boundary problem*

$$y''' + 2A(x)\,y' + [A'(x) + b(x)]\,y = r(x),$$
$$y(a_1) = y(a_2) = y(a_3) = 0, \qquad a_1 \leq a_2 < a_3 \in (-\infty, \infty)$$

has only one solution given by the formula for $m = 3$.

The proof follows from Lemma 4 and 5 and from Theorem 11.

Lemma 6. *Let $n = 3$. Let the assumption of Lemma 5 be satisfied. Then every solution of the differential equation (a) has at most three zero points of the first derivative. If the solution $y(x)$ has exactly three zero points of the first derivative, then $y(x)$ has exactly two zeros.*

Theorem 13. *Let $n = 3$. Let the coefficients of the differential equation (a) fulfill the assumptions of Lemma 5.*

Then the boundary problem

$$y''' + 2A(x)\,y' + [A'(x) + b(x)]\,y = r(x),$$
$$y^{(i)}(a) = y^{(i)}(b), \qquad i = 0, 1, 2; \qquad a < b$$

with periodical boundary conditions has just one solution of the form (6) *for $m = 2$.*

Remark 3. M. Gera of Bratislava in his dissertation devotes his attention to the problems of the higher orders of periodic boundary conditions.

Now let us consider the differential equation (a) for $n = 3$ in case that the coefficients are continuous functions of the parameter $\lambda \in (\Lambda_1, \Lambda_2)$ i.e. in the form

$$(\bar{a}) \qquad y''' + 2A(x, \lambda)\,y' + [A'(x, \lambda) + b(x, \lambda)]\,y = 0.$$

The following oscillatory theorem [2] holds:

Theorem 14. *Let the coefficients of the equation $(\bar{a})$ $A = A(x, \lambda)$, $A' = \dfrac{\partial}{\partial x} A(x, \lambda)$, $b = b(x, \lambda) > 0$ be continuous functions of $x \in (-\infty, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$.*

Further let $|A(x, \lambda)| \leqq k$, $|A'(x, \lambda)| \leqq k$, $k > 0$, for all $x \in (-\infty, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and let $\lim\limits_{\lambda \to \Lambda_2} b(x, \lambda) = +\infty$ uniformly for all $x \in (-\infty, \infty)$.

Let $a < b \in (-\infty, \infty)$ be given numbers and let $y(x, \lambda)$ be a solution of the differential equation $(\bar{a})$ with the property $y(a, \lambda) = 0$. Then with the increasing $\lambda \to \Lambda_2$ increases also the number of zero points of the solution y in (a, b) to the infinity and at the same time the distance of every two neighbouring zero points converges to zero.

Remark 4. G. Samsone [5] proved also the oscillatory theorem which can be formulated for the equation $(\bar{a})$ as follows:

Let $A = A(x, \lambda)$, $A' = \dfrac{\partial}{\partial x} A(x, \lambda)$, $b = b(x, \lambda) \geqq 0$ be continuous functions of $x \in (-\infty, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and $\lim\limits_{\lambda \to \Lambda_2} A(x, \lambda) = +\infty$ hold uniformly for all $x \in (-\infty, \infty)$.

Let $b(x, \lambda) \equiv 0$ do not hold in any interval. Then the statement of the previous theorem holds.

With the help of the oscillatory theorems the existence of eigenvalues and of eigenfunctions of the following boundary problem can be proved:

Theorem 15. *Let the coefficients of the differential equation* ($\bar{a}$) *fulfil the assumptions of oscillatory theorems. Let* $a \leqq b < c \in (-\infty, \infty)$ *be given numbers. Further let* $\alpha(\lambda)$, $\alpha_1(\lambda)$, $\beta(\lambda)$, $\beta_1(\lambda)$ *be continuous functions of the parameter* $\lambda \in (\Lambda_1, \Lambda_2)$ *for which holds* $|\alpha| + |\alpha_1| \neq 0$, $|\beta| + |\beta_1| \neq 0$, *at the same time either* $\beta(\lambda) \equiv 0$, or $\beta(\lambda) \neq 0$ *for all the* $\lambda \in (\Lambda_1, \Lambda_2)$.

Then there exists such a natural number ν *and such a sequence of the parameter* λ *(eigenvalues):*

$$\lambda_\nu,\ \lambda_{\nu+1},\ \ldots,\ \lambda_{\nu+p},\ \ldots$$

to which belongs the functional sequence (eigenfunctions)

$$y_\nu,\ y_{\nu+1},\ \ldots,\ y_{\nu+p},\ \ldots$$

of such property that $y_{\nu+p} = y(x, \lambda_{\nu+p})$ *is the solution of the differential equation which fulfils the following boundary conditions*

$$\begin{gathered} y(a, \lambda_{\nu+p}) = 0, \\ \alpha_1(\lambda_{\nu+p})\, y(b, \lambda_{\nu+p}) - \alpha(\lambda_{\nu+p})\, y'(b, \lambda_{\nu+p}) = 0, \\ \beta_1(\lambda_{\nu+p})\, y(c, \lambda_{\nu+p}) - \beta(\lambda_{\nu+p})\, y'(c, \lambda_{\nu+p}) = 0 \end{gathered}$$

and $y(x, \lambda_{\nu+p})$ *has in* (a, c) *exactly* $\nu + p$ *points of zero.*

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