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ASYMPTOTIC FORMULAS FOR THE SOLUTIONS OF THE EQUATION $(py')' + qy = 0$

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In the course of preceding EQUADIFF I had the honour of holding the lecture on asymptotic formulas for the solutions of the equation

$$y'' + q(x)\, y = 0 \tag{1}$$

in $J = \langle a, \infty)$ derived by means of the transformation of (1) into

$$Y'' + Q(x)\, Y = 0, \tag{2}$$

and by means of the method of perturbation. The formulas under discussion were of the form

$$y = \varphi(x)\, \{[c_1 + \varepsilon_1(x)]\, U[\Phi(x)] + [c_2 + \varepsilon_2(x)]\, V[\Phi(x)]\}, \tag{3}$$

where U, V denote linear independent solutions of (2), φ, Φ are functions which satisfy certain conditions and ε_1, ε_2 are continuous functions converging to zero for $x \to \infty$.

From the numerical point of view it is necessary to estimate the speed with which the functions ε_1, ε_2 converge to zero or, in particular, to deduce asymptotic formulas concerning these functions or, at last, to approximate the solution y with a given exactness on the entire interval J. One can find in the literature the estimations of functions taken into consideration in very special cases, especially in the case $U = \sin x$, $V = \cos x$. E.g. G. ASCOLI dealt with the majorisation of functions ε_i [1], [2], V. RICHARD derived asymptotic formulas of these functions; a very fine result has also been introduced in the book by G. SANSONE, where the solutions of the equation $y'' + [1 + q(x)]\, y = 0$ were given under suppositions $\int_x^\infty |q(x)|\, dx < \infty$, $\lim_{x\to\infty} q(x) = 0$ by means of infinite series that make possible to approximate the solution with an arbitrary exactness on J. The same problem can be solved under more general suppositions for the equation

$$(p(x)\, y')' + q(x)\, y = 0 \tag{4}$$

when using FUBINI's and PEANO—BAKER's method [8].

(*) $\begin{cases} \textit{Assume } q, Q \in C_0, p, P \in C_1, p > 0, P > 0 \text{ on } J. \textit{ Let } \varphi, \Phi \textit{ be functions} \\ \textit{satisfying the conditions } \varphi, \Phi \in C_2, \varphi > 0, \Phi > 0, \Phi' > 0. \end{cases}$

Let be U, V two independent solutions of the equation

$$(P(x)\, Y')' + Q(x)\, Y = 0 \tag{5}$$

$W = UV' - U'V$. *Put*

$$k = \frac{1}{W(\Phi)} \left[\log \frac{P(\Phi)}{p\varphi^2\Phi'}\right]',$$

$$l = \frac{1}{p\varphi\Phi' W(\Phi)} [(p\varphi')' + q\varphi - p\varphi\Phi'^2 Q(\Phi)\, P^{-1}(\Phi)],$$

$$A = \begin{pmatrix} V(\Phi)\,[l\dot{U}(\Phi) - kU(\Phi)], & V(\Phi)\,[lV(\Phi) - k\dot{V}(\Phi)] \\ U(\Phi)\,[k\dot{U}(\Phi) - lU(\Phi)], & U(\Phi)\,[k\dot{V}(\Phi) - lV(\Phi)] \end{pmatrix}$$

and assume

$$\int_a^\infty |k|\, \{|U(\Phi)| + |V(\Phi)|\}\, \{|\dot{U}(\Phi)| + |\dot{V}(\Phi)|\} < \infty,$$

$$\int_a^\infty |l|\, \{U^2(\Phi) + V^2(\Phi)\} < \infty.$$

Then Equation (4) *has the general solution of the form*

$$y = \varphi(x)\, (U[\Phi(x)],\ V[\Phi(x)]) \sum_0^\infty (-1)^n R^n(x)\, c, \tag{6}$$

$$y' = (\{\varphi(x)\, U[\Phi(x)]\}', \{\varphi(x)\, V[\Phi(x)]\}') \sum_0^\infty (-1)^n R^n(x)\, c$$

where c denotes a constant vector and

$$R^0(x) = \begin{pmatrix} 1,0 \\ 0,1 \end{pmatrix}, \qquad R^n(x) = \int_x^\infty AR^{n-1}.$$

A suitable choice of functions φ, Φ enables us to find not only asymptotic formulas derived by means of these methods, but also to make full use of formulas (6) for the solution of numerical problems mentioned above, and to study asymptotic properties in connexion with the transformation of equation (4).

Equations (4) and (5) being given, one can find a constant $\xi \geq a$ and functions φ, Φ in such a way that all the elements of the matrix A are equal to zero, supposing only, these equations are simultaneously either oscillatory or non-oscillotory on J. yn this case the functions φ, Φ are solutions of the non-linear system

$$\varphi^2\Phi' = \frac{P(\Phi)}{p},$$

$$(p\varphi')' + \left[q - \frac{Q(\Phi)}{P(\Phi)} p\Phi'^2\right] \varphi = 0$$

and formulas (6) have, for $x \geq \xi$, a simple form

$$y = \varphi(x) \{c_1 U[\Phi(x)] + c_2 V[\Phi(x)]\},$$
$$y' = c_1\{\varphi(x) U[\Phi(x)]\}' + c_2\{\varphi(x) V[\Phi(x)]\}'.$$

In concluding I would like to introduce some applications of the preceding theorem when substituing equation (5) by the equation $y'' + \varepsilon y = 0, \varepsilon = 0, \pm 1$ and omitting in formulas (6) all the members from $n = 2$ up to infinity.

In what follows, we suppose that the conditions (*) are valid.

1. *Put*

$$k = (\log p\varphi^2\Phi')', \qquad l = \frac{1}{p\varphi\Phi'} [p\varphi\Phi'^2 - (p\varphi')' - q\varphi], \qquad h = \sqrt{k^2 + l^2}.$$

Let ψ be a function defined by means of the equations

$$\sin \psi = \frac{k}{h}, \qquad \cos \psi = \frac{l}{h} \quad \textit{for } h \neq 0, \qquad \psi = 0 \quad \textit{for } h = 0.$$

Assume furthermore $\int_a^\infty |h| < \infty$. *Then equation* (5) *has the general solution of the form*

$$y = \lambda\varphi(x) \{\sin [\Phi(x) + \alpha] - \int_x^\infty h(t) \sin [\Phi(x) - \Phi(t)] \sin [\Phi(t) - \psi(t) + \alpha] \,\mathrm{d}t\} + \eta_1(x),$$

$$|\eta_1(x)| \leq |\lambda| \varphi(x) \varkappa^2(x) e^{\varkappa(x)}, \qquad \varkappa(x) = 4 \int_x^\infty h.$$

2. *Put*

$$k = \frac{1}{2} (\log p\varphi^2\Phi')', \qquad l = - \frac{1}{2p\varphi\Phi'} [(p\varphi')' + q\varphi + p\varphi\Phi'^2],$$

and suppose

$$\int_a^\infty (|k| + |l|)e^{2\Phi} < \infty.$$

Then equation (4) *has the general solution of the form*

$$y = c_1\varphi(x) \{e^{\Phi(x)} + \int_x^\infty e^{-\Phi(x)}[e^{2\Phi(x)} - e^{2\Phi(t)}] [k(t) - l(t)] \,\mathrm{d}t\} +$$

$$+ c_2\varphi(x) \{e^{-\Phi(x)} + \int_x^\infty e^{\Phi(x)}[e^{-2\Phi(x)} - e^{-2\Phi(t)}] [k(t) + l(t) \,\mathrm{d}t\} + \eta_2(x),$$

$$|\eta_2(x)| \leq 4\, ||c||\, \varphi(x)\, \varkappa^2(x)\, e^{2\varkappa(x) - 3\Phi(x)}, \qquad \varkappa(x) = \int_x^\infty (|k| + |l|)\, e^{2\Phi}.$$

3. *Put*

$$k = (\log p\varphi^2 \Phi')', \qquad l = -\frac{1}{p\varphi\Phi'}\, [(p\varphi')' + q\varphi],$$

and suppose

$$\int_a^\infty (|k|\, \Phi + |l|\, \Phi^2) < \infty.$$

Then equation (4) *has the general solution of the form*

$$y = c_1\varphi(x)\, \{\Phi(x) + \int_x^\infty [\Phi(x) - \Phi(t)]\, [k(t) - l(t)\, \Phi(t)]\, \mathrm{d}t\} +$$

$$+ c_2\varphi(x)\, \{1 - \int_x^\infty [\Phi(x) - \Phi(t)]\, l(t)\, \mathrm{d}t\} + \eta_3(x),$$

$$|\eta_3(x)| \leq 2\, ||c||\, \varphi(x)\, \Phi^{-1}(x)\, \varkappa^2(x) \exp\, \{2\varkappa(x)\, \Phi^{-1}(x)\}\, [1 + \Phi^{-1}(x)],$$

$$\varkappa(x) = \int_x^\infty (|k|\, \Phi + |l|\, \Phi^2).$$

In all the cases **1.**, **2.**, **3.** analogues formulas for the derivatives of solutions are valid.

REFERENCES

[1] Ascoli G., *Sulla forma degli integrali dell'equazione differenziale $y'' + A(x)\, y = 0$ in un caco noterole di stabilità, Revista Universidad nacional del Tucumán. Facultad de ciencias exactas y aplicadas. Tucumán. Ser. A. Matemáticas y fisica teórica 2 (1941), p. 131—140.*

[2] Ascoli G., *Sopra un caso di stabilità per la equazione differenziale $y'' + A(x)\, y = 0$, Annali di Matematica Pura ed Applicata (4) 26 (1947), p. 199—206.*

[3] Richard U., *Sulla risoluzione asintotico-numerica dell'equazione differenziale $(py')' + qy = 0$, Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali 97 (1962/63), p. 857—890.*

[4] Sansone G., *Equazioni differenziali nel campo reale, Bologna 1949.*

[5] Fubini G., *Studi asintotici per alcune equazioni differenziali, Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali, 6 (26) (1937), p. 253—259.*

[6] Peano G., *Integrazione per serie delle equazioni differenziali lineari, Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali 22 (1887), p. 293—302.*

[7] Baker H. F., *On the integration of linear differential equations, Proceedings of the London Mathematical Society 35 (1903), p. 333/378.*

[8] Ráb M., *Les développements asymptotiques des solutions de l'équation $(py')' + qy = 0$, Arch. Math. (Brno), T2 (1966), 1—17.*