EQUADIFF 2

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ON ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF HIGH ORDER

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We shall briefly report here on the ordinary linear differential equations of high order whereby we suppose that the differential equations are of the normal form, and that their coefficients are integrable and bounded, respectively.

If a function satisfies a sequence of such differential equations with increasing order, then this is the property of the function, which is, apart from the differentiability of all orders connected in a certain way with the regularity of the function and it is so in the case of real as well as in the case of complex functions.

Let us begin with the real case and let us suppose the differential equation is in its normal form:

$$y^{(n)} + \sum_{i=0}^{n-1} \sigma_i y^{(i)} = \varphi \tag{1}$$

where the σ_i and φ are L-integrable, for example on the interval $[0, L]$, $L > 0$.

Without loss of generality, we can now limit ourselves to the solution y with

$$y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0, \tag{2}$$

otherwise there is only the right side of (1) correspondingly to modify.

We begin with an estimate:

Let on the interval $[0, L]$

$$S_i = \sup |\sigma_i|, \qquad \Phi = \sup |\varphi|$$

and let L be so small that

$$\sum_{i=0}^{n-1} S_i \frac{L^{n-i}}{(n-i)!} < 1.$$

Then for the solution y with (2) *holds the estimate*[1)]

$$(3) \qquad |y(x)| \leq \frac{\Phi L \dfrac{x^{n-1}}{(n-1)!}}{1 - \sum_{i=0}^{n-1} S_i \dfrac{L^{n-i}}{(n-i)!}} \; .$$

Now for a fixed X is the quotient $\dfrac{X^{n-1}}{(n-1)!} \to 0$ for $n \to \infty$; thus if the σ_i are equibounded for all $i = 1, 2, \ldots,$

$$|\sigma_i| < K$$

and if

$$K(e^L - 1) < 1$$

then for sufficiently great n the solution y with (2) is arbitrarily small. Therefore is holds:

Any differentiable function $f \in C^\infty$ on $[0, L]$ with $f^{(i)}(0) = 0$ for $i = 0, \ldots,$ which is not identically 0 *can not be represented as the solution of linear differential equations of the corresponding high order and of coefficients $< K$.*[2)]

Naturally such a function is not regular.

We can extend this theorem for the functions $f \in C^\infty$ on $[0, L]$ which are not regular with arbitrary initial values $f^{(i)}(0)$.

There are functions $f \in C^\infty$ with $f^{(i)}(0)$ groowing so that

$$\overline{\lim} \sqrt[n]{\left|\frac{f^{(n)}(0)}{n!}\right|} = \infty.$$

For such (surely not regular) functions it can be easily shown that for an increasing sequence of natural numbers $(n_1, n_2, \ldots)$ they can not satisfy a sequence of linear differential equations of the order n_k if k is sufficiently great and the coefficients remain bounded for $x = 0$.

Let us consider therefore only the case of a function $f \in C^\infty$ on $[0, L]$ for which

$$\overline{\lim} \sqrt[n]{\left|\frac{f^{(n)}(0)}{n!}\right|} < \infty.$$

[1)] Proc. Amer. Math. Soc. 17 (1966), 321—324.

[2)] l. c.

Let f satisfy the differential equation

$$y^{(n)} + \sum_{i=0}^{n-1} \sigma_i y^{(i)} = \varphi.$$

If we put

$$P(x) = \sum_{i=0}^{n-1} f^{(i)}(0) \frac{x^{(i)}}{i!}$$

then $\overline{f}(x) = f(x) - P(x)$ satisfies the differential equation

$$\bar{y}^{(n)} + \sum_{i=0}^{n-1} \sigma_i \bar{y}^{(i)} = \varphi - \sum_{i=0}^{n-1} \sigma_i P^{(i)} = \bar{\varphi}$$

with $\bar{y}(0) = \bar{y}'(0) = \ldots = \bar{y}^{(n-1)}(0) = 0$.

Now we can apply to our differential equation with $\bar{y}, \bar{f}$ our above mentioned estimation and we see that $\bar{y}$ by a corresponding $L > 0$ becomes arbitrarily small for all x and sufficiently great n.

We have supposed f to be non-regular at $x = 0$, therefore is for every natural $m > 0$

$$\sup_x \left| f(x) - \sum_0^{m-1} f^{(i)}(0) \frac{x^i}{i!} \right| = h_m > 0$$

and naturally also

$$\inf h_m = h > 0$$

(otherwise f would be represented as a polynomial or a power series).

Then there exists for every m a point x_m with

$$\left| f(x_m) - \sum_0^{m-1} f^{(i)}(0) \frac{x_m^i}{i!} \right| \geq h.$$

On the other hand $\bar{f}$ is arbitrarily small for $n \to \infty$.

So the inequalities

$$|\sigma_i| < K, \qquad |\varphi| < K \qquad \text{and} \qquad n > n(K)$$

can not exist simultaneously for a $n(K)$.

Then f surely does not satisfy a differential equation with bounded coefficients for a sufficiently great order n.[3]

A simple consequence follows from this:

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[3] l. c. 1.) S. 321.

If $f \in C^\infty$ is on the interval $[0, L]$ and if there exists such a sequence of linear differential equations for every sufficiently great order with equibounded coefficients that f satisfies every one of these differential equation, then f is regular at every point.

We will now deal with the regular functions f and we shall consider them either in the real domain or in the complex plane.

There are simple examples of regular functions which can be represented as the solutions of differential equations of an arbitrary high order with bounded coefficients, and there are other examples of such functions which can not be represented as the solutions of such differential equations.

It is known now, that for the regular functions a number can be determined upon which the behaviour relative to the representation as a solution of differential equations of high order depends. This number depends — likewise as the order of an entire function — only on the behaviour of the power series which represents this function.

First we introduce this number:[4]

Let f be regular at the point 0 and let

$$\sup_{n\geq 1} \sqrt[n]{|f^{(n)}(0)|} < \infty,$$

then

$$f(x) = \sum_{0}^{\infty} f^{(n)}(0) \frac{z^n}{n!}$$

is the entire function of an order ≤ 1.

If we put for $n > 0$

$$\sup_{i\geq n} \sqrt[i]{|f^{(i)}(0)|} = A_n(0),$$

then $A_n(0) \geq 0$ is decreasing and thus the sequence $\{A_n(0)\}$ converges.

For $i > 0$ and an arbitrary z

$$\sqrt[i]{|f^{(i)}(z)|} \leq \sqrt[i]{\sum_{n=0}^{\infty} [A_i(0)]^{n+i} \frac{|z|^n}{n!}} = A_i(0) \, . \, e^{\frac{1}{i} . A_i(0) . |z|} \, .$$

Then exists also

$$\sup_{i\geq n} \sqrt[i]{|f^{(i)}(z)|} = A_n(z)$$

[4] Monatshefte für Math. 70 (1966), S. 330—336.

for an arbitrary z and $n > 0$ and

$A_n(z) \leq A_n(0)e^{\frac{1}{n}A_n(0).|z|}$ holds.

For every z exists also

$$\lim A_n(z) = A(z)$$

and following the above mentioned

$$A(0) = A(z)$$

is constant and we denote the constant

$$\lim_n \sup_{i \geq n} \sqrt[i]{|f^{(i)}(z)|} = ||f||$$

as the degree of convergence of f.

For an arbitrary function f regular at the point 0

$$\sup_n \sqrt[n]{|f^{(n)}(0)|} = \infty$$

(this expression is then for every z infinit!)
we put correspondingly $||f|| = \infty$.

Now our problem is: *when can a regular function be represented as a solution of a linear ordinary differential equation*

$$y^{(n)} + \sum_{i=0}^{n-1} \sigma_i y^{(i)} = \varphi$$

of a high order.

We put for the function f

$$S_i = \sqrt[i]{|f^{(i)}(0)|} \qquad i = 1, 2, \ldots \qquad \text{and} \qquad S_0 = |f(0)|$$

and

$$\frac{S_n^n}{1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1}} = q_n$$

We see at once: when the coefficients σ_i and φ at the point $z = 0$ are all absolute $< q_n$

$$|\sigma_i| < q_n, \qquad |\varphi| < q_n \qquad \text{for } z = 0$$

then f does not satisfy our differentialbe equation, because there would be for $z = 0$

$$S_n^n = q_n[1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1}] > |\varphi| + \sum_{i=0}^{n-1} |\sigma_i| \, S_i^i .$$

Now is for our function f, resp. for the sequence of numbers S_i is

$$\overline{\lim}\, q_n = q > 0$$

then there exists a sequence of natural numbers $(n_1, n_2, \ldots)$ with

$$q_{n_k} \rightarrow q$$

and f is surely not the solution of a differential equation

$$y^{(n_k)} + \sum_{i=0}^{n_k-1} \sigma_i^{(n_k)} y^{(i)} = \varphi^{(n_k)}$$

if the coefficients

$$|\sigma_i^{(n_k)}| < \frac{q}{2}, \qquad |\varphi^{(n_k)}| < \frac{q}{2} \qquad \text{for } z = 0$$

for sufficiently great k.

And now we can show:

If the degree of convergence is for our function f, $||f|| > 1$ (also for $||f|| = \infty$) then it is always

$$q = \overline{\lim}\, q_n = \overline{\lim} \frac{S_n^n}{1 + S_1^1 + \ldots + S_{n-1}^{n-1}} > 0.$$

If this would not be the case, then the

$$\frac{S_n^n}{1 + S_1^1 + \ldots + S_{n-1}^{n-1}} = \varepsilon_n \rightarrow 0$$

would converge, but then would be for an n

$$S_n^n = \varepsilon_n(1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1})$$
$$S_{n+1}^{n+1} = \varepsilon_{n+1}(1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1})(1 + \varepsilon_n)$$
$$S_{n+2}^{n+2} = \varepsilon_{n+2}(1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1})(1 + \varepsilon_n)(1 + \varepsilon_{n+1})$$
$$S_{n+m}^{n+m} = \varepsilon_{n+m}(1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1})(1 + \varepsilon_n) \ldots (1 + \varepsilon_{n+m-1})$$

Because of $\varepsilon_K \rightarrow 0$ then it would be for a sufficiently great n

$$S_{n+m}^{n+m} < (1 + S_1^1 + \ldots + S_{n-1}^{n-1})(1 + \eta)^m \cdot \varepsilon_{n+m}$$

with an arbitrary small $\eta > 0$.

On account of $||f|| > 1$ is at least for one sequence $(m_1, m_2, \ldots)$ with an $\varepsilon > 0$ arbitrarily small:

$$S_{n+m_k}^{n+m_k} > (||f|| - \varepsilon)^{m_k}$$

which can not hold with $||f|| > 1$ after what has been mentioned above.

Then it holds: *If $||f|| > 1$, then there exists a sequence of natural numbers $(n_1, n_2, \ldots)$ so that f is surely not the solution of a linear differential equation, of which the coefficients for $z = 0$ absolute $< \frac{q}{2}$ are with an order n_k for every sufficiently great k[5].*

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5) l. c. S. 336.

If finally $||f|| < 1$, *then for a given and sufficiently great* n

$$\sqrt[n]{|f^{(n)}(z)|} < q < 1 \qquad \textit{hence}$$
$$f^{(n)}(z) \to 0 \qquad \text{for} \qquad n \to \infty.$$

Then f *always satisfies a differential equation with arbitrarily small coefficients of sufficiently high order.*

We have for $||f|| = 1$ simple examples as $f(z) = e^z$, for which f satisfies a differential equation of any high order with bounded coefficients, but I do not know an example with $||f|| = 1$, in which this would not be the case.